Examples of Liapunov Functionals for Non-differentiated Equations

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Abstract. Investigators have had little success in applying Liapunov's direct method to integral and other non-differential equations. In this paper we construct Liapunov functionals for four different types of such equations and show how such constructions originate.

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1. Introduction

Liapunov's direct method has proved to be a very natural and effective way to determine qualitative properties of solutions of ordinary, functional, and partial differential equations. But it has proved to be very elusive for functional equations, including integral equations. The problem is that there does not seem to be a natural way to link the Liapunov functional to the functional equation.

In this paper we give several examples and present a first tentative basis for a theory of Liapunov functionals for functional equations. We will show that appropriate Liapunov functionals have, in fact, been right before our eyes for decades, but we did not recognize them.

There are four considerations which are fundamental in the theory and application of Liapunov's direct method for differential equations. All of these seem to be absent to some degree for non-differentiated equations. Those considerations will now be briefly discussed.

(I) Suppose that $x(t)$ is an (unknown) solution of

$$x' = f(t, x) \tag{1}$$

and that $V(t, x)$ is an arbitrary differentiable scalar function. Then $V(t, x(t))$ is a scalar function of t and, by the chain rule,

$$\frac{dV}{dt} = \text{grad } V \cdot x' + \frac{\partial V}{\partial t} = \text{grad } V \cdot f + \frac{\partial V}{\partial t}.$$

Hence, V and f are connected in a natural way. The hope is that properties of V can be determined and then connected back to the solution of (1); that will be (II) below.

Problem. If $x(t)$ is an (unknown) solution of

$$x(t) = a(t) + \int_0^t F(t, s, x(s))ds \tag{2}$$

and if $H(t, x(\cdot))$ is a scalar functional, is there a natural way to connect (2) to the derivative of H?

(II) In the theory of Liapunov's direct method for (1), we seek a function $V(t, x)$ and increasing scalar functions W_i such that

(i) $W_1(|x|) \le V(t, x), V(t, 0) = 0,$

(ii) $V'_{(1)}(t, x) \le -W_2(|x|),$

or

(iii) $V'_{(1)}(t, x) \le p(t, V).$

The crucial property here is (i); it is through $W_1(|x(t)|) \le V(t, x(t))$ that we relate V back to the solution $x(t)$. Either (ii) or (iii) can yield some type of boundedness or stability properties for V, and these are related to $x(t)$ through (i). We will construct Liapunov functionals for (2) which satisfy (ii) very well, but (i) is satisfied only in the most limited sense.

(III) When (II)(ii) holds, then we know that V is bounded. Thus, if $W_1(r) \to \infty$ as $r \to \infty$, then $x(t)$ is bounded (and, hence, defined to ∞) so we can conclude that $\int^\infty W_2(|x(s)|)ds < \infty$; but this will not drive $x(t)$ to zero. The classical device is to ask that $|f(t, x)|$ be bounded for x bounded; this will imply that any bounded solution of (1) is Lipschitz and, hence, by the integral condition, must tend to zero. But boundedness of the right-hand-side of (2) sheds no light on consequences of $\int^\infty W_2(|x(s)|)ds$. Thus, Liapunov theory for (2) must be modified.

(IV) There are very effective Liapunov functionals for (1) which have been used for a century. More importantly, there are converse theorems characterizing stability in terms of Liapunov functions and yielding important perturbation results. But for (2) we seem to have never constructed a Liapunov functional and we do not know what properties converse theorems should address.

In this paper we construct Liapunov functionals for a collection of examples and show how to circumvent some of the problems raised here. We feel that this is a first approximation and look forward to improvements that other investigators will offer.

2. Examples

Each example listed here is given in at least two parts. First we present a functional differential equation with a well-known Liapunov functional, $V(t, x(\cdot))$. From this we extract a functional $H(t, x(\cdot))$ which is a Liapunov functional for an associated functional equation.

Example 1.a. Let r be a positive constant and $b : R \to R$ be continuous with

$$|b(t+r)| - 1 \leq -\alpha < 0$$

for some constant α. Then

$$x'(t) = -x(t) + b(t)x(t-r)$$

has the functional

$$V(t, x_t) := |x(t)| + \int_{t-r}^{t} |b(s+r)x(s)|ds =: |x(t)| + H(t, x_t)$$

and along any solution we have

$$V'(t, x_t) \leq -\alpha|x(t)|.$$

In summary,

(i) $|x(t)| \leq V(t, x_t) \leq (1+r)\|x_t\|$,
(ii) $V'(t, x_t) \leq -\alpha|x(t)|$,

and

(iii) $|f(t, x_t)|$ is bounded for x_t bounded.

Classical results ([7; p. 191]) imply that all solutions are bounded and that $x = 0$ is globally uniformly asymptotically stable.

Proof. We have

$$\begin{aligned} V'(t, x_t) &\leq -|x(t)| + |b(t)x(t-r)| + |b(t+r)x(t)| - |b(t)x(t-r)| \\ &\leq -\alpha|x(t)|, \end{aligned}$$

as required.

Example 1.b. Let $r > 0$, $a : [0, \infty) \to R$ be $L^1[0, \infty)$, and $b(t)$ satisfy the conditions of Example 1.a. If $x(t)$ is any solution of the functional equation

$$x(t) = a(t) + b(t)x(t-r)$$

then the functional

$$H(t, x_t) = \int_{t-r}^{t} |b(s+r)x(s)|ds$$

satisfies

$$H'(t, x_t) \leq |a(t)| - \alpha|x(t)|$$

so $x(t) \in L^1[0, \infty)$.

Proof. From the equation for $x(t)$ we have

$$|x(t)| \leq |a(t)| + |b(t)x(t-r)|$$

so that

$$-|b(t)x(t-r)| \leq |a(t)| - |x(t)|.$$

Next,

$$\begin{aligned} H'(t, x_t) &= |b(t+r)x(t)| - |b(t)x(t-r)| \\ &\leq |b(t+r)x(t)| + |a(t)| - |x(t)| \\ &\leq -\alpha|x(t)| + |a(t)|, \end{aligned}$$

as required. Since $H \geq 0$, if $x(t)$ exists on an interval $[t_0, \infty)$, then $\int_{t_0}^{\infty} |x(t)|dt < \infty$.

Remark. This example has an extremely simple form. Clearly, $H(t, x_t) \to 0$; but how can this be used to show that $x(t)$ behaves in some manner. Certainly, the problem is easy to evaluate in other ways, but this provides a good example of the types of difficulties which must be faced in advancing Liapunov theory to functional equations.

Example 2. In both parts 2.a and 2.b below, we suppose that there is a continuous function $C : R \times R \to R$ and an $M < 1$ with $\int_0^{\infty} |C(u+t, t)|du \leq M$, while $\int_0^t \int_{t-s}^{\infty} |C(u+s, s)|du\, ds$ exists for $t \geq 0$. Suppose also that there is a constant $k > 1$ with $Mk < 1$. Finally, let $g : R \to R$ be continuous with $xg(x) > 0$ if $x \neq 0$. Work of this sort is found in [1], for example, for Part a.

Example 2.a. Let

$$x'(t) = -g(x) + \int_0^t C(t, s)g(x(s))ds$$

and

$$\begin{aligned} V(t, x(\cdot)) :&= |x(t)| + k \int_0^t \int_{t-s}^{\infty} |C(u+s, s)|du|g(x(s))|ds \\ &=: |x(t)| + H(t, x(\cdot)). \end{aligned}$$

Along any solution $x(t)$ we have

$$V'(t,x(\cdot)) \le (-1+Mk)|g(x)| - (k-1)\int_0^t |C(t,s)g(x(s))|ds$$

so that $x(t)$ is bounded, $\int^\infty |g(x(s))|ds < \infty$, and $x(t) \to 0$ as $t \to \infty$.

Proof. We have

$$\begin{aligned} V'(t,x(\cdot)) &\le -|g(x)| + \int_0^t |C(t,s)g(x(s))|ds \\ &\quad + k\int_0^\infty |C(u+t,t)|du|g(x)| - k\int_0^t |C(t,s)g(x(s))|ds \\ &\le (-1+Mk)|g(x)| - (k-1)\int_0^t |C(t,s)g(x(s))|ds \\ &\le -\gamma[|g(x)| + |x'(t)|] \end{aligned}$$

for some $\gamma > 0$, while

$$|x| \le V(t,x(\cdot)).$$

Thus, $x(t)$ is bounded since $V' \le 0$, and $\int^\infty |g(x(t))|dt < \infty$ since $V \ge 0$. As $x'(t) \in L^1[0,\infty)$, it follows that $x(t)$ has finite arc length and, since $\int^\infty |g(x(s))|ds < \infty$, it follows that $x(t) \to 0$.

Remark. If we ask that $\int_0^t |C(t,s)|ds$ is bounded for $t \ge 0$, then bounded solutions are Lipschitz and we would not need to use $\int^\infty |x'(t)|dt < \infty$ to prove that $x(t) \to 0$. In that case, $k = 1$ suffices.

Remark. In Example 2.b we require that $|g(x)| \le |x|$ solely because it is the only way we can see how to relate H' to the integral equation. Is there a clue here to our Problem in (I) of the Introduction? Have we chosen H poorly or is there another way to look at the problem?

Example 2.b. Let $a : R \to R$, $a(t) \in L^1[0,\infty)$, $|g(x)| \le |x|$,

$$x(t) = a(t) + \int_0^t C(t,s)g(x(s))ds,$$

and

$$H(t,x(\cdot)) = k\int_0^t \int_{t-s}^\infty |C(u+s,s)|du|g(x(s))|ds.$$

Then there exists $\delta > 0$ such that

$$H'(t,x(\cdot)) \le -\delta\left[|g(x)| + \int_0^t |C(t,s)g(x(s))|ds\right] + |a(t)| \qquad (*)$$

so $|g(x(t))|$ and $|x(t)|$ are $L^1[0,\infty)$.

Proof. We have

$$|g(x)| \le |x| \le |a(t)| + \int_0^t |C(t,s)g(x(s))|ds$$

and

$$\begin{aligned} H'(t,x(\cdot)) &= k\int_0^\infty |C(u+t,t)|du|g(x)| - k\int_0^t |C(t,s)g(x(s))|ds \\ &\le kM|g(x)| - (k-1)\int_0^t |C(t,s)g(x(s))|ds + |a(t)| - |g(x)| \end{aligned}$$

so (*) holds. Also, $H'(t,x(\cdot)) \le -\delta|x| + (1+\delta)|a(t)|$ from which the conclusion follows.

We now give a corollary which relates this example to familiar aspects of Liapunov's direct method.

Corollary.

(a) If $a(t)$ and $C(t,s)$ satisfy a local Lipschitz condition in t (uniform in s when $0 \le s \le t$) and if $C(t,s)$ is bounded for $0 \le s \le t < \infty$, then $x(t) \to 0$ as $t \to \infty$.

(b) If there is a continuous function $\Phi : [0,\infty) \to [0,\infty)$ with $|C(t,s)| \le \Phi(t-s)$ for $0 \le s \le t < \infty$ and if $a(t)$ and $\Phi(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$.

(c) If there exists $\beta_1 > 0$ such that $|C(t,s)| \ge \beta_1 \int_{t-s}^\infty |C(u+s,s)|du$ for $t \ge 0$, then there exists $\gamma > 0$ with $H'(t,x(\cdot)) \le -\gamma H(t,x(\cdot)) + |a(t)|$.

(d) If there exists $\beta_2 > 0$ such that $|C(t,s)| \le \beta_2 \int_{t-s}^\infty |C(u+t,s)|du$ for $t \ge 0$, then along any solution $(k/\beta_2)\big[|x| - |a(t)|\big] \le H(t,x(\cdot))$.

Proof. If the conditions in (a) hold and if $x(t) \not\to 0$, then there exists $\epsilon > 0$ and $\{t_n\} \uparrow \infty$ with $|x(t_n)| \ge \epsilon$. Let K be the Lipschitz constant for $a(t)$ and $C(t,s)$, $|C(t,s)| \le B$, and consider a sequence $\{s_n\}$ with $t_n \le s_n \le t_n + L$ where L is fixed, but yet to be determined. Then

$$\begin{aligned} |x(t_n) - x(s_n)| &\le |a(t_n) - a(s_n)| \\ &\quad + \Big|\int_0^{t_n} C(t_n,s)g(x(s))ds - \int_0^{s_n} C(t_n,s)g(x(s))ds\Big| \\ &\quad + \Big|\int_0^{s_n} C(t_n,s)g(x(s))ds - \int_0^{s_n} C(s_n,s)g(x(s))ds\Big| \\ &\le K|t_n - s_n| + B\int_{t_n}^{s_n} \big|g(x(s))\big|ds + \int_0^{s_n} K|t_n - s_n|\,|g(x(s))|ds \\ &\le B\int_{t_n}^{s_n} |g(x(s))|ds + K|t_n - s_n|\left(1 + \int_0^{s_n} |g(x(s))|ds\right). \end{aligned}$$

As $n \to \infty$,

$$|x(t_n) - x(s_n)| \le B\epsilon_n + D|t_n - s_n|$$

where $\epsilon_n \to 0$. Hence, there is an $N > 0$ and $L > 0$ such that $n \geq N$ and $|t_n - s_n| \leq L$ imply that $|x(t_n) - x(s_n)| < \epsilon/2$; thus, $|x(t)| \geq \epsilon/2$ for $t_n \leq s \leq t_n + L$, a contradiction to $x \in L^1[0, \infty)$. This proves (a).

To prove (b), note that

$$|x(t)| \leq |a(t)| + \int_0^t \Phi(t-s)|x(s)|ds.$$

The integral is the convolution of an L^1-function ($|x(t)|$) with a function tending to zero; hence, the integral tends to zero.

To prove (c), we note from (*) that

$$\begin{aligned} H'(t, x(\cdot)) &\leq -\delta\beta_1 \int_0^t \int_{t-s}^\infty |C(u+s, s)|du|g(x(s))|ds + |a(t)| \\ &\leq -(\delta\beta_1/k)H(t, x(\cdot)) + |a(t)|, \end{aligned}$$

as required.

We prove (d) by noting that

$$\begin{aligned} H(t, x(\cdot)) &= k \int_0^t \int_{t-s}^\infty |C(u+s, s)|du|g(x(s))|ds \\ &\geq (k/\beta_2) \int_0^t |C(t, s)g(x(s))|ds \\ &\geq (k/\beta_2)\big[|x| - |a(t)|\big] \end{aligned}$$

as required.

Example 3. Throughout all parts of this example and in Theorem 4 we suppose that $D : R \times R \to R$, $D_s(t, s) \geq 0$, $D_{st}(t, s) \leq 0$, $D(t, t-h) = 0$. It is supposed that there is a bounded and continuous initial function $\phi : [t_0 - h, t_0] \to R$. This example is a variant of one by Levin and Nohel [5] for a differential equation which we now review.

Example 3.a. If $x(t)$ is a solution of

$$x'(t) = -\int_{t-h}^t D(t, s)g(x(s))ds$$

then

$$\begin{aligned} V(t, x(\cdot)) &= \int_0^x g(s)ds + \frac{1}{2}\int_{t-h}^t D_s(t, s)\left(\int_s^t g(x(v))dv\right)^2 ds \\ &=: \int_0^x g(s)ds + H(t, x(\cdot)) \end{aligned}$$

satisfies

$$V'(t,x(\cdot)) \le \frac{1}{2}\int_{t-h}^{t} D_{st}(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \le 0.$$

Hence, $x = 0$ is locally stable. If, in addition, $\int_0^x g(s)ds \to \infty$ as $|x| \to \infty$, then there is a wedge with $W(|x|) \le V(t,x)$, $W(r) \to \infty$ as $r \to \infty$; hence, all solutions are bounded.

Proof. We have

$$\begin{aligned} V'(t,x(\cdot)) = &-g(x)\int_{t-h}^{t} D(t,s)g(x(s))ds \\ &+\frac{1}{2}\int_{t-h}^{t} D_{st}(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \\ &-\frac{1}{2}D_s(t,t-h)\left(\int_{t-h}^{t} g(x(v))dv\right)^2 \\ &+\int_{t-h}^{t} D_s(t,s)g(x(t))\int_s^t g(x(v))dv\,ds. \end{aligned}$$

Integration of the last term by parts yields

$$\begin{aligned} g(x(t))&\left[D(t,s)\int_s^t g(x(v))dv\Big|_{s=t-h}^{s=t} + \int_{t-h}^{t} D(t,s)g(x(s))ds\right] \\ &= g(x(t))\int_{t-h}^{t} D(t,s)g(x(s))ds \end{aligned}$$

which cancels the first term in V'.

Example 3.b. If $a(t) \in L^1[0,\infty)$ and is bounded, then for any solution $x(t)$ of

$$x(t) = a(t) - \int_{t-h}^{t} D(t,s)g(x(s))ds$$

the functional

$$H(t,x(\cdot)) = \frac{1}{2}\int_{t-h}^{t} D_s(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds$$

satisfies

$$(a(t)-x(t))^2 \le 2H(t,x(\cdot))\int_{t-h}^{t} D_s(t,s)ds$$

and there are positive constants c_1 and M with

$$H'(t,x(\cdot)) \le -c_1 x(t)g(x(t)) + M|a(t)|.$$

Hence,

$$\int^{\infty} x(t)g(x(t))dt < \infty$$

and if $\int_{t-h}^{t} D_s(t,s)ds$ is bounded then $x(t)$ is bounded. If, in addition, $\int_{t-h}^{t} D_s(t,s)(t-s)ds$ is bounded, and if for each $M > 0$ there is a $J > 0$ such that $|x| \le M$ implies that $|g(x)| \le J|x|$, then $x(t) \to a(t)$ as $t \to \infty$.

Proof. From the previous calculation

$$\begin{aligned} H'(t,x(\cdot)) &\le g(x(t)) \int_{t-h}^{t} D(t,s)g(x(s))ds \\ &= g(x(t))[a(t) - x(t)]. \end{aligned}$$

By the boundedness of $a(t)$ we can find the required positive constants c_1 and M. To obtain the lower bound on H we note that by the Schwarz inequality we have

$$\begin{aligned} &\left(\int_{t-h}^{t} D_s(t,s) \int_s^t g(x(v))dv\, ds \right)^2 \\ &\qquad \le \int_{t-h}^{t} D_s(t,s)ds \int_{t-h}^{t} D_s(t,s) \left(\int_s^t g(x(v))dv \right) ds \\ &\qquad = 2H(t,x(\cdot)) \int_{t-h}^{t} D_s(t,s)ds. \end{aligned}$$

Integration of the term on the left of this inequality by parts yields

$$\begin{aligned} &\left(D(t,s) \int_s^t g(x(v))dv \Big|_{s=t-h}^{s=t} + \int_{t-h}^{t} D(t,s)g(x(s))ds \right)^2 \\ &\qquad = (a(t) - x(t))^2, \end{aligned}$$

as required. From the stated inequality for H', an integration yields H bounded, which, together with $a(t)$ bounded, yields $x(t)$ bounded by considering the lower bound on H. Thus, $|g(x(t))| \le J|x(t)|$ for some $J > 0$. Hence,

$$\int^{\infty} x(t)g(x(t))dt < \infty \text{ yields } \int^{\infty} g^2(x(t))dt < \infty.$$

This means that $\int_{t-h}^{t} g^2(x(v))dv \to 0$ as $t \to \infty$, which we now use. We have (by Schwarz's inequality)

$$\begin{aligned} H(t,x(\cdot)) &\le \int_{t-h}^{t} D_s(t,s)(t-s) \int_s^t g^2(x(v))dv \\ &\le \int_{t-h}^{t} D_s(t,s)(t-s)ds \int_{t-h}^{t} g^2(x(v))dv \end{aligned}$$

which tends to zero since $\int_{t-h}^{t} D_s(t,s)(t-s)ds$ is bounded. This yields the conclusion.

In the next example we will see an easier way to drive $x(t)$ to $a(t)$. Problems of this sort frequently arise in a circuitous route from

$$x'' + f(t,x,x')x' + g(x) = e(t)$$

or

$$u_{tt} = g(u_x(t,x))_x - f(t,x,u,u_t)u_t + e(t).$$

The function $D(t,s)$ is

$$\exp - \int_s^t f(v,x(v),x'(v))dv$$

where $f \geq \beta > 0$. Then, in the first case,

$$D_s(t,s) = f(s,x(s),x'(s))D(t,s)$$

and

$$\begin{aligned} D_{st}(t,s) &= -f(s,x(s),x'(s))f(t,x(t),x'(t))D(t,s) \\ &\leq -\beta D_s(t,s). \end{aligned}$$

Remark. It is to be emphasized that the work in Example 3.c is purely formal. No claim is made that a solution exists. If a solution exists with u_x continuous, then all calculations will hold.

Example 3.c. Let $f : R \times [0,1] \to R$ be C^1, $f(t,0) = f(t,1) = 0$,

$$\begin{cases} u(t,x) = \displaystyle\int_{t-h}^{t} D(t,s)g(u_x(s,x))_x ds + f(t,x), \\ u(t,0) = u(t,1) = 0, \end{cases}$$

and

$$H(t,u(\cdot)) = \int_0^1 \int_{t-h}^{t} D_s(t,s) \left(\int_s^t g(u_x(v,x))_x dv \right)^2 ds\, dx.$$

Then for any solution $u(t,x)$ on an interval $[t_0,\alpha)$ we have

$$\begin{aligned} H'(t,u(\cdot)) \leq &- \int_0^1 g(u_x(t,x))[u_x(t,x) - f_x(t,x)] \\ &+ \int_0^1 \int_{t-h}^{t} D_{st}(t,s) \left(\int_s^t g(u_x(v,x))_x dv \right)^2 ds\, dx \end{aligned}$$

and

$$\int_0^1 (u(t,x) - f(t,x))^2 dx \leq H(t,u(\cdot)) \int_{t-h}^{t} D_s(t,s)ds.$$

Hence, if $\int_0^1 |f_x(t,x)|dx \in L^1[0,\infty)$ and $f_x(t,x)$ is bounded, then $\int_0^1 u_x(t,x)g(u_x(t,x))dx \in L^1[t_0,\alpha)$. Also:

(a) If, in addition, $f(t,x)$ is bounded, then $\int_0^1 u^2(t,x)dx$ is bounded.

(b) If, in addition, $\alpha = \infty$, $D_{st}(t,s) \le -\beta D_s(t,s)$ for some $\beta > 0$, then $\int_0^1 (u(t,x) - f(t,x))^2 dx \to 0$ as $t \to \infty$.

Proof. We have

$$
\begin{aligned}
H'(t,u(\cdot)) \le & \int_0^1 \int_{t-h}^t D_{st}(t,s) \left(\int_s^t g(u_x(v,x))_x dv \right)^2 ds\, dx \\
& + \int_0^1 \int_{t-h}^t D_s(t,s)\, 2 \int_s^t g(u_x(v,x))_x dv\; g(u_x(t,x))_x ds\, dx \\
\le & \int_0^1 \int_{t-h}^t D_{st}(t,s) \left(\int_s^t g(u_x(v,x))_x dv \right)^2 ds\, dx \\
& + \int_0^1 2g(u_x(t,x))_x \left[D(t,s) \int_s^t g(u_x(v,x))_x dv \Big|_{s=t-h}^{s=t} \right. \\
& \left. + \int_{t-h}^t D(t,s) g(u_x(s,x))_x ds \;\right] dx \\
= & \int_0^1 \int_{t-h}^t D_{st}(t,s) \left(\int_s^t g(u_x(v,x))_x dv \right)^2 ds\, dx \\
& + \int_0^1 \int_{t-h}^t 2D(t,s) g(u_x(s,x))_x ds\; g(u_x(t,x))_x dx \\
= & \int_0^1 \int_{t-h}^t D_{st}(t,s) \left(\int_s^t g(u_x(v,x))_x dv \right)^2 ds\, dx \\
& + \int_0^1 2g(u_x(t,x))_x [u(t,x) - f(t,x)] dx.
\end{aligned}
$$

If we integrate the last term by parts and use the boundary condition on both u and f we obtain the stated form for H'. We then find positive constants c_1 and M with

$$
\begin{aligned}
H'(t,u(\cdot)) \le & -c_1 \int_0^1 g(u_x(t,x))u_x(t,x)dx + M \int_0^1 |f_x(t,x)|dx \\
& + \int_0^1 \int_{t-h}^t D_{st}(t,s) \left(\int_s^t g(u_x(v,x))_x dv \right)^2 ds\, dx.
\end{aligned}
$$

As $H \ge 0$ and $D_{st} \le 0$, an integration yields $\int_0^1 g(u_x(t,x))u_x(t,x)dx \in L^1[t_0,\alpha)$.

Next, for the lower bound on H we have

$$\begin{aligned}
\int_0^1 \left(\int_{t-h}^t D_s(t,s) \int_s^t g(u_x(v,x))_x dv\, ds \right)^2 dx& \\
\leq \int_0^1 \int_{t-h}^t D_s(t,s) ds \int_{t-h}^t D_s(t,s) \left(\int_s^t g(u_x(v,x))_x dv \right)^2 ds\, dx& \\
= H(t,u(\cdot)) \int_{t-h}^t D_s(t,s) ds.&
\end{aligned}$$

Integrating the left side of this inequality by parts yields

$$\int_0^1 \left[\int_{t-h}^t D(t,s) g(u_x(s,x))_x ds \right]^2 dx = \int_0^1 (u(t,x) - f(t,x))^2 dx,$$

as required. If (b) holds, then $H'(t,u(\cdot)) \leq -\beta H + \gamma(t)$ where $\gamma \in L^1[0,\infty)$. It follows that $H(t,u(\cdot)) \to 0$ and the result is proved.

3. Concluding Remarks

These examples are intended as an introduction to Liapunov functionals for non-differentiated equations. All of them can be taken farther. At our lecture at the World Congress we also showed that these Liapunov functionals generate a priori bounds for periodic solutions and we presented examples parallel to 3.b and 3.c with infinite delay. That work will be presented in [2]. We believe that an interesting and useful theory of Liapunov's direct method for non-differentiated equations can be constructed and we look forward to watching and participating in its evolution.

Liapunov functions have certainly been applied to integral equations before, but we are unaware that it has been done in the non-differentiated form. Miller [6; p. 337] and Lakshmikantham and Leela [4] apply the theory after differentiating the equation. Gripenberg et al [3; p. 426] may have some such work in mind, for they say that it has no practical interest. We believe that evaluation to be premature.

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