LIÉNARD EQUATIONS, DELAYS, AND HARMLESS PERTURBATIONS

ledahp.tex

T. A. BURTON
Northwest Research Institute
732 Caroline St.
Port Angeles, WA 98362
taburton@olypen.com

Abstract. In this paper we introduce certain expressions as harmless perturbations of stable equations. Using these expressions we show how delays can be ignored, whether they are pointwise, distributed, or infinite. The ideas are illustrated with three delayed Liénard equations and two delayed equations with variable delays and variable coefficients.

Key Words and Phrases: Liénard equation, delays, perturbations
2000 Mathematics Subject Classification: 34K20

1. INTRODUCTION

In this paper we present a technique which allows us to remove the delay in a functional differential equation and replace it with a stable term without delay plus a harmless perturbation. We conjecture that this will be effective in a wide class of problems and we illustrate it here on three delayed Liénard equations, as well as two related equations.

There is a large literature concerning functional differential equations in which each constant function is a solution and each solution approaches a constant. Such equations frequently take the form of

$$x'(t) = g(x(t)) - g(x(t-L)) = \frac{d}{dt}\int_{t-L}^{t} g(x(u))du$$

and distributed delays can also be included, both finite and infinite. These problems were motivated by a study of Cooke and Yorke [10] more than thirty years ago. We pointed out in a recent paper [8] that many such problems are easily solved by means of contraction mappings.

Since the solutions are so nearly constant, it is natural to ask, then, if a properly restricted term

$$g(x(t)) - g(x(t-L))$$

is a harmless perturbation in a general stable equation. To fix ideas let us look at the scalar equation

$$x' = -a(t)x, \quad a(t) \geq 0.$$

The zero solution is stable and all solutions are bounded. Now, let $g : R \to R$ and let $L, K > 0$ with

$$|g(x) - g(y)| \leq K|x - y|, \quad 2LK < 1,$$

and consider

$$x' = -a(t)x + g(x) - g(x(t-L)) = -a(t)x + \frac{d}{dt}\int_{t-L}^{t} g(x(s))ds.$$

It is easy to show using contraction mappings that the zero solution is stable. Moreover, if $\int_0^\infty a(s)ds = \infty$, then all solutions tend to zero. Notice that L is not particularly small and g has neither sign condition nor monotonicity condition. These are conditions frequently found in the literature concerning attempts to ignore the delay. As a concrete example, the weak term $-\frac{x(t)}{t+1}$ is able to overpower the robust terms $.4x(t) - .4x(t-1)$ and bring all solutions of

$$x' = -\frac{x(t)}{t+1} + .4x(t) - .4x(t-1)$$

to zero.

Using this idea we may frequently be able to ignore the delay in an equation for stability analysis. For example, we might be able to study

$$x' = -g(x(t-L))$$

by studying

$$x'(t) = -g(x(t-L)) + g(x(t)) - g(x(t)) = -g(x(t)) + \frac{d}{dt}\int_{t-L}^{t} g(x(s))ds$$

and ignore all except

$$x'(t) = -g(x(t)).$$

The idea for transforming the equation into a neutral equation seems to go back at least to [5] and was used throughout [6]. But here we hope to actually ignore the delay whenever the neutral term gives rise only to solutions approaching constants.

Recently [9] we began testing this idea by studying eight classical scalar equations using contraction mappings. In the present paper we focus on three Liénard type equations with constant delays, distributed delays, and infinite delays, as well as two problems with variable coefficients and variable delays. In contrast to the aforementioned paper, this work is based on Liapunov's direct method.

The basic thesis here is that in all of the papers [1-3,8,10-16] following the original problem of Cooke and Yorke there are functions which are parallel to $g(x) - g(x(t-L)$, and such expressions may very well be harmless perturbations. This paper is a second demonstration of those ideas.

2. LIÉNARD EQUATIONS WITH DELAY.

We consider three Liénard equations with delays and show that they can be treated in a unified manner using the idea from the Cooke-Yorke work. Consider the equations

$$x'' + f(x)x' + g(x(t-L)) = 0, \tag{1}$$

$$x'' + f(x)x' + \int_{t-L}^{t} p(s-t)g(x(s))ds = 0, \tag{2}$$

with

$$\int_{-L}^{0} p(s)ds = 1, \quad \int_{-L}^{0} |p(s)|ds =: K,$$

and

$$x'' + f(x)x' + \int_{-\infty}^{t} q(s-t)g(x(s))ds = 0, \tag{3}$$

with

$$\int_{-\infty}^{0} q(s)ds = 1, \quad \int_{-\infty}^{0}\int_{-\infty}^{v} |q(u)|dudv =: D$$

where L, K, D are all finite positive numbers. A change of variable will show that each of these equations is an autonomous functional differential equation; hence, stability implies uniform stability and asymptotic stability implies uniform asymptotic stability. We generally think of f as being the damping and g as being the restoring force. We ask that

$$f(x) \geq 0, \quad xg(x) > 0 \quad \text{if} \quad x \neq 0 \tag{4}$$

and denote

$$G(x) = \int_{0}^{x} g(s)ds, \quad F(x) = \int_{0}^{x} f(s)ds. \tag{5}$$

Existence theory is found in Chapter 3 of [7]. The conditions given here are adequate for existence for a given continuous initial function.

The literature concerning (1) is massive when $L = 0$, while much work has also been done when $L > 0$, particularly by Zhang [17-20]. Some of his results are both necessary and sufficient for boundedness and stability.

It is assumed that f and g are continuous and in (3) it is convenient we ask that g be just a bit more than continuous so as to satisfy the following two conditions.

$$\text{If} \quad G(x) \quad \text{is bounded for} \quad x \geq 0, \quad \text{then} \quad g(x) \quad \text{is bounded for} \quad x \geq 0. \tag{6}$$

$$\text{If} \quad G(x) \quad \text{is bounded for} \quad x \leq 0, \quad \text{then} \quad g(x) \quad \text{is bounded for} \quad x \leq 0. \tag{7}$$

While these conditions are mild, they will prove to simplify analysis of (3) and to be of interest in themselves. We had asked that (5) and (6) also hold for (1) and (2), but Professor Bo Zhang privately communicated an idea which made them unnecessary.

It is known that when (4) holds with $f(0) > 0$ and $L = 0$ then the zero solution of (1) is globally asymptotically stable if and only if

$$\int_{0}^{\pm\infty} [f(x) + |g(x)|]dx = \pm\infty. \tag{8}$$

That condition was derived in [4] and has played a central role in investigations of all these equations.

3. SETTING UP THE PROBLEMS.

We begin with (1) and put it into the context discussed in the introduction. Subtract and add $g(x)$ to (1) so that we can write it as

$$x'' + f(x)x' - \frac{d}{dt}\int_{t-L}^{t} g(x(s))ds + g(x) = 0.$$

Then write it as the Liénard system

$$\begin{aligned} x' &= y - F(x) + \int_{t-L}^{t} g(x(s))ds \\ y' &= -g(x). \end{aligned}$$

Next, write

$$\begin{aligned} x' &= y - F(x) + Lg(x) - Lg(x) + \int_{t-L}^{t} g(x(s))ds \\ y' &= -g(x). \end{aligned}$$

We are now in a position to identify the terms discussed in the introduction. Separate that system into two parts as follows. The first part consists of

$$\begin{aligned} x' &= y - F(x) + Lg(x) \\ y' &= -g(x). \end{aligned}$$

Note that under (4) the zero solution is asymptotically stable in case there is a $\delta > 0$ such that

$$Lg^2(x) - g(x)F(x) < 0 \quad \text{for} \quad 0 < |x| < \delta, \tag{9}$$

as may be argued without difficulty using the well-known Liapunov function

$$V(x, y) = y^2 + 2G(x).$$

The second part consists of $-Lg(x) + \int_{t-L}^{t} g(x(s))ds$ which has the form of a harmless perturbation, as discussed in the introduction since each constant is a solution of

$$x' = -Lg(x) + \int_{t-L}^{t} g(x(s))ds = -\frac{d}{dt}\int_{-L}^{0}\int_{t+s}^{t} g(x(u))duds$$

and we can use the fixed point techniques of [8] to derive conditions ensuring that each solution approaches a constant.

Our discussion in the introduction now tells us that (1) should be asymptotically stable when (4) and (9) hold. Moreover, the same type of preparation should work for (2) and (3). The perturbation $-Lg(x) + \int_{t-L}^{t} g(x(s))ds$ can be ignored. We had set up eight classical problems in the same way in [9] using fixed point theory. Here, we take advantage of extensive Liapunov theory available.

The reader may compare our first result with work of Zhang [17, 19] concerning (1). He obtained several results on boundedness and stability by asking for an $N > 1$ with $LNg^2(x) - g(x)F(x) < 0$ on certain intervals in order to get stability and boundedness. As mentioned before, some of his results are both necessary and sufficient. But our technique is sharper in that $N = 1$ suffices, while yielding the idea that (9) is the condition needed. Finally, this presentation readily covers the more sophisticated problems (2) and (3), not treated in [17-20].

Actually, (9) can be weakened further throughout the paper. We can change (9) to $Lg^2(x) - g(x)F(x) \le 0$ for $0 < |x| < \delta$ and then argue that the limit set of a solution does not intersect any value of x for which that expression is negative. We avoid such arguments here because of the length.

4. STABILITY RESULTS.

We begin with the result for (1).

Theorem 1. Let (4) hold for (1) and suppose there is a $\delta > 0$ with

$$Lg^2(x) - g(x)F(x) < 0 \quad \text{for} \quad 0 < |x| < \delta. \tag{9}$$

Then the zero solution of (1) is asymptotically stable. If (4), (8), and (9) hold with $\delta = \infty$, then the zero solution of (1) is asymptotically stable in the large.

Proof. As in the last section, subtract and add $g(x)$ to (1) so that we can write it as

$$x'' + f(x)x' - \frac{d}{dt}\int_{t-L}^{t} g(x(s))ds + g(x) = 0.$$

Then write it as the Liénard system

$$\begin{aligned} x' &= y - F(x) + \int_{t-L}^{t} g(x(s))ds \\ y' &= -g(x). \end{aligned}$$

Define the Liapunov functional

$$V(t, x(\cdot), y(t)) = (1/2)y^2 + G(x) + (1/2)\int_{-L}^{0}\int_{t+v}^{t} g^2(x(u))dudv$$

whose derivative along solutions of the system satisfies

$$\begin{aligned} V'(t, x(\cdot), y(t)) &= g(x)y - g(x)F(x) + g(x)\int_{t-L}^{t} g(x(s))ds \\ &\quad + (1/2)\int_{-L}^{0} [g^2(x(t)) - g^2(x(t+v))]dv - g(x)y \\ &\le -g(x)F(x) + (1/2)\int_{t-L}^{t} [g^2(x(t)) + g^2(x(s))]ds \\ &\quad + (L/2)g^2(x(t)) - (1/2)\int_{t-L}^{t} g^2(x(v))dv \\ &= -g(x)F(x) + Lg^2(x) \\ &< 0 \quad \text{for} \quad 0 < |x| < \delta \end{aligned}$$

by (9). From V and V' it follows that the zero solution is stable.

Let $(x(t), y(t))$ be any fixed solution of the system remaining in a region in which $|x(t)| < \delta$. Clearly, $y(t)$ is bounded. Let $V(t) := V(t, x(\cdot), y(t))$ and note that $V'(t) \leq 0$ so $V(t) \to c$, where c is a non-negative constant. If $c = 0$ then the solution tends to $(0, 0)$. If $c > 0$, then it follows readily that $x(t) \to 0$ as $t \to \infty$; thus, $V(t) \to y^2(t)/2 \to c$. Assume that $y(t) \to \sqrt{2c}$. Then for large t we have $x'(t) \geq \sqrt{c}$, a contradiction to $x(t) \to 0$. As V is positive definite, we have shown that the zero solution is asymptotically stable.

Next, if $\delta = \infty$ and if $G(x) \to \infty$ as $|x| \to \infty$ then all solutions are bounded and, hence, converge to zero. Now, if $G(x)$ is bounded on the right and if $x(t)$ is any fixed solution, then it is clear from the Liapunov functional that $y(t)$ is bounded, say $|y(t)| \leq B_1$. By (8) we can conclude that $F(x) \to \infty$ with x. From $y' = -g(x)$ we obtain

$$|\int_{t-L}^{t} g(x(s))ds| \leq |y(t)| + |y(t-L)| \leq 2B_1.$$

Hence, we can find x_1 so large that $x'(t) < 0$ if $x(t) \geq x_1$; therefore, the solution is bounded on the right. A similar argument shows that $x(t)$ is bounded on the left. By the above argument, the solution tends to zero. This completes the proof.

Remark. The next result contains a very interesting assumption. We require $xg(x) > 0$ but the weight $p(t)$ can change sign, so long as it is positive on average.

Theorem 2. Let (4) hold for (2) and suppose there is a $\delta > 0$ such that

$$KLg^2(x) < g(x)F(x) \quad \text{if} \quad 0 < |x| < \delta. \tag{10}$$

Then the zero solution of (2) is asymptotically stable. If, in addition, (8) holds and $\delta = \infty$ then the zero solution of (2) is asymptotically stable in the large.

Proof. Add and subtract $g(x)$ to (2) so that it can be written as

$$x'' + f(x)x' - \frac{d}{dt}\int_{-L}^{0} p(s) \int_{t+s}^{t} g(x(u))duds + g(x) = 0.$$

Then write it as a system

$$\begin{aligned} x' &= y - F(x) + \int_{-L}^{0} p(s) \int_{t+s}^{t} g(x(u))duds \\ y' &= -g(x). \end{aligned}$$

Define a Liapunov functional by

$$V(t, x(t), y(t)) = (1/2)y^2 + G(x) + (K/2)\int_{-L}^{0}\int_{t+v}^{t} g^2(x(u))dudv$$

so that if we denote the last term by Y, then the derivative along a solution of the system satisfies

$$\begin{aligned}
V' &= yg(x) - g(x)F(x) + g(x)\int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))duds - yg(x) + Y' \\
&\leq -g(x)F(x) + |g(x)|\int_{-L}^{0} |p(s)|ds\int_{t-L}^{t} |g(x(u))|du + Y' \\
&\leq -g(x)F(x) + (K/2)\int_{-L}^{0} [g^2(x(t)) - g^2(x(t+v))]dv \\
&+ (K/2)\int_{t-L}^{t} (g^2(x(t)) + g^2(x(u)))du \\
&\leq -g(x)F(x) + KLg^2(x) < 0
\end{aligned}$$

if $0 < |x| < \delta$. The same arguments given in the proof of the last theorem now complete the proof here.

To prepare for (3) we notice that

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)dug(x(s))ds \\
&= \int_{-\infty}^{0} q(u)dug(x) - \int_{-\infty}^{t} q(s-t)g(x(s))ds \\
&= g(x(t)) - \int_{-\infty}^{t} q(s-t)g(x(s))ds.
\end{aligned}$$

While (6) and (7) are not essential, they are mild and greatly simplify the proof of the next result.

Theorem 3. Let (4) hold for (3) and suppose there is a $\delta > 0$ such that

$$-g(x)F(x) + Dg^2(x) < 0 \quad \text{for} \quad 0 < |x| < \delta. \tag{11}$$

Then the zero solution of (3) is asymptotically stable. If, in addition, (6)-(8) hold and $\delta = \infty$, then the zero solution of (3) is globally asymptotically stable.

Proof. Write (3) as

$$x'' + f(x)x' - \frac{d}{dt}\int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)dug(x(s))ds + g(x) = 0$$

and then as the system

$$\begin{aligned}
x' &= y - F(x) + \int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)dug(x(s))ds \\
y' &= -g(x).
\end{aligned}$$

Define

$$V(t,x,y) = (1/2)y^2 + G(x) + (1/2)\int_{-\infty}^{t}\int_{t-s}^{\infty}\int_{-\infty}^{-v}|q(u)|dudvg^2(x(s))ds$$

and call the last term Y. Then the derivative of V along the system satisfies

$$\begin{aligned}
V' &= yg(x) - g(x)F(x) + g(x)\int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)dug(x(s))ds - yg(x) + Y' \\
&\leq -g(x)F(x) + (1/2)\int_{-\infty}^{t}\int_{-\infty}^{s-t}|q(u)|du(g^2(x(t)) + g^2(x(s)))ds + Y' \\
&= -g(x)F(x) + (1/2)g^2(x)\int_{-\infty}^{t}\int_{-\infty}^{s-t}|q(u)|duds + (1/2)\int_{-\infty}^{t}\int_{-\infty}^{s-t}|q(u)|dug^2(x(s))ds \\
&+ (1/2)\int_{0}^{\infty}\int_{-\infty}^{-v}|q(u)|dudvg^2(x(t)) - (1/2)\int_{-\infty}^{t}\int_{-\infty}^{s-t}|q(u)|dug^2(x(s))ds \\
&= -g(x)F(x) + (1/2)(\int_{0}^{\infty}\int_{-\infty}^{-v}|q(u)|dudv + \int_{-\infty}^{t}\int_{-\infty}^{s-t}|q(u)|duds)g^2(x(t)) \\
&= -g(x)F(x) + Dg^2(x) < 0
\end{aligned}$$

for $0 < |x| < \delta$.

The remainder of the proof is just as that of Theorem 1 until we reach the last paragraph. We proceed as follows.

If $\delta = \infty$ and if $G(x) \to \infty$ as $|x| \to \infty$ then all solutions are bounded and, hence, converge to zero. If $G(x)$ is bounded on the right and if $x(t)$ is any fixed solution, then it is clear from the Liapunov functional that $y(t)$ is bounded and by (6) $g(x(t))$ is bounded. By (8) we can then find x_1 so large that $x'(t) < 0$ if $x(t) \geq x_1$; hence the solution is bounded on the right. A similar argument shows that $x(t)$ is bounded on the left. By the argument given in the proof of Theorem 1, the solution tends to zero. This completes the proof.

5. VARIABLE DELAY

.

We look now at problems which are parallel to Liénard equations to see if our results can be extended to similar problems in which we can not locate a harmless perturbation quite so simply. Consider the scalar equation

$$x'' + f(x)x' + a(t)g(x(q(t))) = 0 \tag{11}$$

where $f : R \to [0,\infty)$ is continuous, $q : [0,\infty) \to R$ is continuous, strictly increasing, and $q(t) < t$. Let $h(t)$ be the inverse of q and let $a : [0,\infty) \to (0,\infty)$ be bounded and continuous. Assume that $g : R \to R$ is continuous,

$$xg(x) > 0 \tag{12}$$

for $x \neq 0$,

$$(a(h(t))h'(t))' > 0, \tag{13}$$

there is an $r > 0$ with

$$q(t) \geq t - r, \tag{14}$$

and for $F(x) = \int_0^x f(s)ds$ we suppose that there is a $\delta > 0$ for which

$$-g(x)F(x) + (1/2)\sup_{t\geq 0}[a(h(t))h'(t)r + \int_t^{h(t)} a(s)ds]g^2(x) < 0 \tag{15}$$

if $0 < |x| < \delta$. Note that (15) will require that $a(h(t))h'(t)$ be bounded above. A calculation will show that if $q(t) = t - L$ and if $a(t) = 1$ then (15) and (9) are the same.

Theorem 4. Let (12)-(15) hold. Then the zero solution of (11) is asymptotically stable. If (8) holds and if (15) holds for $\delta = \infty$, then the zero solution is globally asymptotically stable.

Proof. We can rewrite (11) as

$$x'' + f(x)x' + a(h(t))h'(t)g(x) + \frac{d}{dt}\int_{h(t)}^t a(s)g(x(q(s)))ds = 0. \tag{16}$$

Then write it as the system

$$\begin{aligned} x' &= y - F(x) - \int_{h(t)}^t a(s)g(x(q(s)))ds \\ y' &= -a(h(t))h'(t)g(x). \end{aligned}$$

We will define a Liapunov functional in two steps. First, let

$$V_1(t, x, y) = \frac{y^2}{2a(h(t))h'(t)} + G(x) \tag{17}$$

so that the derivative of V along a solution of the system satisfies

$$V' = g(x)y - g(x)F(x) - g(x)\int_{h(t)}^t a(s)g(x(q(s)))ds - g(x)y - \frac{(a(h(t))h'(t))'y^2}{2(a(h(t))h'(t))^2}. \tag{18}$$

We will make a change of variable. Let $w = q(s)$ so that $s = h(w)$, $ds = h'(w)dw$, and $s = h(t)$ implies $w = q(h(t)) = t$, while $s = t$ implies that $w = q(t)$. Thus,

$$\int_{h(t)}^t a(s)g(x(q(s)))ds = \int_t^{q(t)} a(h(w))g(x(w))h'(w)dw.$$

Since $q(t) \geq t - r$ we have

$$\int_{t-r}^t a(h(w))h'(w)g^2(x(w))dw \geq \int_t^{h(t)} a(s)g^2(x(q(s)))ds.$$

Next, define the second part of the Liapunov functional by

$$2V_2(t,x(\cdot)) = \int_{-r}^{0}\int_{t+s}^{t} a(h(w))h'(w)g^2(x(w))dwds \tag{19}$$

so that the derivative of V_2 along a solution of the system is

$$\begin{aligned} 2V_2' &= \int_{-r}^{0} [a(h(t))h'(t)g^2(x(t)) - a(h(t+s))h'(t+s)g^2(x(t+s))]ds \\ &= a(h(t))h'(t)rg^2(x) - \int_{t-r}^{t} a(h(s))h'(s)g^2(x(s))ds. \end{aligned} \tag{20}$$

Hence,

$$\begin{aligned} (V_1+V_2)' &\le -g(x)F(x) + (1/2)\int_{t}^{h(t)} a(s)[g^2(x(t)) + g^2(x(q(s)))]ds \\ &\quad + (1/2)a(h(t))h'(t)rg^2(x) - (1/2)\int_{t-r}^{t} a(h(s))h'(s)g^2(x(s))ds \\ &\le -g(x)F(x) + (1/2)[a(h(t))h'(t)r + \int_{t}^{h(t)} a(s)ds]g^2(x). \end{aligned} \tag{21}$$

By assumption (15), that is a negative definite function of x on $0 < |x| < \delta$. The remainder of the proof is just as before.

6. ANOTHER CLASSICAL PROBLEM.

There is a very interesting classical problem concerning the equation

$$x'' + a(t)x(t) = 0$$

in which $a(t)$ is positive and increases monotonically. It is readily shown that each solution is bounded; there is then the problem of showing that solutions tend to zero. We show that the ideas here will work on considerable more general problems.

Our interest centers on the scalar equation

$$x'' + a(t)g(x(q(t))) = 0 \tag{22}$$

in which $a : [0,\infty) \to (0,\infty)$ is differentiable, $q(t) = t - r(t)$ where $r : [0,\infty) \to (0,\infty)$ is continuous and q is strictly increasing with inverse function $h(t)$. We suppose that

$$(a(h(t))h'(t))' \ge 0 \tag{23}$$

and that for

$$\alpha(t) := \int_{t}^{h(t)} a(s)ds \tag{24}$$

then

$$\int_0^\infty \alpha(s)ds = \beta. \tag{25}$$

It is also supposed that $xg(x) > 0$ for $x \neq 0$, that g is odd and increasing, and sometimes that

$$\int_0^x g(s)ds/\beta g^2(x) \to \infty \quad \text{as} \quad |x| \to \infty. \tag{26}$$

Theorem 5. Let (23) - (26) hold. Then every solution of (22) is bounded. If $g(x) = x$, then (26) can be replaced by $\beta < 1/2$.

Proof. The proof is of Razumikhin type. Write (22) as

$$x'' + \frac{d}{dt}\int_{h(t)}^t a(s)g(x(q(s)))ds + a(h(t))h'(t)g(x(t)) = 0$$

and then as the system

$$\begin{aligned} x' &= y - \int_{h(t)}^t a(s)g(x(q(s)))ds \\ y' &= -a(h(t))h'(t)g(x). \end{aligned}$$

Define a Razumikhin function

$$V(t,x,y) = \int_0^x g(s)ds + \frac{y^2}{2a(h(t))h'(t)} \tag{27}$$

with derivative along a solution of the system satisfying

$$V' = -g(x)\int_{h(t)}^t a(s)g(x(q(s)))ds - \frac{(a(h(t))h'(t))'y^2}{2(a(h(t))h'(t))^2}. \tag{28}$$

For a given solution $(x(t), y(t))$ suppose that t is any value with $|x(t)| \geq |x(s)|$ for $0 \leq s \leq t$. Integrate V', use the monotonicity and oddness of g, and obtain

$$\int_0^{x(t)} g(s)ds \leq V(t) \leq V(0) + g^2(x(t))\beta.$$

By (29) this shows that $x(t)$ is bounded.

Acknowledgment. We thank Professor Bo Zhang for reading the manuscript and making a number of helpful suggestions and corrections.

REFERENCES

[1] O. Arino and M. Pituk, Asymptotic constancy for neutral functional differential equations, *Differential Equations Dynam. Systems* **6**(1998), 261-273.
[2] F. Atkinson and J. Haddock, Criteria for asymptotic constancy of solutions of functional differential equations, *J. Math. Anal. Appl.* **91**(1983), 410-423.
[3] F. Atkinson, J. Haddock, and O. Staffans, Integral inequalities and exponential convergence of differential equations with bounded delay, in "Ordinary and Partial Differential Equations: Proceedings, Dundee, Scotland 1982, "Lecture Notes in Mathematics: Edited by W. Everitt and B. Sleeman, Springer, New York, 1982, pp. 56-68.
[4] T. A. Burton, The generalized Liénard equation, *J. SIAM on Control* **5**(1965), 223-230.
[5] T. A. Burton, Volterra equations with small kernels, *J. Integral Equations* **5**(1983), 271-285.
[6] T. A. Burton, *Volterra Integral and Differential Equations*, Academic Press, Orlando, 1983.
[7] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, 1985.
[8] T. A. Burton, T. A., Fixed points and differential equations with asymptotically constant or periodic solutions, *Electronic J. Qualitative Theory of Diff. Equ.*, **11** (2004), pp. 1-31. www.math.u-szeged.hu/ejqtde/2004/200411.html
[9] T. A. Burton, Fixed points, stability, and harmless perturbations, *Fixed Point Theory and Applications*, to appear.
[10] Kenneth L.Cooke, and James A. Yorke, Some equations modelling growth processes and gonorrhea epidemics, *Math. Biosciences* **16**(1973), 75-101.
[11] J. R. Haddock, Functional differential equations for which each constant function is a solution: A narrative, *Proc. Eleventh Int. Conf. on Nonlinear Oscillations*, M. Farkas, V. Kertesz, and G. Stepan, eds., Janos Bolyai Math. Soc., Budapest, 1987, pp. 86-93.
[12] J. R. Haddock, T. Krisztin, J. Terjeki, and J. H. Wu, An invariance principle of Lyapunov-Razumikhin type for neutral functional differential equations, *J. Differential Equations* **107**(1994), 395-417.
[13] John Haddock and J. Terjeki, On the location of positive limit sets for autonomous functional-differential equations with infinite delay, *J. Differential Equations* **86**(1990), 1-32.
[14] C. Jehu, Comportement asymptotique des solutions de l'equation $x'(t) = -f(t, x(t)) + f(t, x(t-1)) + h(t)$, *Annales de la Societe Scientifique de Bruxelles*, **92**, IV (1978), 263-269.
[15] James L. Kaplan, M. Sorg, and James A. Yorke, Solutions of $x'(t) = f(x(t), x(t-L))$ have limits when f is an order relation, *Nonlinear Analysis* **3**(1979), 53-58.
[16] Tibor Krisztin and Jianhong Wu, Asymptotic behaviors of solutions of scalar neutral functional differential equations, *Differential Equations Dynam. Systems* **4**(1996), 351-366.
[17] Bo Zhang, On the retarded Liénard equation, *Proc. Amer. Math. Soc.* **115** (1992), 779-783.
[18] Bo Zhang, Boundedness and stability of solutions of the retarded Liénard equation with negative damping, *Nonlinear Analysis* **20** (1993), 303-313.
[19] Bo Zhang, Necessary and sufficient conditions for boundedness and oscillation in the retarded Liénard equation, *J. Math. Anal. Appl.* **200** (1996), 453-473.
[20] Bo Zhang, Periodic solutions of the retarded Liénard equation, *Ann. Mat. pura appl.* **CLXXII** (1997), 25-42.