INTEGRAL EQUATIONS, LARGE FORCING, STRONG RESOLVENTS

T. A. BURTON

Northwest Research Institute
732 Caroline St.
Port Angeles, WA 98362
taburton@olypen.com

Dedicated to Prof. I. Rus on his 70th birthday.

ABSTRACT. We consider an integral equation $x(t) = a(t) - \int_0^t C(t,s)x(s)ds$, a resolvent $R(t,s)$, and a variation of parameters formula $x(t) = a(t) - \int_0^t R(t,s)a(s)ds$. We present three general results concerning the behavior of solutions based on the signs of $C(t,s)$ and its derivatives, in conjunction with the magnitude of $a(t)$ or of $a'(t)$. The first result shows that $(x(t) - a(t))^2$ is bounded even when $a(t)$ is unbounded. The second result, under a different set of conditions on the derivatives of C, shows that $x(t)$ is bounded when a' is bounded.

Classical theory favors the idea that $x(t)$ follows $a(t)$ when the kernel is nice. In three recent papers we disputed this, providing many results in which $\int_0^t R(t,s)a(s)ds$ faithfully duplicates $a(t)$, even when $a(t)$ is unbounded, resulting in $x(t)$ being bounded and, most interestingly, bearing absolutely no relation to $a(t)$. Most of this set of results was based on certain smallness conditions on $C(t,s)$. By taking into account sign conditions on C and its derivatives we find that both views can be defended even when $a(t)$ is unbounded.

There is now ample evidence that there are two strikingly different theories about the relation of the solution to $a(t)$. These should be very important and rewarding areas for investigation.

Keywords: Integral equations, resolvents, Liapunov functions

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1. Introduction

We consider a scalar integral equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds \tag{1}$$

where $a(t)$ and $C(t,s)$ are at least continuous, together with a resolvent equation

$$R(t,s) = C(t,s) - \int_s^t R(t,u)C(u,s)du \tag{2}$$

with solution $R(t,s)$, and variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{3}$$

These formulas may be found in Burton [2; Chapter 7] or Miller [11; p. 190], for example. General theory of integral equations from several different points of view is found in Corduneanu [8], Burton [2], Grippenberg-Londen-Staffans [9], and Miller [11], for example. The work of Ritt [13] shows how complicated $R(t,s)$ can be.

We have four long-term projects (see [4, 6, 7]) which we continue to refine and the two which are studied here can be described as follows:

(I) Determine conditions on $C(t,s)$ to ensure that

$$\sup_{t\geq 0} \int_0^t |R(t,s)|ds < \infty. \tag{*}$$

There is a result in Burton [4] showing that (*) holds if and only if every solution of (1) is bounded for every bounded and continuous function $a(t)$. Many sufficient conditions for this are found in [4, 6, 7] and we provide two more in this paper.

(II) Determine conditions on C and corresponding vector spaces of continuous functions $a : [0,\infty) \to R$ for which the solution $x(t)$ of (1) is at least bounded, even when $a(t)$ is unbounded.

The conditions for (II) to hold usually ask that $a' \in L^p$. The applied mathematician will correctly claim that uncertainties and even stochastic elements make this difficult to establish. But those difficulties tend to vanish when we use (I) and (II) together.

The tactic then is as follows. Fix $C(t,s)$ and obtain (*) for a problem of interest. This is done using small functions $a(t)$ which may be quite simple. Now our real problem of interest is $x(t) = b(t) - \int_0^t C(t,s)x(s)ds$, where $b(t)$ may be a large and badly behaved function. Select a nice function, $a(t)$, which is close to $b(t)$ and satisfies one of our subsequent results; that is, the solution of $y(t) = a(t) - \int_0^t C(t,s)y(s)ds$ is at least bounded. We suppose that there is a $K > 0$ with $|a(t) - b(t)| \leq K$ for all $t \geq 0$. Then using the same $R(t,s)$ which depends only on $C(t,s)$ we have

$$x(t) = b(t) - \int_0^t R(t,s)b(s)ds$$

and

$$y(t) = a(t) - \int_0^t R(t,s)a(s)ds.$$

Now,

$$\begin{aligned} |x(t) - y(t)| &\le |a(t) - b(t)| + \int_0^t |R(t,s)||a(s) - b(s)|ds \\ &\le K[1 + \sup_{t\ge 0} \int_0^t |R(t,s)|ds. \end{aligned}$$

Thus, our results here may seem to demand much from $a(t)$, but when (*) holds they apply to a much larger class of functions.

In this paper we will offer new Liapunov techniques to produce conditions to ensure strong qualitative properties of solutions. The following examples from [4, 6, 7] will not only solidify the ideas, but will serve as a start for the present project. The first three results are of classical type. In each of those three results it can be effectively argued that the solution, $x(t)$, follows $a(t)$ in the sense that $|x(t) - a(t)|$ is bounded or in L^p. That sharply changes in the fourth result which shows that $\int_0^t R(t,s)a(s)ds$ faithfully duplicates $a(t)$ so that, on average, for large t we can not distinguish between the two. As $a(t)$ can be unbounded, this means that there is no comparison at all between a and x. Many such results are found in [4, 6, 7]. Those results would tend to make the case that for $a(t)$ unbounded, then x does not follow $a(t)$ in any reasonable sense. But in this work we will offer a new example of contrary type. Qualitative theory demands that we gain an understanding of which conditions make the solution follow $a(t)$ and which conditions make that integral duplicate $a(t)$.

Here, BC denotes the set of bounded continuous functions. The first result is ancient, the second and third are from [6], and the fourth is from [7].

Theorem 1.1. *Let $\int_0^t |C(t,s)|ds \le \alpha < 1$. If $a \in BC$, so is the solution of (1).*

Theorem 1.2. *Let $\int_0^\infty |C(u+t,t)|du \le \alpha < 1$. If $a \in L^1[0,\infty)$, so is the solution of (1).*

Theorem 1.3. *Suppose there are constants $\alpha < 1$ and $\beta < 1$ with*

$$\int_0^t |C(t,s)|ds \le \alpha \text{ and } \int_0^\infty |C(u+t,t)|du \le \beta.$$

If there is an integer n with $a \in L^{2^n}[0,\infty)$, then the solution of (1) is also in L^{2^n}.

Theorem 1.4. *Suppose there is an integer $n > 0$ with $a' \in L^{2n}[0,\infty)$, a constant $\alpha > 0$, and an $N > 0$ with*

$$\frac{2n-1}{2nN^{\frac{2n}{2n-1}}} - C(t,t) + \frac{2n-1}{2n}\int_0^t |C_t(t,s)|ds + \frac{1}{2n}\int_0^\infty |C_1(u+t,t)|du \le -\alpha.$$

Then the solution of (1) is in $L^{2n}[0,\infty)$.

In the last result we allow $a(t) = \sin(t+1)^\beta + (t+1)^\beta$ for $0 < \beta < 1$ and still the integral $\int_0^t R(t,s)a(s)ds$ is duplicating $a(t)$ to within an L^p function. This shows that the integral can duplicate large, small, monotone, or oscillating functions. Thus, x does not follow $a(t)$.

Our next result is not new, but its proof leads us into the present work. It has its roots in Volterra [14] in 1928, in Levin [10] in 1963, and in Burton [5] in 1993. The first part of the theorem is found in Burton [6], but we need the proof here. The second half of the proof could be deduced from [5]. Our earlier work focused more on C and, at most, one derivative. In this project we are taking into account the finer aspects of C through the use of many derivatives.

Theorem 1.5. *If $a : [0,\infty) \to R$ is continuous, while*

$$C(t,s) \ge 0, \quad C_s(t,s) \ge 0, \quad C_t(t,s) \le 0, \quad C_{st}(t,s) \le 0 \tag{4}$$

then along the solution of (1) the functional

$$V(t) = \int_0^t C_s(t,s)\left(\int_s^t x(u)du\right)^2 ds + C(t,0)\left(\int_0^t x(s)ds\right)^2 \tag{5}$$

satisfies

$$V'(t) \le -x^2(t) + a^2(t). \tag{6}$$

(i) If $a \in L^2[0,\infty)$, so are x and $\int_0^t R(t,s)a(s)ds$; moreover, $V(t)$ is bounded.

(ii) If there are constants B and K with

$$\sup_{t\ge 0}\int_0^t C_s(t,s)ds = B < \infty \text{ and } \sup_{t\ge 0} C(t,0) = K < \infty \tag{7}$$

then along the solution of (1) we have

$$\left(\int_0^t R(t,s)a(s)ds\right)^2 = (a(t) - x(t))^2 \le 2(B+K)V(t) \tag{8}$$

where (8) does not require $a \in L^2$. However, if $a \in L^2$ and bounded then both $V(t)$ and x are bounded.

Proof. We have

$$V(t)=\int_0^t C_s(t,s)\left(\int_s^t x(u)du\right)^2 ds + C(t,0)\left(\int_0^t x(s)ds\right)^2$$

and differentiate along the unique solution of (1) to obtain

$$\begin{aligned}V'(t)&=\int_0^t C_{st}(t,s)\left(\int_s^t x(u)du\right)^2 ds + 2x\int_0^t C_s(t,s)\int_s^t x(u)duds\\&\quad + C_t(t,0)\left(\int_0^t x(s)ds\right)^2 + 2xC(t,0)\int_0^t x(s)ds.\end{aligned}$$

We now integrate the third-to-last term by parts to obtain

$$\begin{aligned}&2x\left[C(t,s)\int_s^t x(u)du\Big|_0^t + \int_0^t C(t,s)x(s)ds\right]\\&= 2x\left[-C(t,0)\int_0^t x(u)du + \int_0^t C(t,s)x(s)ds\right].\end{aligned}$$

Cancel terms, use the sign conditions, and use (1) in the last step of the process to unite the Liapunov functional and the equation obtaining

$$\begin{aligned}V'(t)&=\int_0^t C_{st}(t,s)\left(\int_s^t x(u)du\right)^2 ds + C_t(t,0)\left(\int_0^t x(s)ds\right)^2\\&\quad + 2x[a(t)-x(t)]\\&\le 2xa(t)-2x^2(t)\\&\le a^2(t)-x^2(t).\end{aligned}$$

From this we obtain

$$0\le V(t)\le V(0)+\int_0^t a^2(s)ds-\int_0^t x^2(s)ds;$$

when $a\in L^2[0,\infty)$ then $x\in L^2[0,\infty)$ and V is bounded. Moreover, by the Schwarz inequality we have

$$\begin{aligned}\left(\int_0^t C_s(t,s)\int_s^t x(v)dvds\right)^2 &\le \int_0^t C_s(t,s)ds\int_0^t C_s(t,s)\left(\int_s^t x(v)dv\right)^2 ds\\&\le B\int_0^t C_s(t,s)\left(\int_s^t x(v)dv\right)^2 ds + BC(t,0)\left(\int_0^t x(s)ds\right)^2\\&= BV(t).\end{aligned}$$

But

$$\left(\int_0^t C_s(t,s)\int_s^t x(v)dvds\right)^2 = \left(C(t,s)\int_s^t x(v)dv\Big|_0^t + \int_0^t C(t,s)x(s)ds\right)^2$$
$$= \left(-C(t,0)\int_0^t x(v)dv + \int_0^t C(t,s)x(s)ds\right)^2$$
$$= \left(a(t) - x(t) - C(t,0)\int_0^t x(v)dv\right)^2$$
$$\geq (1/2)(a(t)-x(t))^2 - \left(C(t,0)\int_0^t x(v)dv\right)^2.$$

This yields

$$(1/2)(x(t)-a(t))^2 \leq (B+K)V(t).$$

The left side of (8) is the variation of parameters formula. □

Here is the new contribution of this paper. Investigators have always relied on variations of (6) to bound the Liapunov functional and, hence, the solution. We will obtain a relation from (6) and substitute that into (5). In order to let $a(t)$ become large, we will discover three new things. First, we show how to replace $x(t)$ by $a(t)$ in the Liapunov functional. Next, we show how to replace $x(t)$ by $a'(t)$ and allow $a'(t)$ to be bounded and continuous. Finally, we show how to replace $\int_0^t x(s)ds$ by $a(t)$.

2. A bound for the Liapunov functional

We begin by replacing x in the Liapunov functional by $a(t)$. This will give us a condition to ensure that $V(t)$ is bounded. Then notice that when V is bounded and when (8) holds then x is bounded if and only if a is bounded. We will, thereby, obtain a condition showing that $x(t)$ follows $a(t)$ regardless of the behavior of $a(t)$. We believe that this is the only result of that type in the literature.

Moreover, this result is now a companion of Theorem 1.4. First it guarantees (*) so that if a' satisfies Theorem 1.4 and if $|a(t)-b(t)|$ is bounded, then $y(t) = b(t) - \int_0^t C(t,s)y(s)ds$ has $|x(t)-y(t)|$ bounded.

Notice that if $\int_0^t C_s(t,s)(t-s)^2ds + C(t,0)t^2$ is bounded then there is a vector space of functions $a(t)$ satisfying (9). That space includes continuous functions $a = \phi + \psi$ where ϕ is bounded and $\psi \in L^2[0,\infty)$.

Theorem 2.1. *Let $a : [0,\infty) \to R$ be continuous and let (4) and (7) hold. If, in addition, there is a constant M with*

$$\int_0^t C_s(t,s)(t-s)\int_s^t a^2(u)duds + C(t,0)t\int_0^t a^2(s)ds \leq M \tag{9}$$

then $V(t)$ is bounded along the solution of (1), where V is defined in (5). Noting (8), we have that $\left(\int_0^t R(t,s)a(s)ds\right)^2 = (a(t)-x(t))^2$ is bounded so x is bounded if and only if a is bounded. Finally, when (9) holds then $\int_0^t R(t,s)a(s)ds$ is bounded for every $a \in BC$ so $\sup_{t\geq 0}\int_0^t |R(t,s)|ds < \infty$.

Proof. Focus on (6). Suppose that $V(t)$ is not bounded. Then there is a monotone increasing sequence $\{t_n\} \to \infty$ with $V(s) \leq V(t_n)$ for $0 \leq s \leq t_n$. Let t denote any such t_n and let $0 \leq s \leq t$. From (6)

$$0 \leq V(t) - V(s) \leq \int_s^t a^2(u)du - \int_s^t x^2(u)du$$

so that

$$\int_s^t x^2(u)du \leq \int_s^t a^2(u)du.$$

If we use the Schwarz inequality on both integrals of x in (5) we obtain

$$V(t) \leq \int_0^t C_s(t,s)(t-s)\int_s^t a^2(u)duds + C(t,0)t\int_0^t a^2(u)du \leq M$$

by (9). Hence, V is bounded and we apply (8). As $\int_0^t R(t,s)a(s)ds$ is bounded for every $a \in BC$, it follows from Perron's theorem [12] (or Burton [3; p. 116]) that $\sup_{t\geq 0}\int_0^t |R(t,s)|ds < \infty$. □

The following pedestrian example illustrates how much Theorem 1.5 has been extended. There, we had asked after (8) that $a \in L^2$ and be bounded. In this example we can take $a^2(t)$ to be the sum of a bounded function and an L^1-function which could have a sequence of spikes of magnitude going to infinity. By (8) the solution will follow those spikes in a very faithful way, differing only by a fixed bounded function.

Example 2.2. Let $C(t,s) = e^{-(t-s)}$ and $a^2(t) = \gamma + \mu(t)$ where γ is a fixed positive constant and $\mu \in L^1[0,\infty)$. Condition (7) becomes

$$\int_0^t e^{-(t-s)}ds \leq 1 =: B,$$

while $C(t,0) = e^{-t} \leq 1 := K$. Then (9) is

$$V(t) \leq \int_0^t e^{-(t-s)}(t-s)\int_s^t [\gamma + \mu(u)]duds + e^{-t}t\int_0^t [\gamma + \mu(s)]ds$$

which is bounded. Using this in (8) yields $(x(t) - a(t))^2$ bounded so $x(t)$ follows $a(t)$ on those spikes going off to infinity. Note that $C(t,s) = ke^{-(t-s)}$ works for any $k > 0$.

Next, differentiate (1) to obtain

$$x'(t) = a'(t) - C(t,t)x - \int_0^t C_t(t,s)x(s)ds \tag{10}$$

under the assumption that $a'(t)$ and $C_t(t,s)$ are continuous. The resolvent for (10) is that of Becker [1] (or [2; Chapter 7]) and concerns an equation

$$x' = A(t)x(t) + \int_0^t B(t,s)x(s)ds + f(t),\ x(0) = x_0, \tag{11}$$

with f and A continuous on $[0,\infty)$ and B continuous for $0 \le s \le t < \infty$. Then Becker's resolvent equation is

$$y'(t) = A(t)y(t) + \int_s^t B(t,u)y(u)du$$

where $0 \le s \le t < \infty$. Consulting Becker [1] or Chapter 7 of [2] we find that there is a unique solution, the resolvent $Z(t,s)$, for each $s \ge 0$ with $Z(s,s) = I$ and the unique solution of (11) can be expressed as the variation of parameters formula

$$x(t) = Z(t,0)x_0 + \int_0^t Z(t,s)f(s)ds. \tag{12}$$

Let us look at the result below. If we know that (*) holds, for example if $\int_0^t |C(t,s)ds \le \alpha < 1$, then by (ii) if $b(t)$ is any continuous function with $|a(t) - b(t)|$ bounded then the solution of $y(t) = b(t) - \int_0^t C(t,s)y(s)ds$ is bounded. While $a'(t)$ must be bounded, no such requirement is on $b(t)$. Moreover, it asks such different conditions on $a(t)$ than were asked in Theorem 1.4. Finally, we believe that the use of (9) and (14) in this way is a new idea. When (14) holds then there is the vector space of functions $a(t)$ with $a'(t)$ bounded for which $x(t)$ is bounded. Using (*), that space is greatly enlarged.

Theorem 2.3. *Let $H(t,s) := C_t(t,s)$, and suppose there is an $\alpha > 0$ with $C(t,t) \ge \alpha$ and*

$$H(t,s) \ge 0,\ H_t(t,s) \le 0,\ H_s(t,s) \ge 0,\ H_{st}(t,s) \le 0. \tag{13}$$

(i) If $a' \in L^2[0,\infty)$, then any solution of (1) or (10) is in $L^2[0,\infty)$, as is $\int_0^t Z(t,s)a'(s)ds$.

(ii) If a' is bounded and if there is an $M > 0$ with

$$\int_0^t H_s(t,s)(t-s)\int_s^t |a'(u)|^2 duds + H(t,0)t\int_0^t |a'(u)|^2 du \le M, \tag{14}$$

then any solution of (1) or (10) is bounded. Thus, the resolvent for (10) satisfies $\sup_{t\geq 0}\int_0^t |Z(t,s)|ds < \infty$ *and* $Z(t,0)$ *is bounded.*

(iii) If, in addition to the conditions of (ii), we have

$$|C(t,t)| + \int_0^t |C_t(t,s)|ds$$

bounded, then $Z(t,0) \to 0$ *as* $t \to \infty$.

Proof. Define

$$\begin{aligned} V(t) &= x^2 + \int_0^t H_s(t,s)\Big(\int_s^t x(u)du\Big)^2 ds + H(t,0)\Big(\int_0^t x(s)ds\Big)^2 \\ &= x^2 + \int_0^t C_{st}(t,s)\Big(\int_s^t x(u)du\Big)^2 ds + C_t(t,0)\Big(\int_0^t x(s)ds\Big)^2 \end{aligned} \tag{15}$$

so that if $V(t)$ is bounded, so is $x^2(t)$. Next, write

$$x' = a'(t) - C(t,t)x - \int_0^t H(t,s)x(s)ds.$$

Then the derivative of V along a solution is

$$\begin{aligned} V'(t) &= 2a'(t)x - 2C(t,t)x^2 - 2x\int_0^t C_t(t,s)x(s)ds \\ &\quad + \int_0^t H_{st}(t,s)\Big(\int_s^t x(u)du\Big)^2 ds + 2x(t)\int_0^t H_s(t,s)\int_s^t x(u)duds \\ &\quad + H_t(t,0)\Big(\int_0^t x(s)ds\Big)^2 + 2H(t,0)\int_0^t x(s)dsx(t). \end{aligned}$$

We integrate the fifth term on the right by parts and obtain

$$\begin{aligned} &2x(t)[H(t,s)\int_s^t x(u)du\Big|_0^t + \int_0^t H(t,s)x(s)ds] \\ &\qquad = -2x(t)H(t,0)\int_0^t x(u)du + \int_0^t H(t,s)x(s)ds2x(t). \end{aligned}$$

Cancelling terms and taking into account sign conditions yields

$$V'(t) \leq 2a'(t)x - 2C(t,t)x^2 \leq 2a'(t)x - 2\alpha x^2 \leq D|a'(t)|^2 - Ex^2(t)$$

for positive constants D, E.

Note that x is bounded if V is bounded.

Thus, we first see that if $(a'(t))^2 \in L^1[0,\infty)$, so is $x^2(t)$, yielding $V(t)$ bounded and, hence, $x(t)$ bounded. By (12) for (10) we see that $\int_0^t Z(t,s)a'(s)ds \in L^2[0,\infty)$ and is bounded, by taking $x(0) = 0$.

Now, assume $a'(t)$ bounded and let (14) hold; we will bound V and, hence, x. From V' and the boundedness of $|a'|$ we see that there is a $\mu > 0$ such that if $V'(t) > 0$ then $|x(t)| < \mu$. Suppose, by way of contradiction, that V is not bounded. Then there is a sequence $\{t_n\} \uparrow \infty$ with $V'(t_n) \geq 0$ and $V(t_n) \geq V(s)$ for $0 \leq s \leq t_n$; thus, $|x(t_n)| \leq \mu$. If $0 \leq s \leq t_n$ then

$$0 \leq V(t_n) - V(s) \leq -\int_s^{t_n} Ex^2(u)du + D\int_s^{t_n} |a'(u)|^2 du.$$

Using these values in the formula for V, taking $|x(t_n)| \leq \mu$, and applying the Schwarz inequality yields at $t = t_n$ the inequality

$$\begin{aligned} V(t) \leq \mu^2 + \int_0^t H_s(t,s)(t-s)\int_s^t (D/E)|a'(u)|^2 du ds \\ + H(t,0)t(D/E)\int_0^t |a'(u)|^2 du = \mu^2 + (D/E)M \end{aligned}$$

Thus, $V(t)$ and $x(t)$ are bounded.

Now the variation of parameters formula for (10) is

$$x(t) = Z(t,0)x(0) + \int_0^t Z(t,s)a'(s)ds.$$

If $a'(t) \equiv 0$, then $V'(t) \leq -Ex^2(t)$ so $x^2 \in L^1[0,\infty)$ and $V(t)$ is bounded so $x(t)$ is bounded. This means that $Z(t,0)$ is bounded and, hence, $\int_0^t Z(t,s)a'(s)ds$ is bounded for every bounded and continuous $a'(t)$. By Perron's theorem, $\sup_{t\geq 0}\int_0^t |Z(t,s)|ds < \infty$. If $|C(t,t)| + \int_0^t |C_t(t,s)|ds$ is bounded, then $x'(t)$ is bounded so $Z(t,0) \to 0$. □

Remark 2.4. It is a very useful result. The solution is bounded and (14) will yield a computable bound in spite of $a(t)$ being unbounded.

Example 2.5. Let $C(t,s) = 2 - e^{-(t-s)}$ and $a(t) = (t+1)^{1/2}$. Then $C(t,t) = 1 =: \alpha > 0$, $C_t(t,s) = e^{-(t-s)} =: H(t,s)$ so (13) holds. Also, (14) is

$$\int_0^t e^{-(t-s)}(t-s)\int_s^t \frac{1}{4(u+1)} du ds + e^{-t}t\int_0^t \frac{1}{4(u+1)} du$$

which is bounded. By part (ii), $x(t)$ is bounded. Hence, $\int_0^t R(t,s)a(s)ds$ closely follows $a(t)$, but $a(t)$ diverges far from $x(t)$. Consider (ii) and note that $a(t) = 3t$ qualifies and the solution is bounded. Moreover, if $b(t)$ is any continuous function so that $|a(t)-b(t)|$ is bounded and if (*)

holds then the solution of $y(t) = b(t) - \int_0^t C(t,s)y(s)ds$ is bounded. Notice that $x(t) = 3t - \int_0^t R(t,s)3sds$ is bounded. That resolvent has extremely strong properties enabling the integral to closely approximate $3t$. With the (usually) stronger condition on $a'(t)$ seen in Theorem 1.4 we see that integral actually converging to $a(t)$ in an L^p sense. In view of the fact that $R(t,s)$ depends only on $C(t,s)$, one must concede that the resolvent is a remarkable function.

Finally, again consider (1) with $C_s(t,s)$ continuous and integrate by parts so that it can be written as

$$x(t) = a(t) - C(t,t)\int_0^t x(u)du + \int_0^t C_s(t,s)\int_0^s x(u)duds. \tag{16}$$

Define $y = \int_0^t x(s)ds$, write the equation as

$$y' = a(t) - C(t,t)y + \int_0^t C_s(t,s)y(s)ds, \tag{17}$$

let $C_s(t,s) =: -H(t,s)$, and write

$$y' = a(t) - C(t,t)y - \int_0^t H(t,s)y(s)ds.$$

If we show that y is bounded and assume that $C(t,t)$ and $\int_0^t |C_s(t,s)|ds$ are bounded then we have from (17) that x is bounded.

The following result is a companion to Theorem 1.4 in exactly the same way that Theorem 2.1 is.

Theorem 2.6. *Let $C(t,t) \geq \alpha > 0$, $H(t,s) = -C_s(t,s)$, and let*

$$H(t,s) \geq 0,\ H_t(t,s) \leq 0,\ H_s(t,s) \geq 0,\ H_{st}(t,s) \leq 0; \tag{18}$$

if $a \in L^2[0,\infty)$, then any solution y of (17) is bounded; also, $Z(t,0) \in L^2[0,\infty)$ and bounded so by (12) for (17), $\int_0^t Z(t,s)a(s)ds \in L^2[0,\infty)$ and bounded. Define $\lambda(t)$ by

$$\lambda(t) := \int_0^t H_s(t,s)(t-s)\int_s^t a^2(u)duds + H(t,0)t\int_0^t a^2(u)du. \tag{19}$$

If there is an M with $\lambda(t) < M$ and if $a(t)$ is bounded so is every solution of (17). If $C(t,t)$ and $\int_0^t C_s(t,s)ds$ are bounded, so is the solution of (1); in particular, then, $\sup_{t\geq 0}\int_0^t |R(t,s)|ds < \infty$.

Proof. Define V as in the last theorem with x replaced by y and get

$$V'(t) \leq -Ey^2 + Da^2(t)$$

for positive constants D and E. The constant M in (14) is defined with a' replaced by a. The theorem is repeated with y bounded. That means that $\int_0^t x(s)ds$ is bounded. The bound on x, the solution of (1), follows as stated in the theorem. Finally, in that last case x is bounded for every bounded and continuous $a(t)$ so by Perron's theorem (*) holds. $\square$

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