NEUTRAL INTEGRAL EQUATIONS OF RETARDED TYPE

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Abstract. In this paper we study three neutral integral equations of the form

$$y(t) = f(t, y(t-h)) + \int_{-\infty}^{t} Q(s, y(s), y(s-h))C(t-s)ds + p(t), \tag{a}$$

$$x(t) = \alpha x(t-h) - \int_{-\infty}^{t} [\beta x(s) + \gamma x(s-h)]C(t,s)ds + p(t), \tag{b}$$

and

$$x(t) = \alpha x(t-h) - \int_{-\infty}^{t} [\beta x^3(s) + \gamma x^3(s-h)]C(t,s)ds + p(t). \tag{c}$$

Using a contraction mapping theorem on (a) and Liapunov functions on (b) and (c) we find appropriate norms and metrics for the solutions. When the equations are periodic we use a modification of Krasnoselskii's fixed point theorem to prove that there is a periodic solution. These equations arise from the inversion of well-known problems such as neutral logistic equations.

1. Introduction.

Krasnoselski noted that inversion of a perturbed differential operator frequently yields the sum of a contraction and compact operator. When a system of perturbed neutral differential operators are inverted we obtain very interesting and unusual compact operators of advanced and retarded types. In the earlier papers ([1] and [2]) we studied the advanced kinds. Here, we look at retarded types. The first results are parallel to the earlier ones, but then they become strikingly different. We are most interested in studying the appropriate fixed point theorems for such operators.

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These operators can be motivated by inverting a system of neutral predator-prey equations which have now become quite common ([5], [7]).

Consider the system

$$x' = \alpha x'(t-h) + ax - q(t, x, y, x(t-h), y(t-h))$$
$$y' = \beta y'(t-h) - ky + r(t, x, y, x(t-h), y(t-h)),$$

which we write as

$$\left[(x - \alpha x(t-h))e^{-at}\right]' = \left[a\alpha x(t-h) - q(t, x, y, x(t-h), y(t-h)\right] e^{-at},$$

$$\left[(y - \beta y(t-h))e^{kt}\right]' = \left[-\beta k y(t-h) + r(t, x, y, x(t-h), y(t-h))\right] e^{kt}.$$

If we integrate the first equations from t to ∞ and the second from $-\infty$ to t, we obtain a system

$$X(t) = F\Big(t, X(t-h), \quad \int_{-\infty}^{t} G(s, X(s), X(s-h))ds, \int_{t}^{\infty} H(s, X(s), X(s-h))ds\Big).$$

With a view to eventually understanding such systems, in ([1], [2]) we studied scalar equations of the type

$$x(t) = f(t, x(t-h)) + \int_{t}^{\infty} H(s, x(s), x(s-h))ds$$

by means of contraction mappings and by a combination of Liapunov's direct method and Krasnoselskii-type fixed point theorems (See [6] and [9;p. 31].). Here, we offer a parallel study of

$$y(t) = f(t, y(t-h)) + \int_{-\infty}^{t} Q(s, y(s), y(s-h))C(t-s)ds + p(t). \tag{1}$$

2. Small Kernels

We suppose that in

$$y(t) = f(t, y(t-h)) + \int_{-\infty}^{t} Q(s, y(s), y(s-h))C(t-s)ds + p(t) \qquad t \geq 0, \tag{1}$$

there are constants α, k and C_0 so that

$$\alpha \in [0,1) \text{ and } |f(t,x_1) - f(t,x_2)| \leq \alpha|x_1 - x_2| \qquad \text{for } (t, x_i) \in \mathbb{R}^+ \times \mathbb{R}, \tag{2}$$

$$k \in [0,1] \text{ and } |Q(t,x_1,x_2) - Q(t,x_3,x_4)| \leq k|x_1 - x_3| + (1-k)|x_2 - x_4| \text{ for } t, x_i \in \mathbb{R}, \tag{3}$$

$$\int_0^\infty |C(s)|ds =: C_0 < \infty, \tag{4}$$

$$p \in \mathcal{C}([0,\infty), \mathbb{R}),\ C \in \mathcal{C}([0,\infty), \mathbb{R}),\ f \in \mathcal{C}([0,\infty) \times \mathbb{R}, \mathbb{R}),\ Q \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}). \tag{5}$$

From elementary considerations of the method of undetermined coefficients for ordinary differential equations with constant coefficients, we expect a solution of (1) to follow $p(t)$.

Given a $\overline{\varphi} \in \mathcal{C}\big((-\infty, 0], \mathbb{R}\big)$, we seek a solution $y(t, 0, \overline{\varphi})$ satisfying (1) for $t \geq 0$ and $y(t, 0, \overline{\varphi}) = \overline{\varphi}(t)$ for $t < 0$. Then for y to be continuous at $t = 0$, we would need $\overline{\varphi}(0) = y(0)$, but

$$y(0) = f(0, \overline{\varphi}(-h)) + \int_{-\infty}^{0} Q(s, \overline{\varphi}(s), \overline{\varphi}(s-h))C(-s)ds + p(0), \tag{6}$$

so unless $\overline{\varphi}$ is such that

$$\overline{\varphi}(0) = f(0, \overline{\varphi}(-h)) + \int_{-\infty}^{0} Q(s, \overline{\varphi}(s), \overline{\varphi}(s-h))C(-s)ds + p(0), \tag{7}$$

there will be a discontinuity in $y(t, 0, \overline{\varphi})$ at $t = 0$, and hence at $t = nh$.

Remark.. We could show, as in [3], that for a $\overline{\varphi}$ as above satisfying

$$I(t, \overline{\varphi}) := \int_{-\infty}^{0} Q(s, \overline{\varphi}(s), \overline{\varphi}(s-h))C(t-s)ds \in \mathcal{C}([0,\infty), \mathbb{R}), \tag{8}$$

there is a φ^* arbitrarily near $\overline{\varphi}$ for which $y(t,0,\varphi^*)$ is continuous. In fact, this will be necessary if we are to use Liapunov functions requiring integration by parts on problems with initial functions. But for Theorem 1 no such continuity is required.

These last two paragraphs tell us the main properties of solutions and how to find them.

First, examine $p(t)$ and in order to construct a weight for a norm on a Banach space, find a $D \in \mathcal{C}([-h,\infty),(0,\infty))$ with

$$\text{(9)} \qquad \sup_{t\geq 0}\frac{|Q(t,0,0)|}{D(t)} < \infty, \qquad \sup_{t\geq 0}\frac{|f(t,0,0)|}{D(t)} < \infty, \qquad \sup_{t\geq 0}\frac{|p(t)|}{D(t)} < \infty$$

$$\text{and} \qquad \sup_{t\geq 0}\frac{D(t-h)}{D(t)} = \hat{k} \qquad \text{for some } \hat{k} > 0.$$

$$\text{(10)} \qquad \sup_{t\geq 0}\int_0^t |C(t-s)|\,\frac{D(s)}{D(t)}ds < \infty.$$

Moreover, for the mapping we use to be a contraction, we ask

$$\text{(11)} \qquad \alpha\hat{k} + (k + (1-k)\hat{k})\sup_{t\geq 0}\int_0^t |\mathcal{C}(t-s)|\frac{D(s)}{D(t)}ds =: \mu < 1.$$

Define the Banach space $(\mathcal{B}, |\cdot|_D)$ where

$$\text{(12)} \qquad |\varphi|_D := \sup_{t\geq -h}\frac{|\varphi(t)|}{D(t)} \quad \text{and}$$

$\mathcal{B} = \{\varphi : [-h,\infty) \to \mathbb{R},\ \varphi$ continuous on $[(n-1)h, nh)$ for $n = 0,1,\ldots,$ $\lim_{t\to nh^-}\varphi(t)$ exists and $|\varphi|_D < \infty\}$.

Theorem 1. *Let (2)-(5) hold and $D(t)$ be such that (9)-(12) hold. Then for each $\overline{\varphi} \in C((-\infty,0],\mathbb{R})$ satisfying (9) and $\sup_{t\geq 0}\frac{|I(t,\overline{\varphi})|}{D(t)} < \infty$, there is a unique function $y : \mathbb{R} \to \mathbb{R}$, $y\big|_{[-h,\infty)} \in \mathcal{B}$, such that $y(t,0,\overline{\varphi}) = \overline{\varphi}(t)$ for $t < 0$, $y(t,0,\overline{\varphi})$ satisfies (1) for $t \geq 0$ and $\sup_{t\geq 0}\frac{|y(t,0,\overline{\varphi})|}{D(t)} < \infty$.*

Proof of Theorem 1. $\mathcal{B}^*$ is the complete metric space of functions $\{\varphi : \mathbb{R} \to \mathbb{R}\big|\varphi(t) = \overline{\varphi}(t)$ for $t < 0, \varphi\big|_{[-h,\infty)} \in \mathcal{B}\}$ with the metric $d(\varphi,\psi) = \big|\varphi_{[-h,\infty)} - \psi_{[-h,\infty)}\big|_D$. Define a mapping $P : \mathcal{B}^* \to \mathcal{B}^*$ by $\varphi \in \mathcal{B}^*$ implies

$$(P\varphi)(t) = \begin{cases} f(t,\varphi(t-h)) + \int_{-\infty}^t Q(s,\varphi(s),\varphi(s-h))C(t-s)ds + p(t) & t \geq 0 \\ \overline{\varphi}(t) & t < 0. \end{cases}$$

Since $\varphi\big|_{[-h,\infty)} \in \mathcal{B}$ and $\overline{\varphi} \in C((-\infty,0],\mathbb{R})$, we have that $\varphi(t-h)$ is right continuous at $t=0$ and then by the continuity properties in (5) and (8), $f(t,\varphi(t-h))$, $\int_{-\infty}^{0} Q(s,\overline{\varphi}(s),\overline{\varphi}(s-h))C(t-s)ds$ and $\int_0^t Q(s,\varphi(s),\varphi(s-h))C(t-s)ds$, and hence $(P\varphi)(t)$, are right continuous at $t=0$.

Similarly, $(P\varphi)(t)$ is continuous on $[(n-1)h, nh)$ for $n=0,1,\ldots$ Moreover, by a similar argument $\lim\limits_{t\to nh^-} P\varphi(t)$ exists for $n=0,1,\ldots$ Since for $t\geq 0$, we have that

$$\Big|\big|f(t,\varphi(t-h))\big| - \big|f(t,0)\big|\Big| \leq \big|f(t,\varphi(t-h)) - f(t,0)\big| \leq \alpha\big|\varphi(t-h)\big|,$$

then by (9)

$$\sup_{t\geq 0}\frac{|f(t,\varphi(t-h)|}{D(t)} \leq \sup_{t\geq 0}\alpha\frac{|\varphi(t-h)|}{D(t-h)}\frac{D(t-h)}{D(t)} + \frac{|f(t,0)|}{D(t)} < \infty.$$

Similarly $|Q(t,\varphi(t),\varphi(t-h))| \leq k|\varphi(t)| + (1-k)|\varphi(t-h)| + |Q(t,0,0)|$, and $\sup\limits_{t\geq 0}\dfrac{|Q(t,\varphi(t),\varphi(t-h))|}{D(t)} < \infty$.

So

$$\begin{aligned}\sup_{t\geq 0}\frac{|(P\varphi)(t)|}{D(t)} \leq \sup_{t\geq 0}\frac{|f(t,\varphi(t-h))|}{D(t)} &+ \int_0^t \frac{|Q(s,\varphi(s),\varphi(s-h))|}{D(s)}\frac{D(s)}{D(t)}|C(t-s)|ds \\ &+ \int_{-\infty}^{0} |Q(s,\overline{\varphi}(s),\overline{\varphi}(s-h))|\frac{|C(t-s)|}{D(t)}ds + \frac{|p(t)|}{D(t)} < \infty\end{aligned}$$

and

$$|P\varphi|_D = \sup_{t\geq -h}\frac{(P\varphi)(t)}{D(t)} = \max\left[\sup_{-h\leq t\leq 0}\frac{|\overline{\varphi}(t)|}{D(t)}, \sup_{t\geq 0}\frac{|P\varphi(t)|}{D(t)}\right] < \infty.$$

Then $P\varphi\big|_{[-h,\infty)} \in \mathcal{B}$ and $P\varphi \in \mathcal{B}^*$.

For $\varphi, \psi \in \mathcal{B}^*$,

$$d(P\varphi, P\psi) = \left|P\varphi|_{[-h,\infty)} - P\psi|_{[-h,\infty)}\right|_D = \sup_{t\geq -h} \frac{|P\varphi(t) - P\psi(t)|}{D(t)} \leq$$

$$\sup_{t\geq 0} \frac{|f(t,\varphi(t-h)) - f(t,\psi(t-h))| + \int_{-\infty}^{t} |\left[Q(s,\varphi(s),\varphi(s-h)) - Q(s,\psi(s),\psi(s-h))\right]| |C(t-s)|ds}{D(t)}$$

$$\leq \sup_{t\geq 0} \alpha \frac{|\varphi(t-h) - \psi(t-h)|}{D(t-h)} \frac{D(t-h)}{D(t)} + \int_0^t \left[k \frac{|\varphi(s) - \psi(s)|}{D(s)} \frac{D(s)}{D(t)}\right.$$

$$\left. + (1-k)\left(\frac{|\varphi(s-h) - \psi(s-h)|}{D(s-h)}\right) \frac{D(s-h)}{D(s)} \frac{D(s)}{D(t)}\right] |C(t-s)|ds$$

$$\leq \alpha\hat{k} \sup_{t\geq -h} \frac{|\varphi(t) - \psi(t)|}{D(t)} + \left[k \sup_{t\geq -h} \frac{|\varphi(t) - \psi(t)|}{D(t)} + (1-k)\hat{k} \sup_{t\geq 0} \frac{|\varphi(t) - \psi(t)|}{D(t)}\right] \cdot \int_0^t \frac{D(s)}{D(t)} |C(t-s)|ds$$

$$\leq d(\varphi,\psi) \left[\alpha\hat{k} + \left(k + (1-k)\hat{k}\right) \sup_{t\geq 0} \int_0^t \frac{D(s)}{D(t)} |C(t-s)|ds\right] = \mu d(\varphi,\psi).$$

Thus P is a contraction on $\mathcal{B}^*$ and has a unique fixed point in $\mathcal{B}^*$.

Now consider (1) on the whole axis

$$(1)' \qquad y(t) = f(t, y(t-h)) + \int_{-\infty}^{t} Q(s, y(s), y(s-h)) C(t-s) ds + p(t) \qquad t \in \mathbb{R}$$

where now there are positive constants T, α, k and μ so that

$$(2)' \qquad \alpha \in [0,1) \text{ and } |f(t,x_2) - f(t,x_2)| \leq \alpha|x_1 - x_2| \qquad \text{for } t \in \mathbb{R},\ x_i \in \mathbb{R}$$

(3)′

$k \in [0,1]$ and $|Q(t,x_1,x_2) - Q(t,x_3,x_4) \leq k|x_1 - x_3| + (1-k)|x_2 - x_4|$ for $t \in \mathbb{R}$, $x_i \in \mathbb{R}$

$$(4)' \qquad \alpha + \int_0^\infty |C(s)|ds =: \mu < 1$$

$$(5)' \qquad p \in C(\mathbb{R},\mathbb{R}), \quad C \in \mathcal{C}(\mathbb{R},\mathbb{R}), \quad f \in \mathcal{C}(\mathbb{R}^2,\mathbb{R}), \quad Q \in C(\mathbb{R}^3,\mathbb{R})$$

$$p(t+T) = p(t), \quad f(t+T,x) = f(t,x) \qquad Q(t+T,x_1,x_2) = Q(t,x_1,x_2) \quad \text{for } t, x_i \in \mathbb{R}$$

Proposition 1. *If (2)′-(5)′ hold, then (1)′ has a continuous T-periodic solution on $\mathbb{R}$.*

Proof of Proposition 1. Let $\mathcal{P}_T := \{\varphi \in C(\mathbb{R},\mathbb{R}), \varphi(t+T) = \varphi(t) \quad t \in \mathbb{R}\}$ and $(\mathcal{P}_T, ||\cdot||)$ denote the Banach space of continuous T-periodic functions on $\mathbb{R}$ with the norm $||\varphi|| = \sup_{0\le t\le T} |\varphi(s)|$.

Define $P : \mathcal{P}_T \to \mathcal{P}_T$ by $\varphi \in P_T$ implies

$$(P\varphi)(t) = f(t, \varphi(t-h)) + \int_{-\infty}^{t} Q(s, \varphi(s), \varphi(s-h))C(t-s)ds + p(t).$$

By $\varphi \in \mathcal{P}_T$ and (5)′,

$$\begin{aligned}(P\varphi)(t+T) &= f(t+T, \varphi(t+T-h)) + \int_{-\infty}^{t+T} Q(s, \varphi(s), \varphi(s-h))C(t+T-s)ds + p(t+T)\\ &= f(t, \varphi(t-h)) + \int_{-\infty}^{t} Q(s'+T, \varphi(s'+T), \varphi(s'+T-h)C(t-s')ds' + p(t)\\ &= (P\varphi)(t), \qquad \text{so } P\varphi \in \mathcal{P}_T.\end{aligned}$$

For $\varphi, \psi \in \mathcal{P}_T$, we have for each $t \in \mathbb{R}$

$$\begin{aligned}|P\varphi(t) - P\psi(t)| &\le |f(t, \varphi(t-h)) - f(t, \psi(t-h))|\\ &+ \int_{-\infty}^{t} |Q(s, \varphi(s), \varphi(s-h)) - Q(s, \psi(s), \psi(s-h))||C(t-s)|ds\\ &\le \alpha|\varphi(t-h) - \psi(t-h)| + \int_{-\infty}^{t} [k|\varphi(s) - \psi(s)|\\ &+ (1-k)|\varphi(s-h) - \psi(s-h)|]|C(t-s)|ds\\ &\le \left(\alpha + \int_{0}^{\infty} |C(s)|ds\right) ||\varphi - \psi||,\end{aligned}$$

so $||P\varphi - P\psi|| \le \mu||\varphi - \psi||$ and P is a contraction on $\mathcal{P}_T$ and so has a unique fixed point in P_T.

Remark. In [1] we had considered the advanced case

$$x(t) = f(t, x(t-h)) + \int_{t}^{\infty} \varphi(s, x(s), x(s-h))C(t-s)ds + p(t)$$

and had obtained a result completely parallel to Theorem 1. An example of the following sort showed that contractions work in the stable case; that is, we now construct a simple example satisfying the conditions of the theorem in which solutions go to zero.

Example 1. Consider

$$\text{(DE)} \qquad y(t) = \alpha y(t-h) + \int_{-\infty}^{t} \gamma \hat{Q}(s, y(s), y(s-h)) e^{-a(t-s)} ds + e^{-\beta t} \qquad t \geq 0,$$

where $\hat{Q} \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}), \sup_{t\geq 0} |\hat{Q}(t,0,0)| e^{\beta(t+h)} < \infty$, and for $k \in [0,1]$, $|\hat{Q}(t,x_1,x_2) - \hat{Q}(t,x_3,x_4)| \leq k|x_1 - x_3| + (1-k)|x_2 - x_4|$ for $t \geq 0, x_i \in \mathbb{R}$, and $\alpha \in [0,1)$, $\beta < a$ and $\alpha e^{\beta h} + (k + (1+k)e^{\beta h}) \frac{|\gamma|}{a-\beta} =: \mu < 1$.

Then for each $\varphi \in C((-\infty, 0], \mathbb{R})$ with $\int_{-\infty}^{0} \hat{Q}(s, \varphi(s), \varphi(s-h)) e^{as} ds < \infty$, there is a unique solution of (DE), $y(t,0,\varphi)$, satisfying $|y(t,0,\varphi)| \leq M e^{-\beta t}$ for $t \geq 0$ and some constant $M(\varphi)$.

Proof. With $Q(t, \varphi(t), \varphi(t-h)) = \gamma \hat{Q}(t, \varphi(t), \varphi(t-h)), f(t, \varphi(t-h)) = \alpha\varphi(t-h), C(t) = e^{-at}$, conditions(2)-(5) of Theorem 1 are satisfied and with $D(t) = e^{-\beta(t+h)}$, conditions (8)-(11) are satisfied:

$$\sup_{t\geq 0} \frac{D(t-h)}{D(t)} = e^{\beta h},$$

$$\sup_{t\geq 0} \int_0^t |C(t-s)| \frac{D(s)}{D(t)} ds \leq \frac{1}{a-\beta},$$

$$\alpha e^{\beta h} + (k + (1-k)e^{\beta h})|\gamma| \frac{1}{a-\beta} = \mu < 1.$$

Then by Theorem 1, $y(t,0,\varphi)$ exists and $\sup_{t\geq 0} \dfrac{|y(t,0,\varphi)|}{D(t)} \leq \overline{M} < \infty$, so for each $t \geq 0$, $|y(t,0,\varphi)| \leq \overline{M} D(t) \leq \overline{M} e^{-\beta(t+h)} \leq M e^{-\beta t}$.

Remark.. We have just used a fixed point theorem to prove an asymptotic stability result.

We now consider a linear equation

$$\text{(13)} \qquad x(t) = \alpha x(t-h) - \int_{-\infty}^{t} [\beta x(s) + \gamma x(s-h)] C(t,s) ds + p(t) \qquad t \geq 0,$$

where

$$\text{(14)} \qquad p \in \mathcal{C}([0,\infty), \mathbb{R}), C \in \mathcal{C}(\Omega, \mathbb{R}) \qquad \Omega = \{(t,s) | -\infty < s \leq t < \infty\},$$

there exists $J > 1$ such that

$$(15) \qquad \sup_{t\geq 0}\{|C(t,t)|\int_{-\infty}^{t}(|C(t,s)| + |C_s(t,s)|(t-s)(t-s+1) + |C(t,s)|^2)ds\} \leq J,$$

$$(16) \qquad C_s(t,s) \geq 0, \qquad C_{st}(t,s) \leq 0 \quad \text{for } (t,s) \in \Omega$$

$$\lim_{s\to-\infty} sC(t,s) = 0 \qquad \text{for } t \in [0,\infty),$$

$$(17) \qquad |\alpha| < 1, \qquad \beta - |\alpha\beta - \gamma| - |\alpha\gamma| > 0.$$

We also consider its periodic analogue on the whole axis

$$(13)' \qquad x(t) = \alpha x(t-h) - \int_{-\infty}^{t}[\beta x(s) + \gamma x(s-h)]C(t,s)ds + p(t) \qquad t \in \mathbb{R}$$

where

$$(14)' \qquad \begin{array}{l} p \in C(\mathbb{R},\mathbb{R}), \mathcal{C} \in C(\Omega,\mathbb{R}) \quad \Omega = \{(t,s)| -\infty < s \leq t < \infty\}, \text{there is a } T > 0 \text{ with} \\ p(t+T) = p(t) \text{ for } t \in \mathbb{R},\ C(t+T,s+T) = C(t,s) \text{ for } (t,s) \in \Omega, \end{array}$$

there exists $J > 1$ such that

$$(15)' \qquad \sup_{t\geq 0}\left\{|C(t,t)|\int_{-\infty}^{t}|C(t,s)| + |C_s(t,s)|(t-s)(t-s+T) + |C(t,s)|^2 ds\right\} \leq J,$$

$$(16)' \qquad C_s(t,s) \geq 0,\ C_{st}(t,s) \leq 0 \text{ for } (t,s) \in \Omega, \quad \lim_{s\to-\infty} sC(t,s) = 0 \text{ for } t \in \mathbb{R},$$

$$(17)' \qquad |\alpha| < 1, \qquad \beta > |\alpha\beta - \gamma| + |\alpha\gamma|.$$

Theorem 2. *Suppose (14)-(17) hold. Then there are constants $c_i \geq 0$ such that whenever a continuous solution $x(t, t_0, \varphi)$ of (13) satisfies $|x(t, t_0, \varphi)| \leq X$ for $t \in (-\infty, t_0]$ for some $t_0 \geq 0$ and $X > 0$, we have that*

(i) *for $t \geq t_0$*

$$\left(\int_{t_0}^{t} x^2(s)ds\right)^{1/2} \leq c_1 X + c_2 \left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2},$$

(Here, the c_i are independent of the solution.)

(ii) *if $t \geq t_0$ and $n = 0, 1, \ldots$ are such that $t - nh \geq t_0$, with*

$$|x(t)| < |x(t-h)| < |x(t-2h)| < \cdots < |x(t-nh)|$$

and with $|x(t-nh)| \geq |x(t-(n+1)h)|$, then

$$|x(t)| \leq \tfrac{1}{1-\alpha}|p(t-nh)| + c_3 X + c_4 \left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2},$$

(iii) *for each $t \geq t_0$, there is an $n = 0, 1 \ldots$ such that $t - nh \geq t_0$ and*

$$|x(t)| \leq \tfrac{1}{1-|\alpha|}|p(t-nh)| + c_5 X + c_6 \left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2},$$

(iv) *for $t \geq t_0$,*

$$|x(t)| \leq \tfrac{1}{1-|\alpha|} \sup_{t_0 \leq s \leq t} |p(s)| + c_5 X + c_6 \left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2}.$$

Theorem 3. *Suppose (14)′-(17)′ hold. Then there are constants $c_i \geq 0$ such that any solution $x \in \mathcal{P}_T$ of (13)′ satisfies*

(i) $\int_0^T x^2(s)ds \leq c_1 \int_0^T p^2(s)ds$

(ii) $\|x\| \leq \frac{1}{1-|\alpha|}\|p\| + c_2 \left(\int_0^T p^2(s)ds\right)^{1/2}.$

Proof of Theorem 2 (i). Using (18), choose $\epsilon_i > 0$ so that $\beta - |\alpha\beta - \gamma| - |\alpha\gamma| + \frac{\epsilon_1 + \epsilon_2}{2} := \frac{\mu}{2} > 0$. Let $x(t, 0, \varphi)$ be a continuous solution of (13). Define the Liapunov functional

$$V(t, x_t) = \int_{-\infty}^{t} C_s(t,s) \left[\int_s^t (\beta x(u) + \gamma x(u-h) du \right]^2 ds$$
$$+ (2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2) \int_{t-h}^{t} x^2(u) du.$$

Differentiating it along the solution $x(t, 0, \varphi)$ of (13) yields

$$\begin{aligned}
V'(t) = & \int_{-\infty}^{t} C_{st}(t,s) \left[\int_s^t (\beta x(u) + \gamma x(u-h))du \right]^2 ds \\
& + (2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2)[x^2(t) - x^2(t-h)] \\
& + 2[\beta x(t) + \gamma x(t-h)] \int_{-\infty}^{t} C_s(t,s) \int_s^t (\beta x(u) + \gamma x(u-h))du\, ds \\
\le & (2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2)[x^2(t) - x^2(t-h)] \\
& + 2\left[\beta x(t) + \gamma x(t-h)\right] \left[C(t,s) \int_s^t (\beta x(u) + \gamma x(u-h) du \bigg|_{-\infty}^{t} \right. \\
& \left. + \int_{-\infty}^{t} C(t,s)[(\beta x(s) + \gamma x(s-h)]ds \right] \\
= & (2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2)[x^2(t) - x^2(t-h)] \\
& + 2\left[\beta x(t) + \gamma x(t-h)\right] \int_{-\infty}^{t} C(t,s)[\beta x(s) + \gamma x(s-h)]ds \\
= & [2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2][x^2(t) - x^2(t-h)] \\
& + 2[\beta x(t) + \gamma x(t-h)][\alpha x(t-h) + p(t) - x(t)] \\
= & [2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2][x^2(t) - x^2(t-h)] \\
& + 2\alpha\beta x(t)x(t-h) + 2\beta x(t)p(t) \\
& - 2\beta x^2(t) + 2\alpha\gamma x^2(t-h) + 2\gamma x(t-h)p(t) - 2\gamma x(t-h)x(t) \\
\le & [2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2][x^2(t) - x^2(t-h)] \\
& - 2\beta x^2(t) + |\alpha\beta - \gamma| \left(x^2(t) + x^2(t-h) \right) + 2|\alpha\gamma| x^2(t-h) \\
& + \epsilon_1 x^2(t) + \frac{\beta^2}{\epsilon_1} p^2(t) + \epsilon_2 x^2(t-h) + \frac{\gamma^2}{\epsilon_2} p^2(t) \\
\le & \left[-2\beta + 2|\alpha\gamma| + 2|\alpha\beta - \gamma| + \epsilon_1 + \epsilon_2\right] x^2(t) + \left(\frac{\beta^2}{\epsilon_1} + \frac{\gamma^2}{\epsilon_2} \right) p^2(t) \\
=: & -\mu x^2(t) + Mp^2(t).
\end{aligned}$$

Then for $t \geq 0$

(18)
$$V'(t) \leq -\mu x^2(t) + Mp^2(t).$$

We next show that there is a $J^* > 0$, independent of t_0 and X so that if $|x(t,0,\phi)| < X$ on $(-\infty, t_0]$, then $V(t_0) \leq J^* X^2$. To see this we have

$$\begin{aligned}
V(t_0) = &\int_{-\infty}^{t_0} C_s(t_0,s) \left[\int_s^{t_0} (\beta x(u) + \gamma x(u-h) du) \right]^2 ds \\
&+ (2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2) \int_{t_0-h}^{t_0} x^2(u) du \\
&\leq \int_{-\infty}^{t_0} C_s(t_0,s)(t_0 - s)^2 (|\beta| + |\gamma|)^2 X^2 ds + (2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2) h X^2 \\
&\leq X^2 \left[(|\beta| + |\gamma|)^2 J + (2|\alpha\gamma| + |\alpha\beta - \gamma| + \epsilon_2) h \right] = J^* X^2,
\end{aligned}$$

where J^* is independent of t_0 and X.

Then integrating (18) yields

$$\int_{t_0}^t V'(s) ds \leq -\mu \int_{t_0}^t x^2(s) ds + \int_{t_0}^t M p^2(s) ds$$

$$V(t) \leq V(t_0) - \mu \int_{t_0}^t x^2(s) ds + \int_{t_0}^t M p^2(s) ds$$

(19)
$$\int_{t_0}^t x^2(s) ds \leq \frac{J^* X^2}{\mu} + \frac{M}{\mu} \int_{t_0}^t p^2(s) ds,$$

and then

(19)′
$$\left(\int_{t_0}^t x^2(s) ds \right)^{1/2} \leq c_1 X + c_2 \left(\int_{t_0}^t p^2(s) \right)^{1/2},$$

where c_1 and c_2 are independent of t_0, X, and the solution. ■

Proof of Theorem 2 (ii).

Suppose $t \geq t_0$ and $n = 0, 1, 2, \ldots$ are such that $t - nh \geq t_0$,

$$|x(t)| < |x(t-h)| < |x(t-2h)| < \cdots < |x(t-nh)|$$

and $|x(t-nh)| \geq |x(t-(n+1)h)|$.

Then from (13)

$$
\begin{aligned}
(1-|\alpha|)|x(t-nh)| &\leq \int_{-\infty}^{t-nh} [\,(|\beta|\,|x(s)|\,+|\gamma|\,|x(s-h)|)\;|C(t-nh,s)|ds+|p(t-nh)| \\
&\leq \int_{-\infty}^{t_0} (\,|\beta|+|\gamma|\,)X|C(t-nh,s)|ds+|p(t-nh)| \\
&+\int_{t_0}^{t-nh} |\beta|\,|x(s)|\,|C(t-nh,s)|ds+\int_{t_0}^{t-nh} |\gamma|\,|x(s-h)|\,|C(t-nh,s)|ds \\
&\leq J(|\beta|+|\gamma|)X+|p(t-nh)|+\left(\int_{t_0}^{t-nh} |\beta|^2|x(s)|^2ds\right)^{1/2}\left(\int_{t_0}^{t-nh} |C(t-nh,s)|^2ds\right)^{1/2} \\
&+\left(\int_{t_0}^{t-nh} |\gamma|^2|x(s-h)|^2ds\right)^{1/2}\left(\int_{t_0}^{t-nh} |C(t-nh,s)|^2ds\right)^{1/2} \\
&\leq J(|\beta|+|\gamma|)X+|p(t-nh)|+\left(\int_{t_0}^{t-nh} |\beta|^2|x(s)|^2ds\right)^{1/2}\left(\int_{t_0}^{t-nh} |C(t-nh,s)^2ds\right)^{1/2} \\
&+\left(\int_{t_0-h}^{t-(n+1)h} |\gamma|^2|x(s)|^2ds\right)^{1/2}\left(\int_{t_0}^{t-nh} |C(t-nh,s)|^2ds\right)^{1/2} \\
&\leq J(|\beta|+|\gamma|)X+|p(t-nh)|+J(|\beta|+|\gamma|)\left(\int_{t_0-h}^{t} |x(s)|^2ds\right)^{1/2} \\
&\leq J(|\beta|+|\gamma|)X+|p(t-nh)|+J(|\beta|+|\gamma|)\left(X^2h+\int_{t_0}^{t} |x(s)|^2ds\right)^{1/2} \\
&\leq (J(|\beta|+|\gamma|)+2h^{1/2})X+|p(t-nh)|+J(|\beta|+|\gamma|)2\left(\int_{t_0}^{t} |x(s)|^2ds\right)^{1/2}.
\end{aligned}
$$

So,

$$
|x(t)| \leq c_3^* X+\tfrac{1}{1-|\alpha|}|p(t-nh)|+c_5\left(\int_{t_0}^{t} |x(s)|^2ds\right)^{1/2}, \tag{20}
$$

and using the results of Theorem 2 (i)

$$
\begin{aligned}
|x(t)| &\leq c_3^* X+\tfrac{1}{1-|\alpha|}|p(t-nh)|+c_5^*\left(c_1X+c_2\left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2}\right) \\
&\leq c_3X+c_4\left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2}+\tfrac{1}{1-|\alpha|}|p(t-nh)|,
\end{aligned} \tag{21}
$$

where the c_i are independent of t_0 and X. ■

Proof of Theorem 2 (iii).

Given $t \geq t_0$, choose $\hat{n} = 0, 1\ldots$ so that $t - \hat{n}h \in [t_0, t_0 + h]$. If $\hat{n}$ is such that $|x(t-\hat{n}h)| > |x(t-(\hat{n}-1)h)| > \cdots > |x(t)|$, then following the proof of Theorem 2(ii),

$$
\begin{aligned}
&|x(t-\hat{n}h)| \leq \\
&|\alpha||x(t-(\hat{n}+1)h)| + \int_{-\infty}^{t-\hat{n}h} (\,|\beta|\,|x(s)| \,+ |\gamma|\,|x(s-h)|\,)\,|C(t-\hat{n}h,s)|ds + |p(t-\hat{n}h)| \\
&\leq |\alpha|X + \int_{-\infty}^{t_0} (\,|\beta| + |\gamma|\,)\,X|C(t-\hat{n}h,s)|ds + |p(t-\hat{n}h)| \\
&+ \int_{t_0}^{t-\hat{n}h} |\beta|\,|x(s)|\,|C(t-\hat{n}h,s)|ds + \int_{t_0}^{t-\hat{n}h} |\gamma|\,|x(s-h)|\,|C(t-\hat{n}h,s)|ds \\
&\leq (|\alpha| + J(|\beta|+|\gamma|) + h^{1/2})X + |p(t-\hat{n}h)| + J(|\beta|+|\gamma|)\left(\int_{t_0}^{t} |x(s)|^2 ds\right)^{1/2}
\end{aligned}
$$

and using Theorem 2(i),

$$
|x(t)| \leq |x(t-\hat{n}h)| \leq c_3^* X + |p(t-\hat{n}h)| + c_5 \left(\int_{t_0}^{t} |x(s)|^2 ds\right)^{1/2}
$$

$$
\begin{aligned}
&\leq c_3^* X + |p(t-\hat{n}h)| + c_5^* \left(c_1 X + c_2 \left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2}\right) \\
&\leq c_3 X + c_4 \left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2} + \tfrac{1}{1-|\alpha|}|p(t-\hat{n}h)|,
\end{aligned}
$$

where the c_i are independent of t_0 and X.

Alternatively, if $\Big\{|x(t-kh)|\Big\}_{k=0}^{k=\hat{n}}$ fails to be a strictly increasing sequence, then for the largest $n \in \{0, \ldots \hat{n}-1\}$ for which $\Big\{|x(t-kh)|\Big\}_{k=0}^{k=n}$ is a strictly increasing sequence, we have that $|x(t-nh)| > |x(t-(n-1)h)| > \cdots > |x(t)|$, and $|x(t-(n+1)h)| \leq |x(t-nh)|$. Using the results of Theorem 2 (i) and (ii) we have

$$
|x(t)| \leq c_3 X + c_4 \left(\int_{t_0}^{t} p^2(s)ds\right)^{1/2} + \tfrac{1}{1-|\alpha|}|p(t-nh)|
$$

where the c_i are independent of t_0 and X. This proves (iii). In particular then (iv) is also true. ■

Proof of Theorem 3.

From the proof of Theorem 2 (i), $V'(t,x_t) \leq -\mu x^2(t) + Mp^2(t)$ and then since $x \in \mathcal{P}_T$ implies $V(t) = V(0)$, we have by integration $\int_0^T V'(s)ds \leq -\mu \int_0^T x^2(s)ds + M \int_0^T p^2(s)ds$ or $\int_0^T x^2(s)ds \leq \frac{M}{\mu} \int_0^T p^2(s)ds$.

Note that $I(u) = [2\beta^2 x^2(u) + 2\gamma^2 x^2(u-h)]$ is T-periodic in u, so that its integral over every interval of length T is given by

$$\int_0^T (2\beta^2 + 2\gamma^2)x^2(u)\, du,$$

and then for any $s, t : -\infty < s \leq t < \infty$ if we denote the greatest integer function by $\lfloor|\frac{(t-s)}{T}|\rfloor$ we have

$$\begin{aligned}
&\int_s^t [2\beta^2 x^2(u) + 2\gamma^2 x^2(u-h)]du = \\
&\quad \sum_{i=1}^{i=\lfloor|\frac{(t-s)}{T}|\rfloor} \int_{s+(i-1)T}^{s+iT} [2\beta^2 x^2(u) + 2\gamma^2 x^2(u-h)]du \\
&\quad + \int_{s+\lfloor|\frac{(t-s)}{T}T|\rfloor}^{t} [2\beta^2 x^2(u) + 2\gamma^2 x^2(u-h)]du \\
&\quad \leq \sum_{i=1}^{i=\lfloor|\frac{(t-s)}{T}|\rfloor} \int_0^T [2\beta^2 + 2\gamma^2]x^2(u)du + \int_{t-T}^{t} [2\beta^2 x^2(u) + 2\gamma^2 x^2(u-h)]du \\
&\quad \leq (\lfloor|\frac{(t-s)}{T}|\rfloor + 1)[2\beta^2 + 2\gamma^2] \int_0^T x^2(u)du \\
&\quad \leq (\frac{t-s}{T} + 1)[2\beta^2 + 2\gamma^2] \int_0^T x^2(u)du \\
&\quad \leq (t - s + T)[2\beta^2 + 2\gamma^2] \frac{1}{T} \int_0^T x^2(u)du.
\end{aligned}$$

Using this inequality, we integrate by parts, use Schwarz' inequality, and then Cauchy's inequality to obtain

$$
\begin{aligned}
&[x(t) - \alpha x(t-h) - p(t)]^2 \\
&\quad \leq \left[\int_{-\infty}^{t} (\beta x(s) + \gamma x(s-h))\, C(t,s)\, ds \right]^2 \\
&\quad \leq C(t,t) \int_{-\infty}^{t} C_s(t,s)(t-s) \int_s^t [2\beta^2 x^2(u) + 2\gamma^2 x^2(u-h)] du\, ds \\
&\quad \leq C(t,t) \int_{-\infty}^{t} C_s(t,s)(t-s)(t-s+T)\, ds [2\beta^2 + 2\gamma^2] \frac{1}{T} \int_0^T x^2(u)\, du,
\end{aligned}
$$

and using Theorem 3(i)

$$
\begin{aligned}
|x(t)| &\leq |\alpha|\, |x(t-h)| + |p(t)| + c \left(\int_0^T x^2(s) ds \right)^{1/2} \\
&\leq |\alpha|\, |x(t-h)| + |p(t)| + \overline{c} \left(\int_0^T p^2(s) ds \right)^{1/2}.
\end{aligned}
$$

Then taking the supremum for $t \in [0, T]$, in the above inequality and rearranging we get Theorem 3(ii). ■

Definition. For $x : [0, \infty) \to \mathbb{R}$, define the norm

$$
|x|_D = \sup_{t \geq 0} \frac{|x(t)| + \left(\int_0^t x^2(s) ds \right)^{1/2}}{1 + \sup\limits_{0 \leq s \leq t} |p(s)| + \left(\int_0^t p^2(s) ds \right)^{1/2}},
$$

if it exists, and for $x, p \in \mathcal{P}_T$, the norm

$$
|x|_{\mathcal{P}_T} = \sup_{0 \leq t \leq T} \frac{|x(t)| + \left(\int_0^t x^2(s) ds \right)^{1/2}}{1 + |p(t)| + \left(\int_0^t p^2(s) ds \right)^{1/2}}.
$$

Theorem 4. *Suppose (14)′-(17)′ hold. Then there are constants $c_i \geq 0$ such that*

(i) *any solution $x \in \mathcal{P}_T$ of (13)′ satisfies*

$$
|x|_{P_T} \leq \tfrac{1}{1-|\alpha|} \sup_{t \geq 0} |p(t)| + c_1 \left(\int_0^T p^2(s) ds \right)^{1/2},
$$

(ii) *any continuous solution* $x(t,0,\varphi)$ *of (13) with bounded* ϕ *satisfies*

$$|x|_D \le c_6||\varphi|| + \tfrac{1}{1-|\alpha|}\sup_{t\ge 0}|p(t)| + c_7\left(\int_0^T p^2(s)ds\right)^{1/2},$$

where $||\varphi|| = \sup_{t\le 0}|\varphi(s)|$.

Proof of Theorem 4.

(i) Using the result of Theorem 3,

$$|x|_{\mathcal{P}_T} = \sup_{0\le t\le T}\frac{|x(t)| + \left(\int_0^t x^2(s)ds\right)^{1/2}}{1+|p(t)| + \left(\int_0^t p^2(s)ds\right)^{1/2}} \le \sup_{0\le t\le T}\frac{\frac{1}{1-|\alpha|}|p(t)| + c_3\left(\int_0^T p^2(s)ds\right)^{1/2}}{1+|p(t)| + \left(\int_0^T p^2(s)ds\right)^{1/2}}$$

$$\le \frac{1}{1-|\alpha|}\sup_{0\le t\le T}|p(t)| + c_4\left(\int_0^T p^2(s)ds\right)^{1/2}.$$

(ii) Using the results of Theorem 2 (iv),

$$|x|_D = \sup_{t\ge 0}\frac{|x(t)| + \left(\int_0^t x^2(s)ds\right)^{1/2}}{1+\sup_{0\le s\le t}|p(s)| + \left(\int_0^t p^2(s)ds\right)^{1/2}} \le \sup_{t\ge 0}\frac{\frac{1}{1-|\alpha|}\sup_{0\le s\le t}|p(s)| + c_1||\varphi|| + c_2\left(\int_0^t p^2(s)ds\right)^{1/2}}{1+\sup_{0\le s\le t}|p(s)| + \left(\int_0^t p^2(s)ds\right)^{1/2}}$$

$$\le \frac{1}{1-|\alpha|}\sup_{t\ge 0}|p(t)| + c_6||\varphi|| + c_7\left(\int_0^t p^2(s)ds\right)^{1/2}. \quad \blacksquare$$

Remark. There are other forms for Theorem 4. Clearly, $|x|_{P_T} \le c_3 + \frac{1}{1-|\alpha|}$; but that does not reflect the fact that $|x|_{P_T} \to 0$ as $\|p\| \to 0$. Also, $|x|_D \le \frac{1}{1-|\alpha|} + c_6\|\phi\|$; but that also fails to reflect smallness related to p.

Theorem 5. *Suppose* $p \in L^2[0,\infty)$ *and* $x(t,0,\varphi)$ *is a continuous solution of (14), and* $V(t)$ *is the Liapunov function in the proof of Theorem 2 (i).*

Then $x \in L^2[0,\infty)$, $V(\infty) = \lim_{t\to\infty} V(t)$ *exists and*

$$\int_t^\infty x^2(s)ds \le \frac{1}{\mu}[V(t) - V(\infty)] + \frac{M}{\mu}\int_t^\infty p^2(s)ds.$$

Proof of Theorem 5. By the inequality in the proof of Theorem 2(i)

$$V'(t) \leq -\mu x^2(t) + Mp^2(t) \qquad t > 0,$$

so for any $\rho > 0$

$$\int_t^{t+\rho} V'(s)ds \leq -\mu \int_t^{t+\rho} x^2(s)ds + M \int_t^{t+\rho} p^2(s)ds$$

and taking the $\lim_{\rho \to \infty}$ yields

$$V(\infty) + \mu \int_t^\infty x^2(s)ds \leq V(t) + M \int_t^\infty p^2(s)ds. \quad \blacksquare$$

We now use a combination of results of Krasnoselskii [5] and Schaefer [7].

Theorem (Burton-Kirk[3]). *Let $(\mathcal{S}, ||\cdot||)$ be a Banach space, $A, B : \mathcal{S} \to \mathcal{S}$, B a contraction, A continuous and compact. Either*

(i) *$x = \lambda B(x/\lambda) + \lambda Ax$ has a solution for $\lambda = 1$ or*

(ii) *the set of all such solutions, $0 < \lambda < 1$, is unbounded.*

Proposition 2. *If (14)′-(17)′ hold, then (13)′ has a continuous T-periodic solution on $\mathbb{R}$.*

Proof of Proposition 2. Let $(\mathcal{P}_T, ||\cdot||)$ denote the Banach space of T-periodic continuous functions on $\mathbb{R}$ with the sup norm.

Define $A, B : \mathcal{P}_T \to \mathcal{P}_T$ by $x \in \mathcal{P}_T$ implies

$$\begin{aligned}(Ax)(t) &= -\int_{-\infty}^t [\beta x(s) + \gamma x(s-h)]C(t,s) + p(t) \\ (Bx)(t) &= \alpha x(t-h).\end{aligned}$$

Then B is a contraction on $\mathcal{P}_T$.

If $x_n \to x$ in $\mathcal{P}_T$, then

$$\begin{aligned}|(Ax_n)(t) - (Ax)(t)| &= |\int_{-\infty}^t \beta[x_n(s) - x(s)] + \gamma[x_n(s-h) - x(s-h)]C(t,s)ds| \\ &\leq J(|\beta| + |\gamma|)||x_n - x||;\end{aligned}$$

thus, $||Ax_n - Ax|| \to 0$ if $||x_n - x|| \to 0$, and A is continuous in $\mathcal{P}_T$. Let $x_i \in \mathcal{P}_T$, then $|(Ax_1)(t) - (Ax_2)(t)| \le J(|\beta| + |\gamma|)||x_1 - x_2||$ or $||(Ax_1)(t) - (Ax_2)(t)|| \le C||x_1 - x_2||$ so A takes $\mathcal{P}_T$ into an equicontinuous set. If K is a bounded set in $\mathcal{P}_T$ and $x \in K$ with $||x|| \le M$, then $||Ax|| \le J(|\beta| + |\gamma|)||x|| + ||p|| \le M'$ so AK is uniformly bounded and equicontinuous, hence precompact in P_T. So A is a compact map.

Consider then

$$x(t) = \alpha x(t-h) - \lambda \left[\int_{-\infty}^{t} [\beta x(s) + \gamma x(s-h)] C(t,s) ds + p(t) \right] \quad t \in \mathbb{R}$$

and for A and B defined as in the proof of Proposition 2 we write this as

$$(13)'_\lambda \qquad x = \lambda B(x/\lambda) + \lambda A x.$$

For $x \in \mathcal{P}_T$, $\lambda \in (0,1)$ and ϵ_2 as in the proof of Theorem 2 (i), define

$$V_\lambda(t) = \int_{-\infty}^{t} C_s(t,s) \left[\int_s^t (\lambda\beta x(u) + \lambda\gamma x(u-h)) du \right]^2 ds + (2\alpha\lambda\gamma + |\alpha\lambda\beta - \lambda\gamma| + \lambda\epsilon_2) \int_{t-h}^{t} x^2(u) du$$

so $V'_\lambda(t) \le -\mu\lambda x^2(t) + M\lambda p^2(t)$ and

$$\int_0^t V'_\lambda(s) ds \le -\mu\lambda \int_0^t x^2(s) ds + M\lambda \int_0^T p^2(s) ds.$$

But $V(t) = V(0)$ since $x \in \mathcal{P}_T$ and so for $x \in \mathcal{P}_T$

$$(22) \qquad \int_0^T x^2(s) ds \le \frac{M}{\mu} \int_0^T p^2(s) ds := \hat{M}.$$

By the same argument used in the proof of Theorem 3, for any $s, t : -\infty < s \le t < \infty$ we have

$$\int_s^t [2\beta^2 x^2(u) + (\gamma^2 + 1) x^2(u-h)] du \le (\lfloor |t-s| \rfloor + T)[2\beta^2 + \gamma^2 + 1] \frac{1}{T} \int_0^T x^2(u) du.$$

Then by $(13)'_\lambda$

$$
\begin{aligned}
(x(t)-\alpha x(t-h)-\lambda p(t))^2 &\le \lambda^2\left[\int_{-\infty}^t [\beta x(s)+\gamma x(s-h)]C(t,s)ds\right]^2\\
&\le \lambda^2\left[-C(t,s)\int_s^t \beta x(u)+\gamma x(u-h)du\Big|_{-\infty}^t+\int_{-\infty}^t C_s(t,s)\int_s^t \beta x(u)+\gamma x(u-h)du\,ds\right]^2\\
&\le \lambda^2\left[\int_{-\infty}^t C_s(t,s)\int_s^t \beta x(u)+\gamma x(u-h)du\,ds\right]^2\\
&\le \lambda^2\left[\int_{-\infty}^t \sqrt{C_s(t,s)}\sqrt{C_s(t,s)}\int_s^t \beta x(u)+\gamma x(u-h)du\,ds\right]^2\\
&\le \lambda^2\int_{-\infty}^t C_s(t,s)ds\int_{-\infty}^t C_s(t,s)\left[\int_s^t \beta x(u)+\gamma x(u-h)du\right]^2 ds\\
&\le \lambda^2 C(t,t)\int_{-\infty}^t C_s(t,s)\int_s^t 1^2du\int_s^t [\beta x(u)+\gamma x(u-h)]^2du\,ds\\
&\le \lambda^2 C(t,t)\int_{-\infty}^t C_s(t,s)(t-s)\int_s^t [2\beta^2x^2(u)+2\gamma^2x^2(u-h)]du\,ds\\
&\le \lambda^2 C(t,t)\int_{-\infty}^t C_s(t,s)(t-s)(t-s+T)[2\beta^2+2\gamma^2]ds\,\frac{1}{T}\int_0^T x^2(u)\,du\le Y^2,
\end{aligned}
$$

for some Y.

So, $|x-\alpha x(t-h)-\lambda p(t)|\le Y$ and $|x(t)|\le \alpha|x(t-h)|+Y+\lambda||p||\le \alpha|x(t-h)|+\delta$, and for $0\le t\le T$, the same argument can be applied as in Theorem 3 to establish that $\lim_{n\to\infty}|x(t+nh)|=\delta/(1-|\alpha|)$. Then by periodicity, $|x(t)|\le \delta/(1-|\alpha|)$ for all $t\in\mathbb{R}$; that is solutions of $x=\lambda B(x/\lambda)+\lambda Ax$ for $0<\lambda<1$ are bounded. Then (ii) of the Burton-Kirk theorem cannot happen and since all the conditions are present, (i) must be fulfilled; that is, $(13)'_\lambda$ has a solution in P_T for $\lambda=1$. This means that $(13)'$ has a T-periodic continuous solution on $\mathbb{R}$. ■

Remark. In Theorem 2 the inclusion of t_0 as the lower limit of the integrals, instead of $\int_0^t$ is crucial to understanding the behavior of the solution. If we take $t_0=0$, then a solution $x(t)$ could remain very small and $\int_0^t x^2(s)ds$ could remain very small for a very long time so that $\int_0^t p^2(s)ds$ could become very large, even if $|p(t)|$ remains very small. Suddenly,

$|x(t)|$ could become very large. With the inclusion of t_0, we can see that the growth of $|x(t)|$ is controlled by $|p(t)|$.

Equation (19) will define a weighted norm on a solution space and can be used to obtain existence results. But this weight ceases to make a norm in the nonlinear case, as we now show.

It is now interesting to see how nonlinearities affect (19). We consider an equation

$$x(t) = \alpha x(t-h) - \int_{-\infty}^{t} [g(x(s)) + r(x(s-h))]C(t,s)ds + p(t).$$

The inequalities which we encounter work when

$$g(x) = \sum_{n=0}^{\infty} a_n x^{2n+1} \qquad a_n \geq 0,$$

and $r(x)$ is dominated by $g(x)$, but the details became cumbersome. All of the ideas emerge in the equation

$$(23) \qquad x(t) = \alpha x(t-h) - \int_{-\infty}^{t} [\beta x^3(s) + \gamma x^3(s-h)]C(t,s)ds + p(t) \qquad t \geq 0,$$

$$(24) \qquad p \in C\,[\,[0,\infty), \mathbb{R}\,], \quad C \in \mathcal{C}(\Omega, \mathbb{R}) \quad \Omega = \{(t,s) | -\infty < s \leq t < \infty\},$$

there is a $J > 1$ such that

$$(25) \qquad \sup_{t\geq 0} |C(t,t)| \int_{-\infty}^{t} |C(t,s)| + C^4(t,s) + |C_s(t,s)|(t-s+h)^{1/3}(t-s+1)ds \leq J,$$

$$(26) \qquad C_s(t,s) \geq 0,\ C_{st}(t,s) \leq 0 \ \text{ for } (t,s) \in \Omega \quad \lim_{s\to\infty} sC(t,s) = 0 \text{ for } t \in [0,\infty),$$

$$(27) \qquad |\alpha| < 1 \qquad \beta > |\alpha\beta| + |\gamma| + |\alpha\gamma|.$$

We also consider its periodic analogue on the whole axis

(23)′ $$x(t) = \left[\alpha x(t-h) - \int_{-\infty}^{t} [\beta x^3(s) + \gamma x^3(s-h)]ds + p(t)\right]\lambda, \qquad t \in \mathbb{R}$$

where

(24)′
$p \in \mathcal{C}(\mathbb{R}, R),\ C \in \mathcal{C}(\Omega, R) \quad \Omega = \{(t,s) | -\infty < s \le t < \infty\}$ and there is a $T > 0$ with

$p(t+T) = p(t) \quad t \in \mathbb{R},\ C(t+T, s+T) = C(t,s)$ for $(t,s) \in \Omega$

there is $J > 1$ such that

(25)′ $$\sup_{t\ge 0} |C(t,t)| \int_{-\infty}^{t} |C(t,s)| + C^4(t,s) + |C_s(t,s)|(t-s+h)^{1/3}(t-s+T)ds \le J,$$

(26)′ $$C_s(t,s) \ge 0,\ C_{st}(t,s) \le 0 \quad \text{for } (t,s) \in \Omega \quad \lim_{s\to-\infty} sC(t,s) = 0 \text{ for } t \in [0,\infty),$$

and

(28)′ $$|\alpha| < 1 \qquad \beta > |\alpha\beta| + |\gamma| + |\alpha\gamma|.$$

Theorem 6. *Suppose (24)-(27) hold. Then there are constants $c_i \ge 0$ such that whenever a continuous solution $x(t, t_0, \varphi)$ of (23) satisfies $|x(t, t_0, \varphi)| \le X$ for $t \in (-\infty, t_0]$ for some $t_0 \ge 0$ and $X > 1$, we have*

(i) *for $t \ge t_0$*

$$\left(\int_{t_0}^{t} x^4(s)ds\right)^{1/4} \le c_1 X^{3/2} + c_2 \left(\int_{t_0}^{t} p^4(s)ds\right)^{1/4},$$

(ii) *if $t \ge t_0$ and $n = 0, 1\ldots$ are such that $t - nh \ge t_0$,*

$$|x(t)| < |x(t-h)| < \cdots < |x(t-nh)|$$

and $|x(t-nh)| \ge |x(t-(n+1)h)$, then

$$|x(t)| \le \tfrac{1}{1-\alpha}|p(t-nh)| + c_3 X^{9/2} + c_4 \left(\int_{t_0}^{t} p^4(s)ds\right)^{3/4},$$

(iii) *for each* $t \geq t_0$*, there is an* $n = 0, 1 \ldots$ *such that* $t - nh \geq t_0$ *and*

$$|x(t)| \leq \tfrac{1}{1-|\alpha|}|p(t-nh)| + c_5 X^{9/2} + c_6 \left(\int_{t_0}^{t} p^4(s)ds \right)^{3/4},$$

(iv) *for* $t \geq t_0$*,*

$$|x(t)| \leq \tfrac{1}{1-|\alpha|} \sup_{t_0 \leq s \leq t} |p(s)| + c_5 X^{9/2} + c_6 \left(\int_{t_0}^{t} p^4(s)ds \right)^{3/4}.$$

Proof of Theorem 6(i). Using (27), choose $\epsilon_i > 0$ so that $|\beta| - |\alpha\beta| - |\gamma| - |\alpha\gamma| + \frac{\epsilon_1+\epsilon_2}{2} = \frac{\mu}{2} > 0$. Let $x(t, 0, \varphi)$ be a continuous solution of (23).

Define

$$V(t, x_t) = \int_{-\infty}^{t} C_s(t,s) \left[\int_s^t (\beta x^3(u) + \gamma x^3(u-h)du \right]^2 ds + \left[\frac{|\alpha\beta|}{2} + 2|\alpha\gamma| + \frac{3}{2}|\gamma| + 2\epsilon_2 \right] \int_{t-h}^{t} x^4(u)du.$$

Then

$$\begin{aligned}
V'(t) &\le \int_{-\infty}^{t} C_{st}(t,s)\left[\int_s^t (\beta x^3(u)+\gamma x^3(u-h)du\right]^2 ds \\
&+ 2\int_{-\infty}^{t} C_s(t,s)[\beta x^3(t)+\gamma x^3(t-h)]\int_s^t (\beta x^3(u)+\gamma x^3(u-h)du\,ds \\
&+ \left(\frac{|\alpha\beta|}{2}+2|\alpha\gamma|+\frac{3}{2}|\gamma|+2\epsilon_2\right)[x^4(t)-x^4(t-h)] \\
&\le \left(\frac{|\alpha\beta|}{2}+2|\alpha\gamma|+\frac{3}{2}|\gamma|+2\epsilon_2\right)[x^4(t)-x^4(t-h)] \\
&+ 2[\beta x^3(t)+\gamma x^3(t-h)]\Bigg[C(t,s)\int_s^t (\beta x^3(u)+\gamma x^3(u-h)du\Bigg|_{-\infty}^{t} \\
&+ \int_{-\infty}^{t} (\beta x^3(s)+\gamma x^3(s-h))C(t,s)ds\Bigg] \\
&\le \left[\frac{|\alpha\beta|}{2}+2|\alpha\gamma|+\frac{3}{2}|\gamma|+2\epsilon_2\right][x^4(t)-x^4(t-h)] \\
&+ 2[\beta x^3(t)+\gamma x^3(t-h)][\alpha x(t-h)+p(t)-x(t)] \\
&\le \left[\frac{|\alpha\beta|}{2}+2|\alpha\gamma|+\frac{3}{2}|\gamma|+2\epsilon_2\right][x^4(t)-x^4(t-h)] \\
&+ 2[\beta\alpha x^3(t)x(t-h)+\beta x^3(t)p(t)-\beta x^4(t)+|\gamma\alpha|x^4(t-h) \\
&+ \gamma x^3(t-h)p(t)-\gamma x^3(t-h)x(t)] \\
&\le \left[\frac{|\alpha\beta|}{2}+2|\alpha\gamma|+\frac{3}{2}|\gamma|+2\epsilon_2\right][x^4(t)-x^4(t-h)] \\
&+ 2\left[|\alpha\beta|\left[\frac{3}{4}x^4(t)+\frac{1}{4}x^4(t-h)\right]+\epsilon_1 x^4(t)\right] \\
&+ \frac{3^{3/4}}{4^{7/4}}\,\frac{\beta^{1/4}}{\epsilon_1^{3/4}}\,p^4(t)-\beta x^4(t)+|\gamma\alpha|x^4(t-h)+\epsilon_2 x^4(t-h) \\
&+ \frac{3^{3/4}}{4^{7/4}}\,\frac{\gamma^{1/4}}{\epsilon_2^{3/4}}p^4(t)+|\gamma|\left(\frac{x(t)^4}{4}+\frac{3}{4}x^4(t-h)\right) \\
&\le 2x^4\left[\frac{3}{4}|\alpha\beta|+\epsilon_1-\beta+\frac{|\gamma|}{4}\right]+x^4(t-h)\left[\frac{|\gamma\beta|}{4}+|\gamma\alpha|+\epsilon_2+\frac{3}{4}|\gamma|\right] \\
&+ p^4(t)\hat{M}+\left(\frac{|\alpha\beta|}{2}+2|\alpha\gamma|+\frac{3}{2}|\gamma|+2\epsilon_2\right)[x^4(t)-x^4(t-h)] \\
&= 2[-\beta+|\alpha\beta|+|\gamma|+\epsilon_1+\epsilon_2+|\alpha\gamma|\,]|x(t)|^4+\hat{M}|p^4(t)|.
\end{aligned}$$

So

$$V'(t) \le -\mu x^4(t)+\hat{M}p^4(t). \tag{28}$$

Next, we note that there is a $J^* > 1$ with

$$V(t_0) \le \int_{-\infty}^{t_0} C_s(t,s)[|\beta| + |\gamma|]^2 X^6 (t_0 - s)^2 + \left[\frac{|\alpha\beta|}{2} + 2|\alpha\gamma| + \frac{3}{2}|\gamma| + 2\epsilon_2\right] hX^4$$
$$\le J[\,|\beta| + |\gamma|\,]^2 X^6 + \overline{c}X^4 \le J^* X^6.$$

Then integrating (28)

$$\int_{t_0}^{t} V'(s)ds \le -\mu \int_{t_0}^{t} x^4(s)ds + \hat{M} \int_{t_0}^{t} p^4(s)ds,$$

so

$$0 \le V(t) \le V(t_0) - \mu \int_{t_0}^{t} x^4(s)ds + \hat{M} \int_{t_0}^{t} p^4(s)ds,$$

$$\int_{t_0}^{t} x^4(s)ds \le \overline{c}_1 X^6 + \overline{c}_2 \int_{t_0}^{t} p^4(s)ds \tag{29}$$

and

$$\left(\int_{t_0}^{t} x^4(s)ds\right)^{1/4} \le c_1 X^{3/2} + c_2 \left(\int_{t_0}^{t} p^4(s)ds\right)^{1/4}. \quad \blacksquare \tag{29$'$}$$

Proof of Theorem 6 (ii). Suppose $t \ge t_0$ $n = 0, 1, \ldots$ are such that $t - nh \ge t_0$,

$$|x(t)| < |x(t-h)| < \cdots < |x(t-nh)|$$

and $|x(t-nh)| \geq |x(t-(n+1)h)|$. Then from (23)

$$
\begin{aligned}
(1-|\alpha|)|x(t-nh)| &\leq \int_{-\infty}^{t-nh} [|\beta|\,|x(s)|^3 + |\gamma|\,|x(s-h)|^3]|C(t-nh,s)|ds + |p(t-nh)| \\
&\leq \int_{-\infty}^{t_0} (|\beta|+|\gamma|)X^3|C(t-nh,s)|ds \\
&+ \int_{t_0}^{t-nh} [|\beta|\,|x(s)|^3]|C(t-nh,s)|ds \\
&+ \int_{t_0}^{t-nh} |\gamma|\,|x(s-h)|^3|C(t-nh,s)|ds + |p(t-nh)| \\
&\leq |p(t-nh)| + (|\beta|+|\gamma|)X^3 J \\
&+ \left(\int_{t_0}^{t-nh} |x(s)|^4 ds\right)^{3/4} \left(\int_{t_0}^{t-nh} |\beta|^4 C^4(t-nh,s)ds\right)^{1/4} \\
&+ \left(\int_{t_0-h}^{t-nh-h} |\gamma|^4|C^4(t,s+h)|ds\right)^{1/4} \left(\int_{t_0-h}^{t-(n+1)h} |x(s)|^4 ds\right)^{3/4} \\
&\leq |p(t-nh)| + (|\beta|+|\gamma|)X^3 J + |\beta| J \left(\int_{t_0}^{t} |x(s)^4 ds\right)^{3/4} \\
&+ |\gamma| J \left(\int_{t_0-h}^{t} |x(s)^4 ds\right)^{3/4} \leq |p(t-nh)| + (|\beta|+|\gamma|)X^3 J \\
&+ |\beta| J \left(\int_{t_0}^{t} |x(s)^4 ds\right)^{3/4} + |\gamma| J \left(X^4 h + \int_{t_0}^{t} |x(s)|^4 ds\right)^{3/4} \\
&\leq |p(t-nh)| + \overline{c}_3 X^3 + \overline{c}_4 \left(\int_{t_0}^{t} |x(s)|^4 ds\right)^{3/4} \\
&\leq |p(t-nh)| + \overline{c}_3 X^3 + \overline{c}_4 \left(\overline{c}_1 X^6 + \overline{c}_2 \int_{t_0}^{t} |p(s)|^4 ds\right)^{3/4}.
\end{aligned}
$$

Then

$$
|x(t)| < |x(t-nh)| \leq \tfrac{1}{1-|\alpha|}|p(t-nh)| + c_3 X^{9/2} + c_4 \left(\int_{t_0}^{t} |p^4(s)|ds\right)^{3/4}. \qquad \blacksquare
$$

Proof of Theorem 6(iii). Given $t \geq t_0$, choose $\hat{n} = 0, 1\ldots$ so that $t - \hat{n}h \in [t_0, t_0+h]$. If $\hat{n}$ is such that $|x(t-\hat{n}h)| > |x(t-(\hat{n}-1)h)| > \cdots > |x(t)|$, then following the proof of

Theorem 6(ii),

$$
\begin{aligned}
&|x(t-\hat{n}h)| \leq \\
&\quad |\alpha||x(t-(\hat{n}+1)h)| + \int_{-\infty}^{t-\hat{n}h} \left(|\beta|\,|x(s)|^3 + |\gamma|\,|x(s-h)|^3 \right) |C(t-\hat{n}h,s)|ds + |p(t-\hat{n}h)| \\
&\quad \leq |\alpha|X + \int_{-\infty}^{t_0} \left(|\beta| + |\gamma| \right) X^3 |C(t-\hat{n}h,s)|ds + |p(t-\hat{n}h)| \\
&\quad + \int_{t_0}^{t-\hat{n}h} |\beta|\,|x(s)|^3\,|C(t-\hat{n}h,s)|ds + \int_{t_0}^{t-\hat{n}h} |\gamma|\,|x(s-h)|^3\,|C(t-\hat{n}h,s)|ds \\
&\quad \leq (|\alpha| + J(|\beta|+|\gamma|))X^3 + |p(t-\hat{n}h)| \\
&\quad \left(\int_{t_0}^{t-\hat{n}h} |x(s)|^4 ds \right)^{3/4} \left(\int_{t_0}^{t-\hat{n}h} |\beta|^4 C^4(t-nh,s) ds \right)^{1/4} \\
&\quad + \left(\int_{t_0-h}^{t-\hat{n}h-h} |\gamma|^4 |C^4(t,s+h)| ds \right)^{1/4} \left(\int_{t_0-h}^{t-(\hat{n}+1)h} |x(s)|^4 ds \right)^{3/4} \\
&\quad \leq |p(t-\hat{n}h)| + (|\alpha| + J(|\beta|+|\gamma|))X^3 + |\beta|J \left(\int_{t_0}^{t} |x(s)^4 ds \right)^{3/4} \\
&\quad + |\gamma|J \left(\int_{t_0-h}^{t} |x(s)^4 ds \right)^{3/4} \leq |p(t-\hat{n}h)| + \overline{c}_3 X^3 \\
&\quad + |\beta|J \left(\int_{t_0}^{t} |x(s)^4 ds \right)^{3/4} + |\gamma|J \left(X^4 h + \int_{t_0}^{t} |x(s)|^4 ds \right)^{3/4} \\
&\quad \leq |p(t-\hat{n}h)| + \overline{c}_3 X^3 + \overline{c}_4 \left(\int_{t_0}^{t} |x(s)|^4 ds \right)^{3/4} \\
&\quad \leq |p(t-\hat{n}h)| + \overline{c}_3 X^3 + \overline{c}_4 \left(\overline{c}_1 X^6 + \overline{c}_2 \int_{t_0}^{t} |p(s)|^4 ds \right)^{3/4},
\end{aligned}
$$

by (29). Thus,

$$
|x(t)| \leq |x(t-\hat{n}h)| \leq c_3^* X^{9/2} + |p(t-\hat{n}h)| + c_5 \left(\int_{t_0}^{t} p^4(s) ds \right)^{3/4}
$$

$$
\leq c_3 X^{9/2} + \tfrac{1}{1-|\alpha|} |p(t-\hat{n}h)| + c_4 \left(\int_{t_0}^{t} p^4(s) ds \right)^{3/4}
$$

,

where the c_i are independent of t_0 and X.

Alternatively, if $\left\{|x(t-kh)|\right\}_{k=0}^{k=\hat{n}}$ fails to be a strictly increasing sequence, then for the largest $n \in \{0, \dots \hat{n}-1\}$ for which $\left\{|x(t-kh)|\right\}_{k=0}^{k=n}$ is a strictly increasing sequence, we have that $|x(t-nh)| > |x(t-(n-1)h)| > \cdots > |x(t)|$, and $|x(t-(n+1)h)| \le |x(t-nh)|$. Using the results of Theorem 6 (i) and (ii) we have

$$|x(t)| \le c_3 X^{9/2} + c_4 \left(\int_{t_0}^{t} p^4(s)ds \right)^{3/4} + \tfrac{1}{1-|\alpha|}|p(t-nh)|$$

where the c_i are independent of t_0 and X. Also, (iv) follows from (iii). ■

Theorem 7. *Suppose (24)′-(27)′ hold. Then there are constants $c_i \ge 0$ such that any solution $x \in P_T$ of (23)′ satisfies*

(i) $\int_0^T x^4(s)ds \le c_1 \int_0^T p^4(s)ds$

(ii) $|x(t)| \le \frac{1}{1-|\alpha|}\|p\| + c_2 \left(\int_0^T p^4(s)ds\right)^{3/4}$.

Proof of Theorem 7. Following the proof of Theorem 6(i), $V'(t) \le -\mu x^4(t) + Mp^4(t)$ and then since $V(t) = V(0)$

$$\int_0^T x^4(s)ds \le \frac{M}{\mu}\int_0^T p^4(s)ds.$$

So

$$\begin{aligned} |x(t) - \alpha x(t-h) - p(t)| &\le \left[\int_{-\infty}^{t} (\beta x^3(s) + \gamma x^3(s-h))C(t,s)ds\right] \\ &\le C^* \left(\int_0^T x^4(s)ds\right)^{3/4} \le \overline{c}\left(\int_0^T p^4(s)ds\right)^{3/4}, \end{aligned}$$

using (25), the arguments of the proof of Theorem 3, and Theorem 7(i). Thus,

$$|x(t)| \le |\alpha||x(t-h)| + |p(t)| + \overline{c}\left(\int_0^T p^4(s)ds\right)^{3/4},$$

so that by taking the supremum we obtain

$$\begin{aligned} |x(t)| &\le \tfrac{1}{1-|\alpha|}\left[\|p\| + \overline{c}\left(\int_0^T p^4(s)ds\right)^{3/4}\right] \\ &\le \tfrac{1}{1-|\alpha|}\|p\| + c_2\left(\int_0^T p^4(s)ds\right)^{3/4}. \end{aligned}$$

■

Definition. For $x : [0, \infty) \to \mathbb{R}$, define

$$|x|_{D_T} = \sup_{t \geq 0} \frac{|x(t)| + \left(\int_0^t x^4(s)ds\right)^{3/4}}{1 + \sup\limits_{0 \leq s \leq t} |p(s)| + \left(\int_0^t p^4(s)ds\right)^{3/4}},$$

if it exists, and for $x \in \mathcal{P}_T$

$$|x|_{D_T} = \sup_{0 \leq t \leq T} \frac{|x(t)| + \left(\int_0^t x^4(s)ds\right)^{3/4}}{1 + |p(t)| + \left(\int_0^t p^4(s)ds\right)^{3/4}}.$$

Theorem 8. *Suppose (24)′-(27)′ hold. Then there are constants $c_i \geq 0$ such that*

(i) *for $x \in P_T$ as a solution of (23)′, then*

$$|x|_{D_T} \leq \tfrac{1}{1-|\alpha|} \sup_{t \geq 0} |p(t)| + c_1 \left(\int_0^T p^4(s)ds\right)^{3/4},$$

(ii) *for $x(t, 0, \varphi)$ a continuous solution of (23),*

$$|x|_D \leq c_6 ||\varphi||^{9/2} + \frac{1}{1 - |\alpha|} \sup_{t \geq 0} |p(t)| + c_7,$$

where $||\varphi|| = \sup\limits_{s \leq 0} |\varphi(s)|$.

Proof of Theorem 8.

(i) Using the result of Theorem 7,

$$|x|_{D_T} = \sup_{0 \leq t \leq T} \frac{|x(t)| + \left(\int_0^t x^4(s)ds\right)^{3/4}}{1 + |p(t)| + \left(\int_0^t p^4(s)ds\right)^{3/4}} \leq \tfrac{1}{1-|\alpha|} \sup_{0 \leq t \leq T} |p(t)| + c_4 \left(\int_0^T p^4(s)ds\right)^{3/4}.$$

(ii) using the results of Theorem 6(iv)

$$|x|_D = \sup_{t \geq 0} \frac{|x(t)| + \left(\int_0^t x^4(s)ds\right)^{3/4}}{1 + \sup\limits_{0 \leq s \leq t} |p(s)| + \left(\int_0^t p^4(s)ds\right)^{3/4}} \leq \frac{1}{1 - |\alpha|} \sup_{t \geq 0} |p(t)| + c_6 ||\varphi||^{9/2} + c_7.$$

■

Proposition 3. *If (24)′-(27)′ hold, then (23)′ has a continuous T-periodic solution on $\mathbb{R}$.*

Proof of Proposition 3. Let $(P_T, ||\cdot||)$ denote the Banach space of T-periodic continuous functions on $\mathbb{R}$ with the sup norm.

Define $A, B : P_T \to \mathcal{P}_T$ by $x \in \mathcal{P}_T$ implies

$$\begin{aligned}(Ax)(t) &= -\int_{-\infty}^{t} [\beta x^3(s) + \gamma x^3(s-h)]C(t,s)ds + p(t)\\ (Bx)(t) &= \alpha x(t-h).\end{aligned}$$

B is a contraction on P_T.

Suppose $x_n \to x$ in $\mathcal{P}_T$. Then with $\hat{M}$ such that $|x_n| \le \hat{M}$ for all n,

$$\begin{aligned}|(Ax_n)(t) - (Ax)(t)| &= \int_{-\infty}^{t} \left[\beta[x_n^3(s) - x^3(s)] + \gamma[x^3(s-h) - x^3(s-h)]\right] C(t,s)ds\\ &\le \int_{-\infty}^{t} \beta[x_n(s) - x(s)]\,[x_n^2(s) + x_n(s)x(s) + x^2(s)]\\ &+ \gamma[x_n(s-h) - x(s-h)]\,[x_n^2(s-h) + x_n(s-h)x(s-h) + x^2(s-h)]C(t,s)ds\\ &\le (|\gamma| + |\beta|)3\hat{M}^2 J||x_n - x||\end{aligned}$$

so $||x_n - x|| \to 0 \Rightarrow ||Ax_n - Ax|| \to 0$, and A is continuous in $\mathcal{P}_T$.

If $\{x_i\} \subseteq \mathcal{P}_T$ is a bounded set with bound M we have

$$||Ax_i|| \le 3J(|\beta| + |\gamma|)||x_i||^3 + ||p|| \le M',$$

so $\{Ax_i\}$ is uniformly bounded in $\mathcal{P}_T$.

By an argument similar to that above, $||Ax_i - Ax_j|| \le \overline{M}||x_i - x_j||$ so $\{Ax_i\}$ is equicontinuous in $\mathcal{P}_T$. Thus, $\{Ax_i\}$ is precompact in $\mathcal{P}_T$ and A is a compact map on bounded sets.

Consider now for each $\lambda \in (0, 1]$

$$x(t) = \alpha x(t-h) - \lambda\left[\int_{-\infty}^{t} \beta x^3(s) + \gamma x^3(s-h)\right] C(t,s)ds + p(t) \quad t \in \mathbb{R}$$

which we write as

$$(23)'_\lambda \qquad x = \lambda B(x/\lambda) + \lambda A x.$$

Following the proof of Theorem 6(i), for $x \in \mathcal{P}_T$, $\lambda \in (0,1)$,and ϵ_2, define

$$V_\lambda(t, x_t) = \int_{-\infty}^{t} C_s(t,s) \left[\int_s^t \left(\lambda\beta x^3(u) + \lambda\gamma x^3(u-h) \right) du \right]^2 ds$$
$$+ \left[\lambda \frac{|\alpha\beta|}{2} + 2\lambda|\alpha\gamma| + \frac{3}{2}\lambda|\gamma| + 2\lambda\epsilon_2 \right] \int_{t-h}^{t} x^4(s)ds.$$

and then

$$(30) \qquad V'_\lambda(t) \le -\lambda\mu x^4(t) + \lambda \hat{M} p^4(t) \qquad \text{for each } \lambda \in (0,1].$$

Integrating from 0 to T and observing that $V(0) = V(T)$ yields

$$(31) \qquad \int_0^T x^4(s)ds \le \frac{\hat{M}}{\mu} \int_0^T p^4(s)ds \le \hat{Y} \qquad \text{for some } \hat{Y} > 0.$$

Note that $I(u) = x^4(u)$ is T-periodic in u, so that its integral over every interval of length T is given by

$$\int_0^T x^4(u)\, du.$$

Then by an argument analagous to that in Theorem 3, for any $s, t : -\infty < s \le t < \infty$ we have

$$\int_{s-h}^{t} x^4(u)du \le (t - s + h + T)\frac{1}{T} \int_0^T x^4(u)du.$$

Using this inequality in $(23)'_\lambda$ yields

$$
\begin{aligned}
&|x(t) - \alpha x(t-h) - \lambda p(t)| \\
&\leq \lambda \left[\int_{-\infty}^{t} [\beta x^3(s) + \gamma x^3(s-h))|C(t,s)| ds \right] \\
&\leq -C(t,s) \int_{s}^{t} \beta x^3(u) + \gamma x^3(u-h) du \Big|_{-\infty}^{t} + \int_{-\infty}^{t} C_s(t,s) \int_{s}^{t} \beta x^3(u) + \gamma x^3(u-h) du\, ds \\
&\leq \int_{-\infty}^{t} C_s(t,s) \int_{s}^{t} \beta x^3(u) + \gamma x^3(u-h) du\, ds \\
&\leq \int_{-\infty}^{t} C_s^{1/4}(t,s) C_s^{3/4}(t,s) \int_{s}^{t} |\beta|\, |x(u)|^3 + |\gamma|\, |x(u-h)|^3 du\, ds \\
&\leq \left(\int_{-\infty}^{t} C_s(t,s) ds \right)^{1/4} \left(\int_{-\infty}^{t} C_s(t,s) \left[\int_{s}^{t} |\beta|\, |x(u)|^3 + |\gamma|\, |x(u-h)|^3 du \right]^{4/3} ds \right)^{3/4} \\
&\leq |C(t,t)|^{1/4} \left(\int_{-\infty}^{t} C_s(t,s) \left[\int_{s}^{t} |\beta|\, |x(u)|^3 du + |\gamma| \int_{s-h}^{t-h} |x^3(u)| du \right]^{4/3} ds \right)^{3/4} \\
&\leq |C(t,t)|^{1/4} \left(\int_{-\infty}^{t} C_s(t,s) [(t-s+T)(|\beta| + |\gamma|) \int_{0}^{T} |x^3(u)| du]^{4/3} ds \right)^{3/4} \\
&\leq |C(t,t)|^{1/4} \left(\int_{-\infty}^{t} C_s(t,s) (|\beta| + |\gamma|)^{4/3} (t-s+T)^{4/3} \left(\int_{0}^{T} (1/T) |x^3(u)| du \right)^{4/3} ds \right)^{3/4} \\
&\leq J(|\beta| + |\gamma|) \left[\frac{1}{T} \int_{0}^{T} x^4(u)\, du \right]^{3/4} \\
&\leq c_8 \hat{Y}^{3/4} := Y
\end{aligned}
$$

Now by an argument similar to that used in the proof of Proposition 2, $|x(t) - \alpha x(t-h) - \lambda p(t)| \leq Y$ implies that there exists $\delta > 0$ so that $|x(t)| \leq \frac{\delta}{1-|\alpha|}$ for every solution $x(t)$ of $(23)'_\lambda$ and each $\lambda \in (0, 1]$.

Then by the Burton-Kirk Theorem, $(23)'$ has a solution in $\mathcal{P}_T$.

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