Fixed Points, Volterra Equations, and Becker's Resolvent

T. A. Burton
Northwest Research Institute
732 Caroline St.
Port Angeles, WA 98362
taburton@olypen.com

Abstract. In a recent paper we derived a stability criterion for a Volterra equation which is based on the contraction mapping principle. It turns out that this criterion has significantly wider application. In particular, when we use Becker's form of the resolvent it readily establishes critical resolvent properties which have been very illusive when investigated by other techniques.

First, it enables us to show that the resolvent is L^1. Next, it allows us to show that the resolvent satisfies a uniform bound and that it tends to zero. These properties are then used to prove boundedness of solutions of a nonlinear problem, establish the existence of periodic solutions of a linear problem, and to investigate asymptotic stability properties. We also apply the results to a Liénard equation with distributed delay and possibly negative damping so that relaxation oscillations may occur.

AMS Subject Classification: 34K20, 34K13, 47H10, 34K11

Key words: Fixed points, Stability, Volterra equations, Resolvent, Liénard equation, Relaxation oscillations.

1. Overview

The purpose of this paper is to present a condition under which we can offer simple, unified, and concise proofs of the three most important properties of the resolvent of a linear Volterra equation. These results rest on Becker's form of the resolvent and very simple contraction mapping arguments.

The resolvent is a function, $R(t,s)$, with which we can express the solution of

$$y'(t) = A(t)y(t) + \int_0^t C(t,s)y(s)ds + p(t), \quad y(0) = y_0$$

as

$$y(t) = R(t,0)y_0 + \int_0^t R(t,s)p(s)ds.$$

If we can show that $\int_0^t |R(t,s)|ds$ is bounded, that $|R(t,s)|$ is bounded, and that $R(t,s) \to 0$ as $t \to \infty$ for fixed s, then we can show boundedness of solutions for bounded p, various stability properties, and a formula for periodic solutions.

Establishing these properties can be challenging and usually requires a variety of *ad hoc* conditions and techniques. The reader will find treatment of the resolvent, together with many references, in [3,5] as well as in the recent papers Eloe-Islam-Zhang [8] and Zhang [13]. In [13] the resolvent and contraction mappings are used.

While the main focus of the paper is on linear equations, we are also interested in what the results will say about equations of the form

$$x'(t) = A(t)h(x(t)) + \int_0^t C(t,s)g(x(s))ds \tag{1}$$

in which A, C, h, and g are continuous real valued functions. The equation is almost linear in the following sense. It is assumed that there is a $K > 0$ so that for all $x \in R$ we have

$$|h(x) - x| \leq K, \qquad |g(x) - x| \leq K. \tag{2}$$

This equation is motivated by a forced Liénard equation with distributed delay

$$x'' + f(x)x' + \int_0^t C(t,s)g(x(s))ds = e(t)$$

which we discuss in the last section of the paper. The conditions in (2) can allow $f(0) < 0$, giving rise to negative damping and relaxation oscillations. Thus, our term "almost linear" refers only to behavior for very large values of the dependent variable.

We will study stability properties of (1) through information obtained from the forced linear equation

$$y'(t) = A(t)y(t) + \int_0^t C(t,s)y(s)ds + p(t), \quad y(0) = y_0 \tag{3}$$

The analysis is by means of fixed point theory and follows that in [6]. But here we focus on the resolvent in Becker's form [1; p. 11] and [2] which holds for R, A, C being matrices:

$$\frac{\partial R(t,\tau)}{\partial t} = A(t)R(t,\tau) + \int_\tau^t C(t,u)R(u,\tau)du, \qquad R(\tau,\tau) = I. \tag{4}$$

As $' = \frac{d}{dt}$, it will cause no confusion if we write (4) also as

$$R'(t,\tau) = A(t)R(t,\tau) + \int_\tau^t C(t,u)R(u,\tau)du, \qquad R(\tau,\tau) = I.$$

For the significance of using Becker's resolvent, see the remark after the proof of Theorem 1.

Under conditions to be given, we can use the stability criterion from [6] (see (15) for the condition) to show that solutions of (3) are bounded for every bounded continuous $p(t)$. If we then write the solution of (3) as

$$y(t) = R(t,0)y_0 + \int_0^t R(t,s)p(s)ds$$

and apply Perron's theorem [10] (or [5 ; p. 114]), we can conclude that there is a positive number W with

$$\int_0^t |R(t,s)|ds \le W \tag{5}$$

for all $t \ge 0$. We can also apply the aforementioned stability criterion to (4) and find a fixed positive number Z such that

$$|R(t,s)| \le Z, \qquad 0 \le s \le t < \infty \tag{6}$$

and that for fixed s

$$|R(t,s)| \to 0 \tag{7}$$

as $t \to \infty$.

After establishing these properties of R, we write (1) as

$$\begin{aligned} x'(t) =& A(t)x(t) + \int_0^t C(t,s)x(s)ds \\ & - A(t)[x(t) - h(x(t))] - \int_0^t C(t,s)[x(s) - g(x(s))]ds \end{aligned} \tag{8}$$

and express its solution as

$$\begin{aligned} x(t) =& R(t,0)x_0 + \int_0^t R(t,s)[A(s)(x(s) - h(x(s)))]ds \\ & + \int_0^t R(t,s) \int_0^s C(s,u)[x(u) - g(x(u))]duds. \end{aligned} \tag{9}$$

We then ask that A and the integral of C be bounded and conclude from (9) that all solutions of (1) are bounded. The result says: In (1) replace $g(x)$ and $h(x)$ by x to obtain

$$y'(t) = A(t)y(t) + \int_0^t C(t,s)y(s)ds. \tag{1a}$$

If $|g(x)-x|$ and $|h(x)-x|$ are bounded and if the stability condition holds then the difference of the solutions of (1) and (1a) are bounded.

By way of application of the theory, we point out that the result on boundedness of solutions of (3) allows us to write down the formula for a periodic solution of the limiting equation for (3) under periodicity assumptions on the coefficients. We also briefly introduce the reader to stability using the technique and, as mentioned above, discuss the Liénard equation with distributed delay.

2. Properties of the resolvent

We ask that $A : [0,\infty) \to R$ and $C : [0,\infty) \times [0,\infty) \to R$ be continuous. Then for each $y_0 \in R$ there is one and only one solution $y(t,0,y_0)$ of (3) satisfying $y(0,0,y_0) = y_0$ and is defined for $t \geq 0$. See [5; p. 175] for details on existence and uniqueness. In Section 4 we will be interested in more general solutions.

We follow [6] and invert (3), transforming it into (16) below, by asking that

$$G(t,s) := -\int_t^\infty C(u,s)du \tag{10}$$

exist and be continuous for $0 \leq s \leq t < \infty$ so that we can write (3) as

$$y'(t) = [A(t) - G(t,t)]y(t) + \frac{d}{dt}\int_0^t G(t,s)y(s)ds + p(t)$$

or for $Q(t) := A(t) - G(t,t)$ then

$$y'(t) = Q(t)y(t) + \frac{d}{dt}\int_0^t G(t,s)y(s)ds + p(t). \tag{11}$$

We further require that there is a $\Gamma > 0$ with

$$\int_0^t e^{\int_s^t Q(u)du} ds \leq \Gamma \qquad \text{and} \qquad \int_0^t Q(s)ds \to -\infty \tag{12}$$

as $t \to \infty$, that there is a $J > 0$ such that $0 \leq t_0 \leq t$ implies that

$$\int_{t_0}^t Q(s)ds \leq J, \tag{13}$$

and for each fixed $T > 0$ we have

$$\int_0^T |G(t,v)|dv \to 0 \tag{14}$$

as $t \to \infty$.

In condition (15) below, if $Q(t) \le 0$ then (15) can be replaced by

$$2 \sup_{t \ge 0} \int_0^t |G(t,v)| dv \le \alpha. \tag{15a}$$

Theorem 1. Let (12), (13), and (14) hold and let G be defined in (10). Suppose there is an $\alpha < 1$ such that

$$\int_0^t [|G(t,v)| + |Q(v)| e^{\int_v^t Q(u)du} \int_0^v |G(v,u)| du] dv \le \alpha \tag{15}$$

for $t \ge 0$. Then every solution of (3) is bounded for every bounded and continuous function p. Moreover, (5), (6), and (7) hold.

Proof. There are four conclusions given and these are proved with one basic argument. For the sake of clarity we will separate them.

Proof of the boundedness. Starting with (11) we apply the variation of parameters formula and for a given $y(0)$ write

$$\begin{aligned} y(t) = e^{\int_0^t Q(s)ds} y(0) + \int_0^t e^{\int_s^t Q(u)du} \frac{d}{ds} \int_0^s G(s,u) y(u) du ds \\ + \int_0^t e^{\int_s^t Q(u)du} p(s) ds \\ = e^{\int_0^t Q(s)ds} y(0) + e^{\int_s^t Q(u)du} \int_0^s G(s,u) y(u) du \Big|_0^t \\ + \int_0^t e^{\int_s^t Q(u)du} Q(s) \int_0^s G(s,u) y(u) du ds \\ + \int_0^t e^{\int_s^t Q(u)du} p(s) ds. \end{aligned}$$

or

$$\begin{aligned} y(t) =& e^{\int_0^t Q(s)ds} y(0) + \int_0^t G(t,u) y(u) du \\ & + \int_0^t e^{\int_s^t Q(u)du} Q(s) \int_0^s G(s,u) y(u) du ds \\ & + \int_0^t e^{\int_s^t Q(u)du} p(s) ds. \end{aligned} \tag{16}$$

Let $(X, \|\cdot\|)$ be the complete metric space of bounded continuous functions $\phi : [0,\infty) \to R$ with the supremum metric, satisfying $\phi(0) = y(0)$. Define a mapping

$P : X \to X$ by $\phi \in X$ and $t \geq 0$ implies

$$
\begin{aligned}
(P\phi)(t) =& e^{\int_0^t Q(s)ds} y(0) + \int_0^t G(t,u)\phi(u)du + \int_0^t e^{\int_s^t Q(u)du} Q(s) \int_0^s G(s,u)\phi(u)duds \\
&+ \int_0^t e^{\int_s^t Q(u)du} p(s)ds. \qquad (17)
\end{aligned}
$$

By (12), (15), and the assumed boundedness of p, if ϕ is bounded so is $P\phi$. To see that P is a contraction, if $\phi, \eta \in X$ then using (15) we have

$$
\begin{aligned}
|(P\phi)(t) - (P\eta)(t)| \leq& \int_0^t |G(t,u)||\phi(u) - \eta(u)|du \\
&+ \int_0^t e^{\int_s^t Q(u)du} |Q(s)| \int_0^s |G(s,u)||\phi(u) - \eta(u)|duds \\
&\leq \alpha \|\phi - \eta\|.
\end{aligned}
$$

This gives a fixed point representing a bounded solution.

Proof of (5). We now write the solution of (3) as

$$
y(t) = R(t,0)y_0 + \int_0^t R(t,s)p(s)ds. \qquad (18)
$$

For $y_0 = 0$ the argument just given shows that y is bounded for every bounded and continuous p; hence, $\int_0^t R(t,s)p(s)ds$ is bounded for every bounded and continuous p. By Perron's theorem ([10] or [5; p. 114]) this says that $\int_0^t |R(t,s)|ds$ is bounded, so (5) holds.

Proof of (6). To see that (6) is true, we invert (4) for an arbitrary $\tau \geq 0$. Letting z denote $R(t,\tau)$ in the scalar case, we write

$$
z'(t) = A(t)z(t) + \int_\tau^t C(t,s)z(s)ds, \qquad z(\tau) = 1, \qquad (19)
$$

and using (10) we have

$$
\begin{aligned}
z'(t) &= [A(t) - G(t,t)]z(t) + \frac{d}{dt}\int_\tau^t G(t,s)z(s)ds \\
&= Q(t)z(t) + \frac{d}{dt}\int_\tau^t G(t,s)z(s)ds
\end{aligned}
$$

so that by the variation of parameters formula we get

$$
\begin{aligned}
z(t) &= e^{\int_\tau^t Q(s)ds} + \int_\tau^t e^{\int_s^t Q(u)du} \frac{d}{ds}\int_\tau^s G(s,u)z(u)duds \\
&= e^{\int_\tau^t Q(s)ds} + e^{\int_s^t Q(u)du} \int_\tau^s G(s,u)z(u)du\bigg|_\tau^t \\
&+ \int_\tau^t e^{\int_s^t Q(u)du} Q(s) \int_\tau^s G(s,u)z(u)duds
\end{aligned}
$$

or

$$z(t) = e^{\int_\tau^t Q(s)ds} + \int_\tau^t G(t,u)z(u)du + \int_\tau^t e^{\int_s^t Q(u)du} Q(s) \int_\tau^s G(s,u)z(u)duds. \tag{20}$$

Let $(M, \|\cdot\|)$ be the space of bounded continuous functions $\phi : [\tau, \infty) \to R$ with $\phi(\tau) = 1$ and having the supremum metric. Use (20) to define a new mapping P on M by

$$\begin{aligned}(P\phi)(t) &= e^{\int_\tau^t Q(s)ds} + \int_\tau^t G(t,u)\phi(u)du \\ &\quad + \int_\tau^t e^{\int_s^t Q(u)du} Q(s) \int_\tau^s G(s,u)\phi(u)duds.\end{aligned}$$

The mapping P will be a contraction if there is an $\alpha < 1$ with

$$\int_\tau^t |G(t,u)|du + \int_\tau^t e^{\int_s^t Q(u)du} |Q(s)| \int_\tau^s |G(s,u)|duds \le \alpha. \tag{21}$$

Clearly, if (15) holds, so does (21).

Moreover, if $P\phi = \phi$ then

$$\|\phi\| \le e^{\int_\tau^t Q(s)ds} + \alpha\|\phi\|$$

or by (13) we have

$$\|\phi\| \le \frac{e^{\int_\tau^t Q(s)ds}}{1-\alpha} \le \frac{e^J}{1-\alpha}, \tag{22}$$

a uniform bound on $|R(t,\tau)|$, so (6) holds.

Proof of (7). Finally, if we add to the space M the condition that $\phi(t) \to 0$, then we can show that $P\phi \to 0$ and so $R(t,\tau) \to 0$ as $t \to \infty$ and (7) holds. Here are the details. If $\phi(t) \to 0$ as $t \to \infty$, then consider $t > T > \tau$ and for G not identically zero write

$$\left| \int_\tau^t G(t,u)\phi(u)du \right| \le \int_\tau^T |G(t,u)\phi(u)|du + \int_T^t |G(t,u)||\phi(u)|du.$$

For a given $\epsilon > 0$, for fixed ϕ, and with (21) in mind find T so that $t \ge T$ implies that $|\phi(t)| \le \epsilon/[2\alpha]$, leaving the last term in the above display bounded by $\epsilon/2$. Then find t so large that, by use of (14), we have

$$\|\phi\| \int_\tau^T |G(t,u)|du < \epsilon/2.$$

Next, with (12) and (15) in mind and $t > T > \tau$ write

$$\begin{aligned}
&\left|\int_\tau^t e^{\int_s^t Q(u)du} Q(s)\int_\tau^s G(s,u)\phi(u)duds\right| \\
&\leq \left|\int_\tau^T e^{\int_s^T Q(u)du} Q(s)\int_\tau^s G(s,u)\phi(u)duds\right| e^{\int_T^t Q(u)du} \\
&+ \left|\int_T^t e^{\int_s^t Q(u)du} Q(s)\int_\tau^s G(s,u)\phi(u)duds\right| \\
&\leq \alpha\|\phi\| e^{\int_T^t Q(u)du} + \alpha \sup_{t\geq T}|\phi(t)|
\end{aligned}$$

which can be made small by taking T large and then t large. This completes the proof.

Remark 1. At this point it is possible to explain the significance of using the Becker resolvent. Notice that in the proof of (6) and (7) we treated the resolvent equation, (19), exactly as we did (3). The stability condition (15) worked on (19) exactly as it did on (3). By contrast, the classical resolvent (see [3; p. 64] or [5; p. 101]) is given by

$$\frac{\partial R(t,s)}{\partial s} = -R(t,s)A(s) - \int_s^t R(t,u)C(u,s)du, \quad R(t,t) = I,$$

and it is totally unclear to us how we can invert that equation and apply (15). Our entire paper rests on Becker's form of the resolvent. Becker's form of the resolvent has also been used effectively in [2], [8], and [12].

Remark 2. One of the important features of analysis by contraction mappings, as compared with Liapunov's direct method, is that forcing functions often simply drop out of the contraction condition, while in Liapunov's direct method they remain as a product with partial derivatives of the Liapunov function.

We return to (1) which we write as

$$\begin{aligned}
x'(t) =& A(t)x(t) + \int_0^t C(t,s)x(s)ds \\
&- A(t)[x(t) - h(x(t))] - \int_0^t C(t,s)[x(s) - g(x(s))]ds.
\end{aligned} \tag{23}$$

Theorem 2. Let the conditions of Theorem 1 and condition (2) hold. Suppose there is a $\Lambda > 0$ with

$$|A(t)| + \int_0^t |C(t,s)|ds \leq \Lambda \tag{24}$$

for $t \geq 0$. Then all solutions of (1) are bounded.

Proof. Let x_0 be given and identify the last two terms of (23) as an inhomogeneous term in (3). We can then use the resolvent discussed in Theorem 1 to write the solution of (23) as

$$\begin{aligned} x(t) &= R(t,0)x_0 + \int_0^t R(t,s)A(s)[h(x(s)) - x(s)]ds \\ &+ \int_0^t R(t,s)\int_0^s C(s,u)[g(x(u)) - x(u)]duds. \end{aligned} \tag{25}$$

The standard existence theorem [5; p. 187] assures us that there is a solution, while this formula shows that it is bounded. In fact, for a given x_0 by (2), (5), and (24) we have

$$\begin{aligned} |x(t)| &\leq |R(t,0)||x_0| + \int_0^t |R(t,s)|K\Lambda ds + \int_0^t |R(t,s)|\Lambda K ds \\ &\leq |R(t,0)||x_0| + 2K\Lambda \int_0^t |R(t,s)|ds \\ &\leq Z|x_0| + 2K\Lambda W \\ &=: J^*. \end{aligned} \tag{26}$$

This completes the proof.

3. Periodic Solutions

Theorem 1 can supply properties for periodic equations which have been very illusive in the literature.

Consider again

$$y'(t) = A(t)y(t) + \int_0^t C(t,s)y(s)ds + p(t) \tag{27}$$

with A, p continuous on $(-\infty, \infty)$ and C continuous for $-\infty < s \leq t < \infty$. If the conditions of Theorem 1 hold then every solution of (27) is bounded. If $y_1(t)$ is a fixed solution of (27) and if $y_2(t)$ is any other solution, then $y(t) =: y_1(t) - y_2(t)$ solves our old Equation (1a)

$$y'(t) = A(t)y(t) + \int_0^t C(t,s)y(s)ds$$

and every solution of this equation tends to zero by Theorem 1. Hence, if $y_1(t)$ is any fixed solution of (27) then every other solution converges to it. Can we say that $y_1(t)$ has "special properties?" For example, if A and p are periodic, might $y_1(t)$ be periodic under some periodic assumptions on C?

This problem is discussed in [5; pp. 92-94]. In fact, $y_1(t)$ can be periodic only if there are very special orthogonal relationships between p and C. The reason for this is that for an arbitrary continuous T-periodic function ϕ, then

$$A(t)\phi(t) + \int_0^t C(t,s)\phi(s)ds + p(t)$$

is not periodic even if $C(t+T,s+T) = C(t,s)$, $A(t+T) = A(t)$, and $p(t+T) = p(t)$. By contrast, under the same conditions

$$A(t)\phi(t) + \int_{-\infty}^t C(t,s)\phi(s)ds + p(t)$$

is T-periodic. We can discover special properties of $y_1(t)$ by examining $y_1(t+nT)$ for $n = 1, 2, ...$ The details are found in [2], [4], and [5; pp. 101-107]. See especially [5; p. 102]. In that work we assumed there that there exists $T > 0$ such that:

$$(28) \qquad A(t+T) = A(t), \qquad p(t+T) = p(t), \qquad C(t+T,s+T) = C(t,s)$$

and that there is a number $J > 0$ such that for each $[a,b]$ if $t \in [a,b]$ then

$$(29) \qquad \lim_{n\to\infty} \int_{-nT}^t |C(t,s)|ds = \int_{-\infty}^t |C(t,s)|ds \le J$$

and $\int_{-\infty}^t |C(t,s)|ds$ is continuous.

Under these conditions, in the aforementioned references we showed that

$$R(t+T,s+T) = R(t,s)$$

and that *if (27) has a bounded solution* then

$$(30) \qquad x(t) = \int_{-\infty}^t R(t,s)p(s)ds$$

is a periodic solution of

$$(31) \qquad x'(t) = A(t)x(t) + \int_{-\infty}^t C(t,s)x(s)ds + p(t).$$

The sequence $y_1(t+nT)$ has a subsequence converging to this periodic solution. We remarked [5; p. 107] that the major difficulty lay in proving that (27) has a bounded solution. Our Theorem 1 gives a simple solution to the boundedness question. We formally state this as follows.

Theorem 3. Let the conditions of Theorem 1 hold, as well as (28) and (29). Then (31) has a T-periodic solution given by (30). Moreover, if y is any solution of (27) then there is a subsequence of $\{y(t+nT)\}$ converging to that T-periodic solution and the convergence is uniform on compact subsets of $(-\infty, \infty)$.

This theorem is just Theorem 1.5.1 (iii) of [5; p. 102] in which we have replaced the hypothesis that (27) have a bounded solution with the assumption that the conditions of Theorem 1 hold.

4. Stability

Theorem 1 can reduce stability questions to a triviality and it can supply very illusive conditions in difficult stability problems.

The reader is referred to [3; p. 34] or [5; p. 118] for definitions and a general introduction to stability theory for Volterra equations.

Equation (3) conceals much of the situation. Generally, we are interested in many solutions besides those starting at $t = 0$. In the general case, there is a given $t_0 \geq 0$ and a given continuous initial function $\phi : [0, t_0] \to R$. Moreover, we take $f(t) = 0$ and we will see a new forcing function emerge. Under our continuity conditions there is then a unique continuous solution $y(t, t_0, \phi)$ which agrees with ϕ on the initial interval $[0, t_0]$ and satisfies

$$y'(t) = A(t)y(t) + \int_{t_0}^{t} C(t,s)y(s)ds + \int_{0}^{t_0} C(t,s)\phi(s)ds \tag{32}$$

on $[t_0, \infty)$.

If the conditions of Theorem 1 are satisfied then the resolvent satisfies (5), (6), and (7). Becker [1; p. 22] notes that we can express the solution of (32) as

$$y(t, t_0, \phi) = R(t, t_0)\phi(t_0) + \int_{t_0}^{t} R(t,u) \int_{0}^{t_0} C(u,s)\phi(s)dsdu. \tag{33}$$

From this we can immediately derive a number of results on stability. The following standard definitions will be employed.

Definition 1. The zero solution of (32) is said to be *Liapunov stable* if for each $\epsilon > 0$ and each $t_0 \geq 0$ there is a $\delta > 0$ such that if

$$\phi : [0, t_0] \to R \quad \text{is continuous and} \quad |\phi(t)| < \delta \quad \text{on} \quad [0, t_0]$$

then $|y(t, t_0, \phi)| < \epsilon$ for $t \geq t_0$.

Definition 2. The zero solution of (32) is said to be *uniformly stable* if it is Liapunov stable and if δ is independent of t_0.

Definition 3. The zero solution of (32) is said to be *asymptotically stable* if it is Liapunov stable and if for each $t_0 \geq 0$ there is a $\gamma > 0$ such that $|\phi(t_0)| < \gamma$ on $[0, t_0]$ implies that $|y(t, t_0, \phi)| \to 0$ as $t \to \infty$.

Definition 4. The zero solution of (32) is said to be *uniformly asymptotically stable* if it is uniformly stable and if there is an $\eta > 0$ such that, for each $\epsilon > 0$, there is a $T > 0$ such that

$$t_0 \geq 0, \quad |\phi(t)| < \eta \quad \text{on} \quad [0, t_0], \quad \text{and} \quad t \geq t_0 + T$$

imply that $|y(t, t_0, \phi)| < \epsilon$.

Theorem 4. If the conditions of Theorem 1 are satisfied and if for each $t_0 \geq 0$

$$\int_0^{t_0} |C(t,s)|ds \tag{34}$$

is bounded for $t_0 \leq t < \infty$, then the zero solution of (32) is stable.

Proof. For a given $\epsilon > 0$ and $t_0 \geq 0$, we find $M > 0$ with $\int_0^{t_0} |C(t,s)|ds \leq M$ for $t \geq t_0$. For δ yet to be determined and for $|\phi(t)| < \delta$, from (5), (6), and (33) we have

$$\begin{aligned} |y(t, t_0, \phi)| &\leq |R(t, t_0)|\delta + M\delta \int_{t_0}^t |R(t,u)|du \\ &\leq Z\delta + M\delta W \\ &< \epsilon \end{aligned}$$

provided that

$$\delta < \epsilon/[Z + MW].$$

Theorem 5. If the conditions of Theorem 1 are satisfied and if there is a uniform bound on (34) for all t_0 and all $t \geq t_0$, then the zero solution of (32) is uniformly stable.

Proof. In the proof of Theorem 4 we have M independent of t_0 and so the uniform δ needs to satisfy the relation just given.

A classical theorem states that the convolution of an L^1-function with a function tending to zero does, itself, tend to zero. Our proof of (7) in the proof of Theorem 1 was a type of extension of this. We need a theorem of this type to show that the last term in (33) tends to zero under the conditions of Theorem 1.

Lemma. Let $R(t,s)$ satisfy (5), (6), and (7). Let $q : [0, \infty) \to R$ be continuous and let $q(t) \to 0$ as $t \to \infty$. Then $\int_0^t R(t,s)q(s)ds \to 0$ as $t \to \infty$.

Proof. For $0 < T < t$ we have

$$\begin{aligned}\left|\int_0^t R(t,s)q(s)ds\right| &\le \left|\int_0^T R(t,s)q(s)ds\right| + \left|\int_T^t R(t,s)q(s)ds\right| \\ &\le \|q\| \int_0^T |R(t,s)|ds + \int_T^t |R(t,s)|ds\|q\|_{[T,\infty)}.\end{aligned}$$

where $\|\cdot\|$ will represent the supremum on $[0,\infty)$, while $\|\cdot\|_{[T,\infty)}$ will represent the supremum on $[T,\infty)$. Use (5) and fix T so that for a given $\epsilon > 0$ we have $\|q\|_{[T,\infty)} < \frac{\epsilon}{2W}$, making the last term bounded by $\epsilon/2$. Then use (7) and take t so large that $\|q\| \int_0^T |R(t,s)|ds < \epsilon/2$. This completes the proof.

With this lemma and (33) we can immediately prove the following result.

Theorem 6. If the conditions of Theorem 1 are satisfied and if for each $t_0 \ge 0$

$$\int_0^{t_0} |C(t,s)|ds \to 0 \quad \text{as} \quad t \to \infty \tag{35}$$

then the zero solution of (32) is asymptotically stable.

Proof. We choose $\gamma = \delta$ in the proof of Theorem 4. Then by (33) and Theorem 4 we have

$$|y(t,t_0,\phi)| \le \delta|R(t,t_0)| + \delta \int_0^t |R(t,u)| \int_0^{t_0} |C(u,s)|dsdu.$$

Apply (7), (5), (35), and the lemma to see that $y(t,t_0,\phi) \to 0$.

For uniform asymptotic stability we refer the reader to several formulations in Zhang [12; pp. 326-7]. Some of the formulations are his, some are formulations of Hino and Murakami, and some are a combination. But all of them require some form of (5). Thus, the conditions of Theorem 1 can be used in place of (5) in any of those formulations as sufficient conditions for uniform asymptotic stability. The following is an example of what can be deduced under the conditions of Theorem 1. It is taken directly from Zhang's Theorem 2.1 on p. 341 where we replace his assumption of (5) with our assumption that the conditions of Theorem 1 hold. Several other results of this type are readily formulated in conjunction with Zhang's paper.

Theorem 7. Suppose that $p(t) = 0$ and that the conditions of Theorem 1 hold. In addition, assume that:

(i) $\sup_{t \ge s \ge 0}\{|A(t)| + \int_0^t |C(t,s)|ds\} < \infty$,

(ii) for any $\sigma > 0$, there exists an $S = S(\sigma) > 0$ such that

$$\int_0^{t-S} |C(t,s)|ds < \sigma \quad \text{for all} \quad t \ge S,$$

and

(iii) $A(t)$ and $C(t, t+s)$ are bounded and uniformly continuous in $(t, s) \in \{(t, s) \in R^+ \times K : -t \leq s \leq 0\}$ for any compact set $K \subset R^-$.
Then the zero solution of (32) is uniformly asymptotically stable.

But (5), (6), and (7) are derived conditions. They follow from (12)-(15). So for difficult problems we would always expect to do more with (12)-(15) than we can ever do with (5), (6), and (7), even though the latter are the standard conditions discussed extensively in the literature.

Returning to (32), a translation $x(t) = y(t+t_0)$ results in

$$\begin{aligned} x'(t) =& y'(t+t_0) = A(t+t_0)x(t) + \int_{t_0}^{t_0+t} C(t_0+t, s)y(s)ds \\ &+ \int_0^{t_0} C(t_0+t, s)\phi(s)ds \end{aligned}$$

or

$$\begin{aligned} x'(t) &= A(t+t_0)x(t) + \int_0^{t} C(t_0+t, s+t_0)x(s)ds \\ &+ \int_0^{t_0} C(t_0+t, s)\phi(s)ds \end{aligned} \tag{36}$$

which we write as

$$x'(t) = A^*(t)x(t) + \int_0^t C^*(t, s)x(s)ds + f^*(t). \tag{37}$$

But this raises a very big question. Will the stability assumptions for (37) be the same as those in Theorem 1 for (3) and (11)?

If this part of the stability investigation is to be optimal, we need to know that the conditions which held for (3) and (11) will also hold for (37) when we transform it into the counterpart of (11). We will check this under the simpler condition (15a).

Starting with (37), we define

$$G^*(t, s) = -\int_t^\infty C(t_0+u, s+t_0)du \tag{38}$$

and write (37) as

$$x'(t) = [A(t+t_0) - G^*(t, t)]x(t) + \frac{d}{dt}\int_0^t G^*(t, s)x(s)ds + f^*(t) \tag{39}$$

Proposition 1. If for (3) and (11) we have $Q(t) \leq 0$ and (15a) holds, then for (37) and (39) we have $Q^*(t) \leq 0$ and (15a) holds for G^*.

Proof. Since we assume that

$$Q(t) = A(t) - G(t,t) = A(t) + \int_t^\infty C(u,t)du \leq 0$$

we now have

$$\begin{aligned} Q^*(t) = A(t+t_0) - G^*(t,t) &= A(t+t_0) + \int_t^\infty C(t_0+u, t+t_0)du \\ &= A(t+t_0) + \int_{t+t_0}^\infty C(u, t+t_0)du \\ &= Q(t+t_0) \leq 0. \end{aligned}$$

Next, for (15a) to hold we have

$$\begin{aligned} \sup_{t\geq 0} \int_0^t |G^*(t,v)|dv &= \sup_{t\geq 0} \int_0^t \left| \int_t^\infty C(t_0+u, v+t_0)du \right| dv \\ &= \sup_{t\geq 0} \int_0^t \left| \int_{t+t_0}^\infty C(u, v+t_0)du \right| dv \\ &= \sup_{t\geq 0} \int_{t_0}^{t_0+t} \left| \int_{t+t_0}^\infty C(u,v)du \right| dv. \end{aligned}$$

If we set $q = t + t_0$, then the last equality can be written as

$$\begin{aligned} &\sup_{q\geq t_0} \int_{t_0}^q \left| \int_q^\infty C(u,v)du \right| dv \\ &\leq \sup_{q\geq 0} \int_0^q \left| \int_q^\infty C(u,v)du \right| dv \\ &\leq \alpha < 1. \end{aligned}$$

This completes the proof.

Next, we need to examine (12), (13), and (14) for (37) since these were assumptions in Theorem 1.

Proposition 2. If $Q(t) \leq 0$ and if (12) and (14) hold for (3) and (11), then $Q^*(t) \leq 0$ and (12) and (14) hold for (37) and (39).

Proof. Assume that (12) holds for (3) and (11). We have

$$Q(t) = A(t) - G(t,t) = A(t) + \int_t^\infty C(u,t)du$$

and from the proof of Proposition 1 we have

$$Q^*(t) = Q(t + t_0)$$

so

$$\int_0^t Q^*(s)ds \to -\infty.$$

Also,

$$\begin{aligned}
\int_0^t e^{\int_s^t Q^*(u)du} ds &= \int_0^t e^{\int_s^t Q(u+t_0)du} ds \\
&= \int_0^t e^{\int_{s+t_0}^{t+t_0} Q(u)du} ds \\
&\leq \int_0^{t+t_0} e^{\int_s^{t+t_0} Q(u)du} ds \\
&\leq \Gamma.
\end{aligned}$$

This means that (12) holds for (37) and (39).

Assume that (14) holds for (3) and (11). Then we can say that

$$\int_0^{T+t_0} |G(t,v)|dv = \int_0^{T+t_0} \left| \int_t^\infty C(u,v)du \right| dv \to 0$$

as $t \to \infty$.

Hence,

$$\begin{aligned}
\int_0^T |G^*(t,v)|dv &= \int_0^T \left| \int_t^\infty C(t_0 + u, v + t_0)du \right| dv \\
&= \int_{t_0}^{T+t_0} \left| \int_{t+t_0}^\infty C(u,v)du \right| dv \\
&\leq \int_0^{T+t_0} \left| \int_{t+t_0}^\infty C(u,v)du \right| dv \to 0
\end{aligned}$$

as $t \to \infty$. This proves Proposition 2.

5. Higher order equations

Theorem 1 applies to important second order classical problems.

All of the work here is for scalar equations. Any linear $n-$th order normalized Volterra equation can be written as a scalar first order equation [3; p. 10]. Many systems of Volterra equations can be reduced to a single $n-$th order Volterra equation. But (15) may not be natural for equations under such reduction. On the other hand, it is routine to do everything here for a system instead of a scalar equation. We did not do so because the straight-forward derivation of (15) for systems yields a condition depending on the dimension. It is a reasonable conjecture that a clever norm can be constructed which will yield a result compatible with (15) and be, largely, independent of the dimension. Thus, we leave this as an open problem.

But many higher order equations of considerable interest can be reduced to (1). Study of stability, boundedness, and periodicity properties of differential equations by fixed point theory depends on exactly one thing: we must invert the equation in such a way that the natural mapping defined by that inversion will map a set containing only functions of the type of the desired solution into itself. Here is an example of this for the Liénard equation with memory in the restoring force which we write as

$$x'' + f(x)x' = -\int_0^t C(t,s)g(x(s))ds + e(t). \tag{40}$$

We suppose that there are positive constants D, H, K such that

$$|g(x) - Hx| \le K, \quad |\int_0^x f(s)ds - Dx| < K \quad \text{for all} \quad x \tag{41}$$

and a differentiable function k such that

$$C(t,s) = \frac{d}{dt}k(t,s), \quad k(s,s) = 0 \tag{42}$$

with

$$\int_t^\infty k(u,s)du \quad \text{is continuous.} \tag{43}$$

Remark 3. Clever choices for k are rewarded. But the pedestrian choice is

$$k(t,s) = \int_{t-s}^\infty C(u+s,s)du - \int_0^\infty C(u+s,s)du,$$

provided the integrals exist. Then (42) is satisfied. With that choice for $k(t,s)$, then to satisfy (43) we need

$$\begin{aligned}\int_t^\infty k(u,s)du &= \int_t^\infty \left[\int_{u-s}^\infty C(v+s,s)dv - \int_0^\infty C(v+s,s)dv\right]du \\ &= -\int_t^\infty \int_0^{u-s} C(v+s,s)dvdu\end{aligned}$$

to be continuous. Examples are readily constructed.

To relate our work here to Theorem 1 we set

$$Q(t) = -D - H \int_t^\infty k(u,t)du \tag{44}$$

and

$$G(t,s) = H \int_t^\infty k(u,s)du. \tag{45}$$

We define

$$F(x) := \int_0^x f(s)ds \quad \text{and} \quad E(t) := \int_0^t e(s)ds. \tag{46}$$

Theorem 8. If (41)-(43) hold, if $E(t)$ is bounded, and if Q and G defined by (44)-(45) satisfy (12)-(15), then (40) has a bounded solution for each $x(0), x'(0)$. If $e(t) = 0$ and if $x'(0) + F(x(0)) = 0$, then the linearization of (4) (see (49)) has a solution tending to zero.

Proof. Write (40) as

$$\begin{aligned}
x'' + f(x)x' &= -\int_0^t C(t,s)g(x(s))ds + e(t) \\
&= -\frac{d}{dt}\int_0^t \int_0^{t-s} C(u+s,s)dug(x(s))ds + e(t) \\
&= -\frac{d}{dt}\int_0^t \int_0^{t-s} \frac{d}{du}k(u+s,s)dug(x(s))ds + e(t)
\end{aligned}$$

or

$$x'' + f(x)x' = -\frac{d}{dt}\int_0^t k(t,s)g(x(s))ds + e(t). \tag{47}$$

Thus,

$$x'(t) + F(x) = -\int_0^t k(t,s)g(x(s))ds + E(t) + x'(0) + F(x(0)). \tag{48}$$

To make clear how Theorem 2 is applied we trivially point out that (48) can be written as

$$x'(t) + D\frac{F(x)}{D} = -\int_0^t Hk(t,s)\frac{g(x(s))}{H}ds + E(t) + x'(0) + F(x(0)).$$

This last equation is parallel to (1) and we write the equation parallel to (3) as

$$x'(t) = -Dx(t) - H\int_0^t k(t,s)x(s)ds + E(t) + x'(0) + F(x(0))$$
$$= -Dx(t) - H\int_t^\infty k(u,t)du\,x(t)$$
$$+ \frac{d}{dt}\int_0^t\int_t^\infty Hk(u,s)du\,x(s)ds + E(t) + x'(0) + F(x(0)). \tag{49}$$

Thus, for each fixed $x(0), x'(0)$ the conditions of Theorem 1 are satisfied and Equation (49) has a bounded solution. By Theorem 2, (40) has a bounded solution. If $e(t) = 0$ and $x'(0) + F(x(0)) = 0$ then we can conclude that there is a solution of (49) tending to zero by using the same argument as in the proof of (7) in Theorem 1.

Our Equation (40) includes the case of negative damping, $f(0) < 0$, which is known to give rise to relaxation oscillations. Much has been written about the Liénard equation with no delay or with a pointwise delay in the restoring force. We refer the reader to Zhang [14-17] and the references therein. We do not expect periodic solutions of (40) because the delay is on $[0,t]$, instead of on $[-\infty,t]$; this is exactly the difficulty cited concerning (27). If we rewrite (40) with the infinite delay and with periodic assumptions on e and C then it is possible that conditions very much like those of Theorem 1 could yield a periodic solution. We have not investigated that possibility. The equations with pointwise delay in the papers [14-17] do not share this lack of natural periodicity property which occurs in equations with a distributed delay on $[0,t)$.

The reader will find fascinating examples of relaxation oscillation problems in Haag [9; pp. 147-171] of both mechanical and electrical type (briefly repeated in [5; pp. 144-148]). The introduction of a distributed delay goes back to Volterra [11] and even to Picard in 1907. An interesting discussion and references is given in Davis [7; pp. 112-113]. It tells of the debate over the question of whether distorted material has a memory, as we have indicated, or if there simply needs to be more variables. Both views can be right as one will see in our Equation (40) by selecting $C(t,s) = e^{t-s}$. For with such a kernel, we can differentiate (40) again and eliminate the integral, at the cost of having a higher order equation. Volterra [11; pp. 138-154] gives a detailed discussion of memory effects on elastic materials and their representation by integrals. If our problem is to represent the mechanical system of Haag, then $g(x)$ would represent the restoring force of the spring and the integral takes into account the effects of previous distortions of the spring.

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