FIXED POINTS, STABILITY, AND EXACT LINEARIZATION

T. A. Burton

Northwest Research Institute
732 Caroline St.
Port Angeles, WA 98362
taburton@olypen.com

Abstract. We study the scalar equation $x'' + f(t, x, x')x' + b(t)g(x(t - L)) = 0$ by means of contraction mappings. Conditions are obtained to ensure that each solution $(x(t), x'(t)) \to (0, 0)$ as $t \to \infty$. The conditions allow f to grow as large as t, but not as large as t^2. This is parallel to the classical result of R. A. Smith in 1961 for the linear equation without a delay.

1. Introduction. Long ago Smith [13] proved a beautiful result on asymptotic stability of a second order linear equation

$$x'' + h(t)x' + k^2x = 0 \tag{1.1}$$

where $h(t)$ is continuous for $t \geq 0$, k^2 is a positive constant, and h is a positive constant with

$$h(t) \geq h. \tag{1.2}$$

He showed that the zero solution is asymptotically stable if and only if

$$\int_0^\infty e^{-H(t)} \int_0^t e^{H(s)} ds dt = \infty \tag{1.3}$$

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where $H(t) := \int_0^t h(s)ds$. It is not hard to show that (1.3) holds for $h(t) = h + t$, but fails for $h(t) = h + t^2$.

In the 1990's a number of papers appeared which were written by Hatvani, Krisztin, and Totik [6-10] and Pucci and Serrin [12] extending the result in several ways. They studied the linear case and also a nonlinear version

$$x'' + f(t, x, x')x' + g(x) = 0 \tag{1.4}$$

with variants of $f(t, x, x') \geq h(t) \geq 0$, obtaining results very close to those of Smith. Other nonlinear results are found in [1, 5, 11, 14]. The condition $h(t) \geq h$ was reduced (by Smith also), usually asking that $h(t)$ be integrally positive but, in any case, asking that the intervals on which $h(t)$ vanishes be bounded by a local Lipschitz constant for g. Those results were dictated in large measure by the technique of proof, often a Liapunov argument and/or differential inequalities. In particular, they did not extend well to the case of time dependent restoring force with a delay. One may find Liapunov treatment of

$$x'' + f(x)x' + g(x(t - L)) = 0$$

in Zhang [15].

In recent years we have investigated several classical stability problems by means of fixed point theory to determine what advances can be made which were not seen, for example, in Liapunov theory. There emerge conditions averaging the various parts of the equation, usually through integral relations. For example, in the problem at hand we readily allow time dependence in the restoring force, together with a delay, and we do not see the intervals on which $h(t)$ may vanish as being tied to g; instead, we see relations between all of the functions involved and over the whole time axis $[0, \infty)$.

Most emphatically we do not claim that the fixed point technique is better than the Liapunov technique, but it does add many new dimensions to the problem and it yields results not seen before.

Here, we use fixed point theory to develop a close counterpart of the sufficient part of Smith's theorem for the delay equation

$$x'' + f(t, x, x')x' + b(t)g(x(t - L)) = 0 \tag{1.5}$$

where $f(t, x, y) \geq a(t)$ for some continuous function a. Like Smith's result, our condition holds for $a(t) = t$ but fails for $a(t) = t^2$. And, like Smith's result again, actually verifying the conditions can be a challenging chore.

Our aim is, of course, to present a strong result for a classical problem of continuing interest. But we have a second purpose. We wish to present evidence that fixed point theory is a very competitive tool in the study of stability. Parallel recent results of this nature are found in [3] and [4] which also contain substantial references.

2. Main result. When we refer to (2.1) below, all propertes in this paragraph are assumed. Consider the scalar equation

$$x'' + f(t, x, x')x' + b(t)g(x(t - L)) = 0 \tag{2.1}$$

in which $b : [0, \infty) \to [0, \infty)$ is continuous and bounded. We require that $g : R \to R$ and $f : R \times R \times R \to [0, \infty)$ be continuous. Also, we ask that there are continuous functions a and $c : [0, \infty) \to [0, \infty)$ and $F : R \times R \to [0, \infty)$ such that for all $t \geq 0, x \in R, y \in R$ we have

$$\begin{gathered} a(t) \leq f(t, x, y) \leq F(x, y)c(t), \quad \int_0^\infty a(t)dt = \infty, \\ \frac{g(x)}{x} \geq \beta > 0, \quad \text{and} \quad \lim_{x \to 0} \frac{g(x)}{x} \quad \text{exists.} \end{gathered} \tag{2.2}$$

We assume also that f satisfies a local Lipschitz condition and that there is a positive constant K with

$$|g(x) - g(y)| \leq K|x - y| \tag{2.3}$$

for all real x, y. Our objective is to give conditions under which each solution of (2.1) satisfies $(x(t), x'(t)) \to (0, 0)$ as $t \to \infty$. Thus, we suppose that there is given an arbitrary continuous initial function $\psi : [-L, 0] \to R$ and a slope $x'(0)$. This is sufficient to establish a unique solution $x_1(t)$. It will be crucial to have such solutions defined for all future time. Thus, we will suppose that either $L > 0$ and $b(t)$ is continuous, or that $L = 0$ and $b(t)$ is differentiable. We will return to the argument in a moment.

But first we need to define certain functions from $x_1(t)$. Let

$$A(t) := f(t, x_1(t), x_1'(t)) \tag{2.4}$$

(noting that $c(t)F(x_1(t), x_1'(t)) \geq A(t) \geq a(t) \geq 0$) and write (2.1) (retaining the initial condition) as

$$\begin{aligned} x' &= y \\ y' &= -A(t)y - b(t)g(x(t-L)). \end{aligned} \tag{2.5}$$

Notice that for the given initial condition, $(x_1(t), y_1(t))$ is the unique solution of (2.5). We have linearized (2.1), but with the initial condition it is exactly the same equation: thus, the term exact linearization of the title of this article.

Concerning continuation of the solution $x_1(t)$, if $L > 0$ then in the second equation in (2.5) we take the last term as a forcing function, apply the variation of parameters formula, and conclude that $y(t)$ is bounded on any interval $[0, T)$ for which the solution is defined. We then use the first equation in (2.5) to see that $x(t)$ is also bounded on that interval. It then follows that the solution is defined for all future time, as may be seen in Chapter 3 of Burton [2]. Next, if $L = 0$ and $b(t)$ is differentiable, write $b(t) = \lambda(t)\mu(t)$ where $\lambda(t)$ is increasing and $\mu(t)$ is decreasing. Then define a function

$$W(t, x, y) = \frac{y^2}{2\lambda(t)} + \mu(t)\int_0^x g(s)ds$$

whose derivative along solutions of (2.5) is non-positive. We readily see that $y(t)$ is bounded on any finite interval. This, in the first equation of (2.5), yields $x(t)$ also bounded on any finite interval. This proves that solutions can be continued for all future time. In particular, the aforementioned $(x_1(t), y_1(t))$ is defined for all future time. Note carefully that this does not yield bounded solutions because $\lambda(t)$ can tend to ∞ and $\mu(t)$ can tend to zero.

Return now to (2.5) and use the variation of parameters formula on the second member to obtain

$$x'(t) = y(t) = x'(0)e^{-\int_0^t A(s)ds} - \int_0^t e^{-\int_u^t A(s)ds} b(u)g(x(u-L))du \tag{2.6}$$

which we now write as

$$x' = B(t) - \int_0^t C(t,u)g(x(u-L))du \tag{2.7}$$

where $B(t)$ is the first term on the right in (2.6). We have reduced (2.1) (retaining its initial condition) to a first order integro-differential equation.

If

$$\int_{t-s}^{\infty} C(u+s,s)du \tag{2.8}$$

exists, then we can write (2.7) as

$$x' = B(t) - g(x(t-L)\int_0^{\infty} C(u+t,t)du + \frac{d}{dt}\int_0^t \int_{t-s}^{\infty} C(u+s,s)dug(x(s-L))ds. \tag{2.9}$$

Taking $D(t) = \int_0^{\infty} C(u+t,t)du$ and $E(t,s) = \int_{t-s}^{\infty} C(u+s,s)du$ we can write (2.9) as

$$\begin{aligned} x' &= B(t) - D(t+L)g(x(t)) + \frac{d}{dt}\int_{t-L}^t D(s+L)g(x(s))ds \\ &+ \frac{d}{dt}\int_0^t E(t,s)g(x(s-L))ds. \end{aligned} \tag{2.10}$$

When $L = 0$ this becomes

$$x' = B(t) - D(t)g(x(t)) + \frac{d}{dt}\int_0^t E(t,s)g(x(s))ds. \tag{2.11}$$

REMARK. Notice that $x_1(t)$ is involved in every one of these new terms:

$$0 \le a(t) \le A(t) := f(t, x_1(t), x_1'(t)) \le F(x_1(t), x_1'(t))c(t), \tag{2.4}$$

$$C(t,u) := e^{-\int_u^t A(s)ds}b(u) \ge 0, \tag{2.12}$$

$$B(t) := x'(0)e^{-\int_0^t A(s)ds}, \tag{2.13}$$

$$D(t) := \int_0^{\infty} C(u+t,t)du = \int_0^{\infty} e^{-\int_t^{u+t} A(s)ds}b(t)du \ge 0, \tag{2.14}$$

and

$$E(t,s) := \int_{t-s}^{\infty} C(u+s,s)du = \int_{t-s}^{\infty} e^{-\int_s^{u+s} A(v)dv} b(s)du \geq 0. \tag{2.15}$$

Our task now is to obtain a stability relation for (2.9) when (2.2)-(2.4) hold, including the initial condition $(\psi, x'(0))$. By (2.2) and (2.14) we can define a continuous function

$$q(t) := \frac{D(t+L)g(x_1(t))}{x_1(t)} \geq 0 \tag{2.16}$$

so that for the given initial condition then from (2.16) and (2.10) we have

$$x' = B(t) - q(t)x + \frac{d}{dt}\int_{t-L}^{t} D(s+L)g(x(s))ds + \frac{d}{dt}\int_0^t E(t,s)g(x(s-L))ds \tag{2.17}$$

with unique solution $x_1(t)$. Again, we have exact linearization.

It will help to understand the next result if we think of (2.18) as stipulating a minimum lower bound on $A(t)$, while (2.19) asks that the growth of $A(t)$ be controlled. Furthermore, the last part of the coming proof involving showing that $P\phi$ tends to zero becomes almost trivial if we replace (2.20) by $A(t) \geq a_0 > 0$.

In the theorem below g enters through K, f enters through $a(t)$ and $c(t)$, and all the functions in the equation are related through these integrals on $0 \leq t < \infty$. While $A(t)$ is approximated through $a(t)$ and $c(t)$, in the linear case

$$x'' + A(t)x' + b(t)x(t-L) = 0,$$

(2.18) and (2.19) with $a(t) = \gamma c(t) = A(t)$ would exactly reflect the equation. Moreover, this linear equation would serve as a guide to a local asymptotic stability result, which we do not formulate here.

THEOREM 2.1. Let the conditions in the paragraph containing (2.1) hold, including (2.2) and (2.3). Suppose also that there is an $\alpha < 1$ such that

$$\begin{aligned} &2K\sup_{t\geq 0}\int_{t-L}^{t}\int_0^{\infty} e^{-\int_{s+L}^{u+s+L} a(v)dv} b(s+L)duds \\ &+ 2K\sup_{t\geq 0}\int_0^t\int_{t-s}^{\infty} e^{-\int_s^{u+s} a(v)dv} b(s)duds \leq \alpha, \end{aligned} \tag{2.18}$$

that for each $\gamma > 0$

$$\int_0^\infty \int_0^\infty e^{-\int_{t+L}^{u+t+L} \gamma c(s)ds} b(t+L)dudt = \infty, \tag{2.19}$$

and that there are numbers $a_0 > 0$ and $Q > 0$ such that for all $t \ge 0$ if $J \ge Q$ then

$$\int_t^{t+J} a(v)dv \ge a_0 J. \tag{2.20}$$

(Note that since $A(t) \ge a(t)$, it also follows that $\int_t^{t+J} A(v)dv \ge a_0 J$.) Then $(x_1(t), y_1(t)) \to (0,0)$ as $t \to \infty$.

Proof. By the variation of parameters formula applied to (2.17) we have

$$\begin{aligned}
x(t) =&\psi(0)e^{-\int_0^t q(s)ds} + \int_0^t e^{-\int_u^t q(s)ds} B(u)du \\
&+ \int_0^t e^{-\int_u^t q(s)ds} \frac{d}{du} \int_{u-L}^u D(s+L)g(x(s))dsdu \\
&+ \int_0^t e^{-\int_u^t q(s)ds} \frac{d}{du} \int_0^u E(u,s)g(x(s-L))dsdu.
\end{aligned}$$

If we integrate that last two terms by parts we obtain

$$\begin{aligned}
&e^{-\int_u^t q(s)ds}\left[\int_{u-L}^u D(s+L)g(x(s))ds + \int_0^u E(u,s)g(x(s-L))ds\right]\Bigg|_0^t \\
&- \int_0^t q(u)e^{-\int_u^t q(s)ds}\left[\int_{u-L}^u D(s+L)g(x(s)) + \int_0^u E(u,s)g(x(s-L))ds\right]du \\
&= \int_{t-L}^t D(s+L)g(x(s))ds + \int_0^t E(t,s)g(x(s-L))ds \\
&- e^{-\int_0^t q(s)ds}\left[\int_{-L}^0 D(s+L)g(\psi(s))ds\right] \\
&- \int_0^t q(u)e^{-\int_u^t q(s)ds}\left[\int_{u-L}^u D(s+L)g(x(s))ds + \int_0^u E(u,s)g(x(s-L))ds\right]du.
\end{aligned}$$

This yields

$$\begin{aligned}
x(t) =&\psi(0)e^{-\int_0^t q(s)ds} + \int_0^t e^{-\int_u^t q(s)ds} B(u)du \\
&+ \int_{t-L}^t D(s+L)g(x(s))ds + \int_0^t E(t,s)g(x(s-L))ds \\
&- e^{-\int_0^t q(s)ds}\left[\int_{-L}^0 D(s+L)g(\psi(s))ds\right] \\
&- \int_0^t q(u)e^{-\int_u^t q(s)ds}\left[\int_{u-L}^u D(s+L)g(x(s))ds + \int_0^u E(u,s)g(x(s-L))ds\right]du.
\end{aligned}$$

Let M be the complete metric space with the supremum metric defined by

$$M=\{\phi:[-L,\infty)\to R|\phi\in C,\phi(t)=\psi(t)\quad\text{for}\quad -L\le t\le 0,\quad \phi(t)\to 0\quad\text{as}\quad t\to\infty\}.$$

Define $P:M\to M$ by $\phi\in M$ implies $(P\phi)(t)=\psi(t)$ on $[-L,0]$, while $t\ge 0$ implies that

$$\begin{aligned}(P\phi)(t)=&\psi(0)e^{-\int_0^t q(s)ds}+\int_0^t e^{-\int_u^t q(s)ds}B(u)du\\ &+\int_{t-L}^t D(s+L)g(\phi(s))ds+\int_0^t E(t,s)g(\phi(s-L))ds\\ &-e^{-\int_0^t q(s)ds}\left[\int_{-L}^0 D(s+L)g(\psi(s))ds\right]\\ &-\int_0^t q(u)e^{-\int_u^t q(s)ds}\left[\int_{u-L}^u D(s+L)g(\phi(s))ds+\int_0^u E(u,s)g(\phi(s-L))ds\right]du.\end{aligned}$$

We will show that $P:M\to M$ and that under (2.18) P is a contraction with unique fixed point $\phi\in M$ which is therefore the unique solution $x_1(t)$, inheriting the properties of M. Then we will show from (2.6) that y_1 also has the desired properties. To ensure that $P:M\to M$ we must examine all terms of $P\phi$ and show that each term tends to zero as $t\to\infty$.

But first we need to find a bound on $F(x_1(t),x_1'(t))$.

LEMMA. Let

$$M^*=\{\phi:[-L,\infty)\to R|\phi\in C,\phi(t)=\psi(t)\quad\text{for}\quad -L\le t\le 0,\quad\phi\quad\text{bounded}\}.$$

Then $P:M^*\to M^*$ and is a contraction under the supremum metric.

Proof. We examine the terms of $P\phi$ for a fixed $\phi\in M^*$.

The first term is bounded since $q(t)\ge D(t+L)\beta\ge 0$.

The absolute value of the second term with $t\ge Q$ is

$$\begin{aligned}|\int_0^t e^{-\int_u^t q(s)ds}B(u)du|&=|x'(0)|\int_0^t e^{-\int_u^t q(s)ds}e^{-\int_0^u A(s)ds}du|\\ &\le|x'(0)|\int_0^t e^{-\int_0^u a(s)ds}du\le|x'(0)|\left[\int_0^Q du+\int_Q^t e^{-a_0u}du\right]\end{aligned}$$

and that is bounded.

The third term is bounded by

$$\begin{aligned}\int_{t-L}^{t} D(s+L)|g(\phi(s))|ds &\le \int_{t-L}^{t}\int_0^\infty e^{-\int_{s+L}^{u+s+L} A(v)dv} b(s+L)du|g(\phi(s))|ds \\ &\le \int_{t-L}^{t}\int_0^\infty e^{-\int_{s+L}^{u+s+L} a(v)dv} b(s+L)du|g(\phi(s))|ds\end{aligned}$$

which is bounded by (2.18). The fourth term is bounded by

$$\int_0^t \int_{t-s}^\infty e^{-\int_s^{u+s} a(v)dv} b(s)|g(\phi(s-L))|duds$$

which can be shown to be bounded using a lengthy argument which will be given in the proof of the theorem itself.

The fifth term is clearly bounded, while the sixth term is bounded for the same reasons given for the third term. The seventh term is like the fourth term.

Next, to see that P is a contraction on M^*, if $\phi, \eta \in M^*$, then

$$\begin{aligned}&|(P\phi)(t) - (P\eta)(t)| \\ &\le 2\sup_{t\ge 0}\int_{t-L}^{t} D(s+L)K|\phi(s)-\eta(s)|ds + 2\sup_{t\ge 0}\int_0^t E(t,s)K|\phi(s)-\eta(s)|ds \\ &\le [2\sup_{t\ge 0}\int_{t-L}^{t}\int_0^\infty e^{-\int_{s+L}^{u+s+L} a(v)dv} b(s+L)duds \\ &+ 2\sup_{t\ge 0}\int_0^t\int_{t-s}^\infty e^{-\int_s^{u+s} a(v)dv} b(s)duds]K\|\phi-\eta\| \le \alpha\|\phi-\eta\|\end{aligned}$$

by (2.18). Hence, P has a unique fixed point $x_1(t) \in M^*$ and it is bounded.

We now return to the proof of the theorem and see that $x_1(t)$ bounded in (2.6) will yield $x_1'(t)$ bounded since $b(u)$ and $g(x_1(u-L))$ are bounded, while $A(s) \ge a(s)$. It suffices to note that for $t > Q$ we have

$$\begin{aligned}\int_0^t e^{-\int_u^t A(s)ds}du &= \int_0^{t-Q} e^{-\int_u^t A(s)ds}du + \int_{t-Q}^t e^{-\int_u^t A(s)ds}du \\ &\le \int_0^{t-Q} e^{-a_0(t-u)}du + Q\end{aligned}$$

which is bounded.

Hence there is a $\gamma > 0$ with

$$a(t) \le A(t) = f(t, x_1(t), x_1'(t)) \le F(x_1(t), x_1'(t))c(t) \le \gamma c(t).$$

If we now consider (2.19) we have

$$\begin{aligned}\int_0^\infty D(t+L)dt &= \int_0^\infty \int_0^\infty e^{-\int_{t+L}^{u+t+L} A(s)ds} b(t+L)dudt \\ &\geq \int_0^\infty \int_0^\infty e^{-\int_{t+L}^{u+t+L} \gamma c(s)ds} b(t+L)dudt = \infty\end{aligned}$$

by (2.19).

Now we are ready to show that $\phi \in M$ implies that $(P\phi)(t) \to 0$ as $t \to \infty$.

The first term of $P\phi$ is

$$\psi(0)e^{-\int_0^t q(s)ds}$$

and since $q(t) \geq D(t+L)\beta$ and $\int_0^\infty D(t+L)dt = \infty$ by (2.19) that term tends to 0.

The second term is

$$\int_0^t e^{-\int_u^t q(s)ds} B(u)du = x'(0)\int_0^t e^{-\int_u^t q(s)ds} e^{-\int_0^u A(s)ds}du.$$

Let $2\epsilon > 0$ be given and find $T > Q$ so that $e^{-a_0 T} < \epsilon a_0$. Since $T < t$ and $0 < Q < T \leq u$,

$$\begin{aligned}\int_T^t e^{-\int_0^u A(s)ds}du &\leq \int_T^t e^{-a_0 u}du \\ &\leq e^{-a_0 T}/a_0 < \epsilon.\end{aligned}$$

Next, for $t > T$ we have

$$\int_0^T e^{-\int_u^t q(s)ds} e^{-\int_0^u A(s)ds} = e^{-\int_T^t q(s)ds}\int_0^T e^{-\int_u^T q(s)ds} e^{-\int_0^u A(s)ds}du.$$

The first factor tends to zero as $t \to \infty$, while the second factor is simply a fixed number.

The third term is

$$\int_{t-L}^t D(s+L)g(\phi(s))ds$$

which, by (2.18), tends to zero if $\phi \to 0$.

The fourth term is

$$\int_0^t E(t,s)g(\phi(s-L))ds = \int_0^t \int_{t-s}^\infty e^{-\int_s^{u+s} A(v)dv} b(s)g(\phi(s-L))duds.$$

Let $2\epsilon > 0$ be given and use boundedness of b and ϕ to find $T > 2Q$ with

$$\int_T^t b(t-w)g(\phi(t-w-L))e^{-wa_0}dw < \epsilon a_0.$$

for $t \geq T$. Then change variable below by $w = t - s$ and for $0 < T < t$ obtain

$$\begin{aligned}
&\int_0^t b(s)g(\phi(s-L)) \int_{t-s}^{\infty} e^{-\int_s^{u+s} A(v)dv} du ds \\
&= \int_0^t b(t-w)g(\phi(t-w-L)) \int_w^{\infty} e^{-\int_{t-w}^{u+t-w} A(v)dv} du dw \\
&= \int_0^T b(t-w)g(\phi(t-w-L)) \int_w^{\infty} e^{-\int_{t-w}^{u+t-w} A(v)dv} du dw \\
&+ \int_T^t b(t-w)g(\phi(t-w-L)) \int_w^{\infty} e^{-\int_{t-w}^{u+t-w} A(v)dv} du dw \\
&=: I_1 + I_2.
\end{aligned}$$

In I_2 take into account that $u \geq w \geq T \geq 2Q$ so that we have

$$\begin{aligned}
I_2 &\leq \int_T^t b(t-w)g(\phi(t-w-L)) \int_w^{\infty} e^{-a_0 u} du dw \\
&= \int_T^t b(t-w)g(\phi(t-w-L))e^{-a_0 w}/a_0 dw \\
&< \epsilon.
\end{aligned}$$

Now,

$$\begin{aligned}
I_1 &= \int_0^T b(t-w)g(\phi(t-w-L)) \int_w^{w+Q} e^{-\int_{t-w}^{u+t-w} A(v)dv} du dw \\
&+ \int_0^T b(t-w)g(\phi(t-w-L)) \int_{w+Q}^{\infty} e^{-\int_{t-w}^{u+t-w} A(v)dv} du dw \\
&\leq \int_0^T b(t-w)g(\phi(t-w-L))Q dw + \int_0^T b(t-w)g(\phi(t-w-L)) \int_{w+Q}^{\infty} e^{-a_0 u} du dw
\end{aligned}$$

since $u \geq w + Q \geq Q$ in that last inner integral. Both tend to zero since $\phi \to 0$ as $t \to \infty$.

The fifth term is

$$e^{-\int_0^t q(s)ds}[\int_{-L}^0 D(s+L)g(\psi(s))ds]$$

which tends to zero by (2.19).

The sixth term is

$$\int_0^t q(u)e^{-\int_u^t q(s)ds} \int_{u-L}^u D(s+L)g(\phi(s))ds$$

and it has the same bound as the third term.

The seventh term is

$$\int_0^t q(u)e^{-\int_u^t q(s)ds}\int_0^u E(u,s)g(\phi(s-L))dsdu$$

and it has the same bound as the fourth term.

This verifies that $P : M \to M$.

To see that P is a contraction, if $\phi, \eta \in M$ then

$$|(P\phi)(t)-(P\eta)(t)| \le 2\sup_{t\ge 0}\int_{t-L}^t D(s+L)K|\phi(s)-\eta(s)|ds+2\sup_{t\ge 0}\int_0^t E(t,s)K|\phi(s)-\eta(s)|ds$$

and the contraction follows from (2.18). It now follows that P has a unique fixed point and it is the function $x_1(t)$. The fixed point tends to zero since it is in M.

Now that we have shown that $x_1(t)$ tends to zero, refer back to (2.6) and apply the above arguments to show that $y_1(t)$ also tends to zero. Here are the details. Let the bound on $b(t)$ be 1, let $G(t) := |g(x_1(t-L))|$, let G^* be the maximum of $G(t)$, and let $2\epsilon > 0$ be given. Fix $T > 2Q$ so that $e^{-a_0 T}G^* < \epsilon a_0$. Take N so large that $TG(t) < \epsilon$ if $t > N$. Then $t > N + T$ implies that

$$\begin{aligned}\int_0^t e^{-\int_u^t A(s)ds}b(u)|g(x_1(u-L))|du &= \int_0^{t-T} e^{-\int_u^t A(s)ds}b(u)G(u)du\\ &\quad+\int_{t-T}^t e^{-\int_u^t A(s)ds}b(u)G(u)du\\ &\le \int_0^{t-T} e^{-a_0(t-u)}b(u)g(u)du+\int_{t-T}^t G(u)du\\ &\le G^*e^{-a_0T}/a_0+\epsilon < 2\epsilon.\end{aligned}$$

This completes the proof.

PROPOSITION 2.1. Consider (2.19). Let $c(t) = h + 2t$ for some $h \ge 0$ and let $0 \le b(t) \le M$ for some $M > 0$. Then the integral condition in (2.19) is satisfied provided that $b(t)$ is large enough that

$$\int_0^\infty \frac{b(t+L)}{(t+L)}dt = \infty.$$

Proof. The computations are virtually unchanged if we take $h = 0$ and $\gamma = 1$. We have

$$\begin{aligned}
\int_0^\infty \int_0^\infty e^{-\int_{t+L}^{u+t+L} 2s ds} b(t+L)dudt &= \int_0^\infty b(t+L) \int_0^\infty e^{-(u+t+L)^2+(t+L)^2} dudt \\
&= \int_0^\infty b(t+L) \int_0^\infty e^{-u^2-2u(t+L)} dudt \\
&\geq \int_0^\infty b(t+L) \int_0^1 e^{-u^2-2u(t+L)} dudt \\
&\geq \int_0^\infty b(t+L)e^{-1} \int_0^1 e^{-2(t+L)u} dudt \\
&= \int_0^\infty \frac{b(t+L)}{-2e(t+L)} e^{-2(t+L)u}\Big|_0^1 dt \\
&= \int_0^\infty \frac{b(t+L)}{2e(t+L)} dt - \int_0^\infty \frac{b(t+L)}{2e(t+L)} e^{-2(t+L)} dt \\
&= \infty,
\end{aligned}$$

as required. In fact, we see that $b(t)$ can vanish on long intervals.

PROPOSITION 2.2. Consider (2.19). If $c(t) = 3t^2$ and if $0 \leq b(t) \leq \beta$ for some $\beta > 0$, then (2.19) is not satisfied.

Proof. We have

$$\begin{aligned}
\int_0^\infty \int_0^\infty e^{-\int_{t+L}^{u+t+L} 3w^2 dw} b(t+L)dudt &= \int_0^\infty \int_0^\infty e^{-(u+t+L)^3+(t+L)^3} b(t+L)dudt \\
&= \int_0^\infty \int_0^\infty e^{-u^3-3u^2(t+L)-3u(t+L)^2} b(t+L)dudt \\
&\leq \int_0^\infty \int_0^\infty e^{-3(t+L)^2 u} b(t+L)dudt \\
&= \int_0^\infty \frac{b(t+L)}{-3(t+L)^2} e^{-3(t+L)^2 u}\Big|_0^\infty dt \\
&= \int_0^\infty \frac{b(t+L)}{3(t+L)^2} dt < \infty,
\end{aligned}$$

as required.

We return to Proposition 2.1 now and see what must be done to satisfy (2.18). In fact, there are three simple ways to do so. We could take a large t_0 as a starting point for our solutions and go back through the calculations replacing 0 by the new t_0. We could take h large; this is the easiest one for we will be able to approximate $a(t)$ by h and the integrals will become of convolution type and very simple to evaluate. Or, we could let $b(t)$ be zero on an interval $[0, N]$ for large N. Since letting $b(t)$ vanish is a condition not seen before in the literature, we select that alternative.

PROPOSITION 2.3. Let $a(t) = 2t$, let $b(t) = 0$ on $[0, N]$ where L and N satisfy

$$K\left[\ln\left(1 + \frac{L}{N}\right) + \frac{2}{N^2}\right] < 1.$$

and let $b(t) \leq 1$. Then (2.18) is satisfied.

Proof. The second integral in (2.18) is

$$\begin{aligned}
\int_0^t \int_{t-s}^\infty e^{-u^2-2us} b(s)duds &\leq \int_N^t \int_{t-s}^\infty e^{-(u+s)u} b(s)duds \\
&\leq \int_N^t \int_{t-s}^\infty e^{-(t-s+s)u} b(s)duds \\
&= \int_N^t \frac{e^{-tu}}{-t}\bigg|_{t-s}^\infty ds \\
&= \frac{e^{-t^2}}{t} \int_N^t e^{ts} ds \\
&\leq 1/t^2.
\end{aligned}$$

Taking the supremum in t we have the value $1/N^2$.

In the computation below, $b(s + L) = 0$ if $s \leq N - L$, so we take $t \geq N$. Moreover, $b(t) \leq 1$ so it exits the sequence of computations. The first integral in (2.18) is

$$\begin{aligned}
\int_{t-L}^t \int_0^\infty e^{-\int_{s+L}^{u+s+L} a(v)dv} b(s+L)duds &= \int_{t-L}^t \int_0^\infty e^{-v^2|_{s+L}^{u+s+L}} b(s+L)duds \\
&\leq \int_{t-L}^t \int_0^\infty e^{-(u+s+L)^2+(s+L)^2} duds \\
&= \int_{t-L}^t \int_0^\infty e^{-u^2-2u(s+L)} duds \\
&\leq \int_{t-L}^t \int_0^\infty e^{-2(s+L)u} duds \\
&= \int_{t-L}^t \frac{1}{2(s+L)} ds \\
&= (1/2)\ln(1 + \frac{L}{t}).
\end{aligned}$$

Because $b(t)$ is zero on $[0, N]$ this quantity has a maximum of $(1/2)\ln(1 + \frac{L}{N})$.

Putting both integrals together, (2.18) will be satisfied if

$$K\left[\ln\left(1+\frac{L}{N}\right)+\frac{2}{N^2}\right]<1.$$

REMARK. There is an intricate relation between $g(s)$, $b(t)$, L, and $a(t)$ on the whole interval $[0,\infty)$. We can (and have) constructed examples such as $a(t)=|\sin 2\pi t|+\sin 2\pi t$ with crude approximations such as

$$\int_t^{t+J} A(s)ds \geq (1/\pi)J$$

to fulfill (2.20). Condition (2.19) is readily satisfied. And the approximation

$$e^{-\int_s^{u+s} a(v)dv} \leq e^{-(1/\pi)u+(1/\pi)}$$

starts us on our way to verifying (2.18), but we are still left with finding bounds on

$$\pi e^{(1/\pi)} \int_{t-L}^{t} b(s+L)ds$$

and

$$\pi e^{(1/\pi)} \int_0^t b(s)e^{-(1/\pi)(t-s)}ds$$

in order to fulfill (2.18). For many tabulated functions $b(t)$, the value of the last integral can be read from Laplace transform tables with the aid of shift theorems.

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