ON THE EXISTENCE OF PERIODIC SOLUTIONS OF SOME NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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1. Introduction. Consider the system of nonlinear integrodifferential equation

$$x'(t) = Dx(t) + \int_{-\infty}^{\infty} [d_s E(t,s)] g(x(t+s)) + f(t, x(t)), \tag{E}$$

where $D \in R^{n\times n}$ is a constant matrix; $E : R \times R \to R^{n\times n}$ is measurable, is continuous to the left (or to the right) in s, has bounded total variation in s for each t and the function

$$Q(t) := \int_{-\infty}^{\infty} |d_s E(t,s)| \qquad (t \in R)$$

is continuous. Here, $|\cdot|$ denotes the matrix norm induced by a norm, also denoted by $|\cdot|$, on R^n. We assume that $g : R^n \to R^n,\ f : R \times R^n \to R^n$ are continuous; $E(\cdot, s)$ and $f(\cdot, x)$ are T-periodic ($T > 0$) for every $s \in R,\ x \in R^n$. This equation is important in applications [1, 2, 4], e.g., in population dynamics [7], in which the special case

$$x'(t) = Dx(t) + \sum_{k=1}^{\infty} A_k(t) g_k(x(t - \tau_k)) + \int_{-\infty}^{0} C(t,s) g(x(t+s)) ds \tag{C}$$

often appears where $A_k : R \to R^{n\times n},\ g_k : R^n \to R^n,\ C : R \times (-\infty, 0] \to R^{n\times n}$ are continuous, $A_k,\ C(\cdot, s)$ are T-periodic for all s, and the functions

$$Q_1(t) := \sum_{k=1}^{\infty} |A_k(t)| < \infty, \qquad Q_2(t) := \int_{-\infty}^{0} |C(t,s)| ds$$

are continuous, and $\overline{g}(u) := \sup\{|g_k(u)| : 1 \le k < \infty\} < \infty$ for all $u \in R^n$; $\overline{g}(u)$ is bounded for u bounded.

Our purpose is to give conditions guaranteeing the existence of T-periodic solutions of equation (E) or (C). We apply the Leray-Schauder continuation method [8, 13, 14]. This method has been developed for FDE's with finite delay [10, 12, 15]. Burton, Eloe and Islam [3] used the Granas version [9] of the continuation method for the special case of (C) when $A_k \equiv 0,\ f(t,x) = h(x) + p(t)$. They have given a general theorem requiring a uniform *a priori* bound on all the periodic solutions of the family of homotopic equations. They applied their theorem to the scalar case of (C) and found the desired *a priori* bound by a Lyapunov functional technique.

We consider (E) as a perturbed equation of the ODE $x' = Dx + f(t,x)$ and try to select the properties of f which give rise to the existence of periodic solutions. When D is hyperbolic (i.e., D has no eigenvalues which are pure imaginary) we give a "sign condition" for f. In this case we also use a Lyapunov *functional* to find an *a priori* bound for the periodic solutions of the family of homotopic equations. For the same purpose, in the general case we can use a very simple Lyapunov *function*. The condition obtained for this case works very well for the stable and totally unstable matrices (i.e., when all the real parts of the eigenvalues of D are of the same sign). Both in the hyperbolic and the general case the conditions are much simpler technically for scalar equations which are, therefore, treated separately.

2. The continuation method for equation (E). We suppose that the reader is familiar with the definitions of the Brouwer degree and the Leray-Schauder degree and with their most basic properties (see, e.g., [5, 6, 8, 14]).

The following notation will be used:

$$BC := \{\phi \in C(R; R^n) : \phi \text{ is bounded on } R\};$$

$$\|\phi\| := \sup\{|\phi(t)| : t \in R\}, \quad (\phi \in BC);$$

$$C_T := \{\phi \in C(R; R^n) : \phi(t+T) \equiv \phi(t) \text{ for } t \in R\};$$

$$B(\rho) := \{\phi \in C_T : \|\phi\| < \rho\}, \quad (\rho > 0).$$

By a solution of (E) or (C) we mean a function $x \in BC \cap C^1(R; R^n)$ satisfying (E) or (C), respectively.

Consider now the linear inhomogeneous ODE

$$x'(t) = Dx(t) + h(t) \qquad (h \in C_T). \tag{2.1}$$

It is well-known [11, 14] that if $2k\pi i/T$ (for $k = 0, \pm 1, \pm 2, \dots$) are not eigenvalues of D, then $\mathcal{K} : C_T \to C_T$ defined by

$$(\mathcal{K}h)(t) := \int_0^T [e^{-AT} - I]^{-1} e^{-As} h(t+s)ds$$

($I \in R^{n\times n}$ is the unit matrix) is a bounded linear operator yielding the unique T-periodic solution of (2.1).

For $\phi \in C_T,\ \lambda \in [0,1]$ we shall use the notation:

$$\begin{aligned}(\mathcal{D}\phi)(t) &:= D\phi(t);\\ (\mathcal{G}\phi)(t) &:= \int_{-\infty}^{\infty} [d_s E(t,s)] g(\phi(t+s));\\ (\mathcal{F}\phi)(t) &:= f(t,\phi(t));\\ \mathcal{M}(\phi,\lambda) &:= \lambda\mathcal{K}\{\mathcal{G}\phi + \mathcal{F}\phi\}.\end{aligned}$$

From our assumptions it follows that $\mathcal{D},\ \mathcal{G},\ \mathcal{F} : C_T \to C_T$ and $\mathcal{M} : C_T \times [0,1] \to C_T$. By the definition of $\mathcal{K}$, a function x_λ is a T-periodic solution of the equation

$$(\mathrm{E}^1_\lambda) \qquad x' = \mathcal{D}x + \lambda\{\mathcal{G}x + \mathcal{F}x\}, \qquad (0 \le \lambda \le 1)$$

if and only if x_λ is a fixed point of $\mathcal{M}_\lambda := \mathcal{M}(\cdot,\lambda)$.

LEMMA 2.1. *Suppose that*

(i) $\det[D - (2k\pi i/T)I] \neq 0, \quad (k = 0, \pm1, \pm2, \dots)$

and that

(ii) *there is a bounded open set $\Omega \subset C_T$ such that $0 \in \Omega$ and for every $\lambda \in (0,1)$, every T-periodic solution x_λ of (E^1_λ) satisfies $x_\lambda \notin \partial\Omega$.*

Then equation (E) has a T-periodic solution in Ω.

PROOF. For any bounded set $H \subset C_T$ the family of functions $\{\mathcal{K}h : h \in H\}$ is uniformly bounded and equicontinuous, so $\mathcal{K}$ is a compact operator by the Arzela-Ascoli theorem. Since, in addition, $\mathcal{G} + \mathcal{F}$ maps bounded sets into bounded sets, $\mathcal{M}$ is compact, too. The Leray-Schauder degree of a compact perturbation of the identity map $\mathcal{I} : C_T \to C_T$ is invariant with respect to homotopy [5, 6, 14]; hence

$$d[\mathcal{I} - \mathcal{M}(\cdot,1), \Omega, 0] = d[\mathcal{I} - \mathcal{M}(\cdot,0), \Omega, 0] = d[\mathcal{I}, \Omega, 0] = 1.$$

By the Kronecker existence theorem [5, 6, 14], $\mathcal{M}(\cdot,1)$ has a fixed point in Ω; i.e., equation (E) has at least one T-periodic solution in Ω. This completes the proof.

We now consider the so-called critical case when condition (i) in Lemma 2.1 is not fulfilled. The following notation is used:

$$\mathcal{P} : BC \to R^n, \quad \mathcal{P}\phi := (1/T)\int_0^T \phi(t)dt;$$
$$C_T^0 := \{\phi \in C_T : \mathcal{P}\phi = 0\}.$$

Let C_T^c denote the n-dimensional subspace of C_T consisting of the constant functions, and use the simple notation $a \in C_T^c$ for the function $\phi \in C_T^c$, $\phi(t) \equiv a \in R^n$. Then define $\Omega^c := \Omega \cap C_T^c$.

LEMMA 2.2. *Suppose that the following conditions are fulfilled:*

(i) *there is a bounded open set $\Omega \subset C_T$ such that $0 \in \Omega$, and for every $\lambda \in (0,1)$, every T-periodic solution x_λ of the equation*

$$(\mathrm{E}^2_\lambda) \qquad x' = \lambda\{\mathcal{D}x + \mathcal{G}x + \mathcal{F}x\}, \quad (0 \le \lambda \le 1)$$

satisfies $x_\lambda \notin \partial\Omega$;

(ii) *if for $a \in R^n$, $h(a) := \mathcal{P}[\mathcal{D} + \mathcal{G} + \mathcal{F}]a = 0$, then $a \notin \partial\Omega$;*

(iii) *the Brouwer degree $d[h, \Omega^c, 0]$ of the function $h : R^n \to R^n$ defined in (ii) is different from zero.*

Then equation (E) has a T-periodic solution in Ω.

PROOF. Since the operator $\mathcal{K}$ can not be used here we need to transform (E) into a fixed point problem. Thus, a new operator $\mathcal{N}$ to replace $\mathcal{M}$ is defined as follows:

$$\mathcal{J} : BC \to C^1(R; R^n), \quad (\mathcal{J}\phi)(t) := \int_0^t \phi(s)ds;$$
$$\mathcal{H} : BC \to C^1(R; R^n), \quad \mathcal{H} := (\mathcal{I} - \mathcal{P})\mathcal{J};$$
$$\mathcal{N} : C_T \times [0,1] \to C_T,$$
$$\mathcal{N}(\phi, \lambda) := \mathcal{P}\phi - \mathcal{P}[\mathcal{D} + \mathcal{G} + \mathcal{F}]\phi + \lambda\mathcal{H}(\mathcal{I} - \mathcal{P})[\mathcal{D} + \mathcal{G} + \mathcal{F}]\phi.$$

We first show that, for $\lambda \in (0,1]$, $x_\lambda \in C_T$ is a solution of (E^2_λ) if and only if it is a fixed point of $\mathcal{N}_\lambda := \mathcal{N}(\cdot, \lambda)$. To see this, suppose that $x_\lambda \in C_T$ is a solution of (E^2_λ). By virtue of $\mathcal{P}x'_\lambda = 0$ and $\lambda \neq 0$ we have

$$\mathcal{P}[\mathcal{D} + \mathcal{G} + \mathcal{F}]x_\lambda = 0. \tag{2.2}$$

Integrating (E^2_λ) and using (2.2) we have

$$x_\lambda - x_\lambda(0) = \lambda\mathcal{J}(\mathcal{I} - \mathcal{P})\{\mathcal{D} + \mathcal{G} + \mathcal{F}\}x_\lambda$$

whence

$$x_\lambda - \mathcal{P}x_\lambda = \lambda\mathcal{H}(\mathcal{I} - \mathcal{P})\{\mathcal{D} + \mathcal{G} + \mathcal{F}\}x_\lambda$$
$$= \mathcal{N}(x_\lambda, \lambda) - \mathcal{P}x_\lambda.$$

Conversely, if x_λ is a fixed point of $\mathcal{N}(\cdot, \lambda)$, then an application of $\mathcal{P}$ to both sides of $x_\lambda = \mathcal{N}(x_\lambda, \lambda)$ yields (2.2). Differentiating the same equality and using (2.2) we obtain

$$x'_\lambda = \lambda(\mathcal{I} - \mathcal{P})[\mathcal{D} + \mathcal{G} + \mathcal{F}]x_\lambda = \lambda[\mathcal{D} + \mathcal{G} + \mathcal{F}]x_\lambda;$$

that is, x_λ is a solution of (E^2_λ).

Obviously, the fixed points of $\mathcal{N}(\cdot, 0)$ are the constant vectors $a \in R^n$ satisfying $h(a) = 0$.

The compactness of $\mathcal{N}$ can be proved by observing that the range of $\mathcal{P}$ is of finite dimension, $\mathcal{D} + \mathcal{G} + \mathcal{F}$ maps bounded sets into bounded sets, and $\mathcal{H}$ is compact by the Arzela-Ascoli theorem.

The Leray-Schauder degree is invariant with respect to homotopy; therefore,

$$d := d[\mathcal{I} - \mathcal{N}(\cdot, 1), \Omega, 0] = d[\mathcal{I} - \mathcal{N}(\cdot, 0), \Omega, 0]$$
$$= d[\mathcal{I} - (\mathcal{P} - \mathcal{P}\{\mathcal{D} + \mathcal{G} + \mathcal{F}\}), \Omega, 0].$$

By the definition of the Leray-Schauder degree [5, 6, 14] we get

$$d = d[\mathcal{I} - (\mathcal{P} - \mathcal{P}\{\mathcal{D} + \mathcal{G} + \mathcal{F}\})|C^c_T, \Omega^c, 0]$$
$$= d[\mathcal{P}\{\mathcal{D} + \mathcal{G} + \mathcal{F}\}|C^c_T, \Omega^c, 0].$$

By condition (iii), $d \neq 0$, which implies the existence of a fixed point of $\mathcal{N}(\cdot, 1)$ according to the Kronecker existence theorem. This completes the proof.

Remark. Lemma 2.1 (Lemma 2.2) generalizes Theorem 3 (Theorem 4) of J. Mawhin [12] for a case of infinite delay.

3. The main theorems. The first theorem is concerned with equation (C) in the hyperbolic case, i.e., when the matrix D has no eigenvalue on the imaginary axis. It is known [1] that in this case there exists a symmetric matrix $L \in R^{n\times n}$ with $D^T L + LD = -I$.

For the functions Q_i defined immediately after (C), assume that

$$Q_1^0 := \sup_{0\le t\le T} Q_1(t) > 0, \quad Q_2^0 := \sup_{0\le t\le T} Q_2(t) > 0$$

and let

$$[a]_+ := (|a| + a)/2, \quad [a]_- := (|a| - a)/2.$$

THEOREM 3.1. *Suppose that D has no eigenvalues on the imaginary axis and the following conditions are fulfilled:*

(i) $\int_{-\infty}^0 \int_0^T |C(t,s)|^2 dtds < \infty$;

(ii) $\int_{-\infty}^t \int_{-\infty}^{u-t} |C(u-s,s)|ds < \infty, \quad (t \in R)$;

(iii) $\int_{-\infty}^0 |C(t-s,s)|ds > 0, \quad (t \in R)$;

(iv) $\sum_{k=1}^{\infty} \int_{t-\tau_k}^t |A_k(u+\tau_k)|du < \infty, \quad (t \in R)$;

(v) *there are $\beta \in (0,1)$ and $M_1 \in R$ such that*

$$\beta\{|x|^2/2 + 2\lambda[f^T(t,x)Lx]_-\} + M_1 \ge$$
$$2\lambda[f^T(t,x)Lx]_+ + 4|L|^2\lambda\{Q_1^0(\overline{g}(x))^2 \sum_{k=1}^{\infty} |A_k(t+\tau_k)|$$
$$+ Q_2^0|g(x)|^2 \int_{-\infty}^0 |C(t-s,s)|ds\}$$

for all $t \in R,\ x \in R^n,\ \lambda \in [0,1]$;

(vi) *there are constants $\gamma_1,\ \gamma_2,\ M_2$ such that*

$$|f(t,x)| \le \gamma_1|x|^2 + \gamma_2[f^T(t,x)Lx]_- + M_2$$

for all $t \in R,\ x \in R^n$.

Then equation (C) has a T-periodic solution.

PROOF. We find a $\rho > 0$ such that for all the T-periodic solutions x_λ of

$$x'(t) = Dx(t) + \lambda\left\{\sum_{k=1}^{\infty} A_k(t)g_k(x(t-\tau_k)) + \right.$$
$$(C_\lambda) \qquad \left.\int_{-\infty}^0 C(t,s)g(x(t+s))ds + f(t,x(t))\right\}$$

are located in $B(\rho)$ for every $\lambda \in (0,1)$. Then we apply Lemma 2.1.

For a T-periodic solution x_λ of (C_λ) define

$$\begin{aligned} V(t, x_\lambda(\cdot)) &:= x_\lambda^T(t) L x_\lambda(t) \\ &\quad + 4Q_1^0 |L|^2 \lambda^2 \sum_{k=1}^{\infty} \int_{t-\tau_k}^{t} |A_k(u+\tau_k)| |g_k(x_\lambda(u))|^2 du \\ &\quad + 4Q_2^0 |L|^2 \lambda^2 \int_{-\infty}^{t} \int_{-\infty}^{u-t} |C(u-s,s)| ds |g(x_\lambda(u))|^2 du. \end{aligned}$$

Then the derivative $V'(t, x_\lambda(\cdot))$ with respect to (C_λ) of V can be estimated as follows:

$$V'(t, x_\lambda(\cdot)) \le I_0 + I_1 + I_2$$

where

$$\begin{aligned} I_0 :&= x_\lambda^T(t)[D^T L + LD] x_\lambda(t) + 2\lambda f^T(t, x_\lambda(t)) L x_\lambda(t) \\ &= -|x_\lambda(t)|^2 + 2\lambda f^T(t, x_\lambda(t)) L x_\lambda(t), \\ I_1 &= 2\lambda |L| \sum_{k=1}^{\infty} |A_k(t)| |g_k(x_\lambda(t-\tau_k))| |x_\lambda(t)| \\ &\quad - 4Q_1^0 |L|^2 \lambda^2 \sum_{k=1}^{\infty} \{ |A_k(t)| |g_k(x_\lambda(t-\tau_k))|^2 \\ &\qquad - |A_k(t+\tau_k)| |g_k(x_\lambda(t))|^2 \} \\ &\le -(1/4Q_1^0) \sum_{k=1}^{\infty} |A_k(t)| \{ |x_\lambda(t)| - 4|L| \lambda Q_1^0 |g_k(x_\lambda(t-\tau_k))| \}^2 \\ &\quad + (1/4)|x_\lambda(t)|^2 + 4Q_1^0 |L|^2 \lambda^2 \sum_{k=1}^{\infty} |A_k(t+\tau_k)| |g_k(x_\lambda(t))|^2, \\ I_2 :&= 2\lambda |L| \int_{-\infty}^{0} |C(t,s)| |g(x_\lambda(t+s))| |x_\lambda(t)| ds \\ &\quad - 4Q_2^0 |L|^2 \lambda^2 \left[\int_{-\infty}^{t} |C(t, u-t)| |g(x_\lambda(u))|^2 du \right. \\ &\quad \left. - \int_{-\infty}^{0} |C(t-s,s)| ds |g(x_\lambda(t))|^2 \right] \\ &\le -(1/4Q_2^0) \int_{-\infty}^{0} |C(t,s)| \{ |x_\lambda(t)| - 4|L| \lambda Q_2^0 |g(x_\lambda(t+s))| \}^2 ds \\ &\quad + (1/4)|x_\lambda(t)|^2 + 4Q_2^0 |L|^2 \lambda^2 \int_{-\infty}^{0} |C(t-s,s)| ds |g(x_\lambda(t))|^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} V'(t,x_\lambda(\cdot)) \le -(x_\lambda(t))^2/2 - 2\lambda[f^T(t,x_\lambda(t))Lx_\lambda(t)]_- \\ + 2\lambda[f^T(t,x_\lambda(t))Lx_\lambda(t)]_+ \\ + 4|L|^2\lambda^2 \left\{ Q_1^0(\overline{g}(x_\lambda(t)))^2 \sum_{k=1}^{\infty} |A_k(t+\tau_k)| \right. \\ \left. + Q_2^0 |g(x_\lambda(t))|^2 \int_{-\infty}^0 |C(t-s,s)|ds \right\}. \end{aligned}$$

From assumption (v) it follows that

$$\begin{aligned} V'(t,x_\lambda(\cdot)) \le -(1-\beta)\{|x_\lambda(t)|^2/2) \\ + 2\lambda[f^T(t,x_\lambda(t))Lx_\lambda(t)]_-\} + M_1 \end{aligned} \tag{3.1}$$

for all $t \in R$. Since the function $v(t) := V(t, x_\lambda(\cdot))$ is T-periodic we get

$$0 = v(t+T) - v(t) \le \int_t^{t+T} V'(s, x_\lambda(\cdot))ds,$$

whence, using (3.1) we obtain

$$\int_t^{t+T} |x_\lambda|^2 \le 2M_1T/(1-\beta); \lambda \int_t^{t+T} [f^T(s,x_\lambda(s))Lx_\lambda(s)]_- ds \le M_1T/2(1-\beta) \tag{3.2}$$

for all $t \in R$.

On the other hand, by equation (C_λ) and the Schwarz inequality

$$\begin{aligned} \int_t^{t+T} |x'_\lambda| \le |D|T^{1/2} \left[\int_t^{t+T} |x_\lambda|^2 \right]^{1/2} + \lambda \int_t^{t+T} |f(s,x_\lambda(s))|ds \\ + \lambda \left[\int_t^{t+T} \left(\sum_{k=1}^{\infty} |A_k(s+\tau_k)| \right) ds \right]^{1/2} \left[\int_t^{t+T} (\overline{g}(x_\lambda(s)))^2 \left(\sum_{k=1}^{\infty} |A_k(s+\tau_k)| \right) ds \right]^{1/2} \\ + \lambda \left[\int_t^{t+T} |g(x_\lambda(s))|^2 ds \right]^{1/2} \int_{-\infty}^0 \left[\int_t^{t+T} |C(s,u)|^2 ds \right]^{1/2} du. \end{aligned}$$

From (v), (vi), (3.2), and the other conditions of the theorem there follows the existence of constants $K_1,\ K_2$ such that

$$\int_t^{t+T} |x_\lambda| \le K_1, \qquad \int_t^{t+T} |x'_\lambda| \le K_2$$

for all $t \in R$; whence, by Sobolev's inequality we get

$$\|x_\lambda\| \le (1/T)K_1 + K_2 =: \rho.$$

The application of Lemma 2.1 with $\Omega = B(\rho)$ completes the proof.

COROLLARY 3.1. *Suppose that conditions (i)-(iv) in Theorem 3.1 are satisfied. If, in addition,*

$$(v') \qquad f(t,x) = o(|x|),\ g(x) = o(|x|),\ \overline{g}(x) = o(|x|),\ (|x| \to \infty),$$

then equation (C) has a T-periodic solution.

In [3] scalar equations are treated. In order to make the results more comparable with one another we develop the method for this case. It is much simpler for scalar equations owing to the fact that if $x \in C^1(R;R) \cap C_T$ then $x'(\tau) = 0$ provided that $|x|$ has a maximum at τ.

Introduce the notation

$$g^*(r) := \max\ \{|g(x)| : x \in R^n,\ |x| \leq r\}.$$

THEOREM 3.2. *Consider equation (E) in the scalar case ($n = 1$), and assume $D \neq 0$. Suppose there exist $\rho, \eta > 0$ with*

$$|D|\rho + \lambda[(\text{sign } D)f(t,\rho)]_+ > \lambda[(\text{sign } D)f(t,\rho)]_- + \lambda Q(t)g^*(\rho),$$

$$|D\eta| + \lambda[(\text{sign } D)f(t,-\eta)]_- > \lambda[(\text{sign } D)f(t,-\eta)]_+ + \lambda Q(t)g^*(\eta)$$

for all $t \in R,\ \lambda \in [0,1]$.

Then equation (E) has T-periodic solutions in $(-\eta, \rho)$.

PROOF. Define $V(x) := Lx^2 = -(1/(2D))x^2$. Then the derivative $V'(t, x_\lambda(\cdot))$ of V with respect to the equation

$$(E^1_\lambda) \qquad x'(t) = Dx(t) + \lambda\{f(t,x(t)) + \int_{-\infty}^{\infty} [d_s E(t,s)] g(x(t+s)) ds\}$$

satisfies the following estimate: if $|x_\lambda(\tau)|$ is the maximal value of $|x_\lambda|$, then

$$\begin{aligned} 0 = V'(\tau, x_\lambda(\cdot)) \leq \\ &- (|x_\lambda(\tau)|/|D|)\{|D||x_\lambda(\tau)| + (\lambda/|x_\lambda(\tau)|)[x_\lambda(\tau)f(\tau, x_\lambda(\tau))/\text{sign } D)]_+ \\ &- (\lambda/|x(\tau)|)[x_\lambda(\tau)f(\tau, x_\lambda(\tau))/\text{sign } D]_- \\ &- \lambda g^*(|x_\lambda(\tau)|) \int_{-\infty}^{\infty} |d_s E(t,s)|. \end{aligned}$$

By the condition of the theorem we obtain

$$x_\lambda \in \Omega := \{\phi \in C^1(R;R) \cap C_+ : -\eta < \phi(t) < \rho \text{ for all } t \in R\}.$$

The statement now follows from Lemma 2.1.

In the following theorem we require nothing of the eigenvalues of D in advance. The method is the same for the case of T-periodically varying matrix D, so we immediately formulate our theorem for the equation

$$(\mathrm{E}_*) \qquad x'(t) = D(t)x(t) + f(t,x(t)) + \int_{-\infty}^{\infty} [d_s E(t,s)]g(x(t+s)),$$

where f, g, E are the same as in (E), and $D : R \to R^n$ is continuous and T-periodic.

THEOREM 3.3. *A) Let $n > 1$ and suppose there exists a $\rho > 0$ such that $|x| = \rho$ implies that*

$$|x^T(D^T(t) + D(t))x + 2f^T(t,x)x| > 2\rho Q(t) g^*(\rho)$$

for all $t \in [0,T]$.

Then equation (E_*) has a T-periodic solution in $B(\rho)$.

B) Let $n = 1$ and suppose that there are $\xi, \eta > 0$ such that

$$|D(t)\xi + f(t,\xi)| > Q(t) g^*(\max\{\xi,\eta\}),$$

$$|D(t)\eta - f(t,-\eta)| > Q(t) g^*(\max\{\xi,\eta\})$$

for all $t \in [0,T]$, and

$$\textit{sign } h(\xi)h(-\eta) < 0$$

where

$$h(x) := (1/T)\int_0^T \{D(t)x + f(t,x) + \int_{-\infty}^{\infty} [d_s\ E(t,s)]g(x)\}dt.$$

Then equation (E_*) has a T-periodic solution in $(-\eta, \rho)$.

PROOF. A) First we find a $\rho > 0$ such that the boundary of $B(\rho)$ does not contain any T-periodic solution x_λ of the equation

$$(\mathrm{E}_2^\lambda) \qquad x'(t) = \lambda\{D(t)x(t) + f(t,x(t)) + \int_{-\infty}^{\infty} [d_s E(t,s)]g(x(t+s))\}, \quad 0 < \lambda < 1.$$

Define the auxiliary function $V(x) := x^T x$. If $x_\lambda \in C_T$ is a solution of (E_2^λ) for some $\lambda \in (0,1)$, and $|x_\lambda(\tau)|$ is a maximal value of $|x_\lambda(\cdot)|$, then $V(x_\lambda(\tau))$ is a maximal value of $V(x_\lambda(\cdot))$ and, consequently, $V'(\tau, x_\lambda(\tau+\cdot)) = 0$. On the other hand,

$$\begin{aligned} 0 = V'(\tau, x_\lambda(\tau+\cdot)) &\leq 2\lambda\{x_\lambda^T(\tau)(D^T(\tau)+D(\tau))x_\lambda(\tau)/2] \\ &+ f^T(\tau, x_\lambda(\tau))x_\lambda(\tau) + |x_\lambda(\tau)|g^*(|x_\lambda(\tau)|)\int_{-\infty}^{\infty}|d_sE(t,s)|\} \end{aligned}$$

By our condition, $|x_\lambda(\tau)| = \rho$ implies that $V'(\tau, x_\lambda(\cdot)) \neq 0$, which means that $x_\lambda \in B(\rho)$; i.e., the first condition in Lemma 2.2 is met with $\Omega := B(\rho)$.

In order to check the fulfilment of the other two conditions in Lemma 2.2 we introduce the function $h : R^n \to R^n$,

$$h(a) := (1/T)\int_0^T \{D(t)a + f(t,a) + \int_{-\infty}^{\infty}[d_sE(t,s)]g(a)\}dt$$

and show that $|a| = \rho$ implies that $h(a) \neq 0$.

Suppose that $|a| = \rho$. Then

$$\begin{aligned} a^T h(a) = (1/T)\int_0^T &\{[a^T(D(t)+D^T(t))a/2] \\ &+ a^T f(t,a) + \int_{-\infty}^{\infty} a^T[d_sE(t,s)]g(a)\}dt. \end{aligned}$$

Since

$$\left|\int_{-\infty}^{\infty} a^T[d_sE(t,s)]g(a)\right| \leq \rho Q(t)g^*(\rho)$$

for all $t \in R$, by the condition of the theorem the integrand has no zero, and $a^T h(a) \neq 0$; therefore, $h(a) \neq 0$.

It remains to be proved that $d[h, B^n(\rho), 0] \neq 0$, where $B^n(\rho) := \{x \in R^n : |x| < \rho\}$. Suppose first $a^T h(a) > 0$ if $|a| = \rho$. Then using the notation $I : R^n \to R^n$ for the identity function ($I(x) = x$ for all $x \in R^n$), and applying the Poincaré-Bohl theorem [6, 14] we get

$$d[h, B^n(\rho), 0] = d[I, B^n(\rho), 0] = 1.$$

If $a^T h(a) < 0$ for all a with $|a| = \rho$, then

$$d[h, B^n(\rho), 0] = d[-I, B^n(\rho), 0] = (-1)^n \neq 0.$$

By Lemma 2.2, the proof is complete.

B) The basic ideas are the same as those of A); the details are omitted.

The condition of Theorem 3.3 works well in the stable or totally unstable case, i.e., when the quadratic form $x^T(D^T(t) + D(t))x$ is negative definite or positive definite.

Remark. Theorem 3.1 can be formulated and proved also for equation (E_*) by using the characteristic exponents of the equation $x' = D(t)x$ instead of the eigenvalues of D.

References

1. T. A. Burton, *Volterra Integral and Differential Equations*, Academic Press, Orlando, Florida, 1983.
2. T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, Florida, 1985.
3. T. A. Burton, P. W. Eloe, and M. N. Islam, *Nonlinear integrodifferential equations with a priori bounds on periodic solutions*, to appear in Ann. Mat. Pura Appl.
4. C. Corduneanu, V. Lakshmikantham, *Equations with unbounded delay: A survey*, Nonlinear Anal. **4** (1980), 831-877.
5. J. Cronin, *Differential Equations*, Dekker, New York, 1980.
6. J. Cronin, *Fixed Points and Topological Degree in Nonlinear Analysis*, Math. Surveys, no. 11, Amer. Math. Soc., Providence, R.I., 1964.
7. J. M. Cushing, *Integrodifferential Equations and Delay Models in Population Dynamics*, Lecture Notes in Biomathematics, vol. 20, Springer, Berlin, 1977.
8. R. E. Gaines, J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, *Lecture Notes in Math.*, vol. 568, Springer, Berlin, 1977.
9. A. Granas, *Sur la méthode de continuité de Poincaré*, C. R. Acad. Sci. Paris **282** no. Ser. A (1976), 983-985.
10. G. B. Gustafson, K. Schmitt, *Periodic solutions of hereditary differential systems*, Differential Equations **13** (1973), 567-587.
11. J. K. Hale, *Ordinary Differential Equations*, Wiley, New York, 1969.
12. J. Mawhin, *Periodic solutions of nonlinear functional differential equations*, J. Differential Equations **10** (1971), 240-261.
13. J. Mawhin, *Functional analysis and boundary value problems*, Studies in Ordinary Differential Equations, J. K. Hale, ed., Math Association America, Wash., DC, 1977.
14. N. Rouche, J. Mawhin, *Ordinary Differential Equations*, Stability and Periodic Solutions, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1980.
15. K. Schmitt, *Periodic solutions of delay-differential equations*, Functional Differential Equations and Approximation of Fixed Points, Heins-Otto Peitgen and Hans-Otto Walther, ed., Lecture Notes in Mathematics, vol. 730, Springer, Berlin, 1979, pp. (pp. 455-469).