EXISTENCE THEOREMS AND PERIODIC SOLUTIONS OF NEUTRAL INTEGRAL EQUATIONS

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Abstract. In this paper we consider the equations

$$x(t) = a(t) + \int_0^t D(t,s,x(s))ds + \int_t^\infty E(t,s,x(s))ds, \quad t \in R^+$$

and

$$x(t) = a(t) + \int_{-\infty}^t D(t,s,x(s))ds + \int_t^\infty E(t,s,x(s))ds, \quad t \in R$$

and discuss the existence of periodic and asymptotically periodic solutions by means of fixed point theorems.

0. Introduction

In this paper we study the equations

$$\text{(1A)} \qquad x(t) = a(t) + \int_0^t D(t,s,x(s))ds + \int_t^\infty E(t,s,x(s))ds, \quad t \in R^+$$

and

$$\text{(1B)} \qquad x(t) = a(t) + \int_{-\infty}^t D(t,s,x(s))ds + \int_t^\infty E(t,s,x(s))ds, \quad t \in R$$

where $R^+ := [0,\infty)$, $R := (-\infty,\infty)$, $a : R \to R^n$, $D : \Delta^- \times R^n \to R^n$, and $E : \Delta^+ \times R^n \to R^n$ are continuous, and where

$$\Delta^- := \{(t,s) : s \le t\} \text{ and } \Delta^+ := \{(t,s) : s \ge t\}.$$

We will suppose that

$$\text{(2A)} \qquad a(t) \text{ is bounded on } R^+$$

or

$$\text{(2B)} \qquad a(t) \text{ is bounded on } R$$

and that for any $J > 0$ there are continuous functions $D_J : \Delta^- \to R^+$ and $E_J : \Delta^+ \to R^+$ such that $|D(t,s,x)| \le D_J(t,s)$ if $(t,s) \in \Delta^-$ and $|x| \le J$, where $|\cdot|$ denotes the Euclidean norm on R^n, $|E(t,s,x)| \le E_J(t,s)$ if $(t,s) \in \Delta^+$ and $|x| \le J$,

$$\text{(3A)} \qquad \int_0^t D_J(t,s)ds + \int_t^\infty E_J(t,s)ds \text{ is bounded on } R^+$$

or

$$\text{(3B)} \qquad \int_{-\infty}^t D_J(t,s)ds + \int_t^\infty E_J(t,s)ds \text{ is bounded on } R,$$

and

$$\text{(4A)} \qquad \int_0^{\max(0,t-\tau)} D_J(t,s)ds + \int_{t+\tau}^\infty E_J(t,s)ds \to 0$$
$$\text{uniformly for } t \in R^+ \text{ as } \tau \to \infty$$

or

$$\text{(4B)} \qquad \int_{-\infty}^{t-\tau} D_J(t,s)ds + \int_{t+\tau}^\infty E_J(t,s)ds \to 0$$
$$\text{uniformly for } t \in R \text{ as } \tau \to \infty.$$

Under various conditions we prove that (1A) and (1B) have solutions and that (1B) has a periodic solution.

As some readers may find these equations unmotivated, we now show that they can represent very common variation of parameters formulae.

As a simple starting place, consider a system

$$y' = Cy + f(t)$$

where we begin with C a real constant $n \times n$ matrix having no root with 0 real part and $f : R \to R^n$ is continuous and T-periodic. We would like to actually display a periodic solution. A general solution is expressed by the variation of parameters formula as

$$y(t) = e^{Ct}y(0) + \int_0^t e^{C(t-s)} f(s)ds;$$

not only is this poorly related to (1A), but we have no way of selecting $y(0)$ to find that periodic solution. But worst of all, we can not control the size of $e^{C(t-s)}$.

Since no root of C has zero real part, there is a matrix P so that the transformation $y = Px$ yields

$$x' = P^{-1}CPx + P^{-1}f(t)$$

where

$$P^{-1}CP = \begin{pmatrix} C_1, & 0 \\ 0, & C_2 \end{pmatrix}$$

and the roots of C_1 have positive real parts, while those of C_2 have negative real parts. Taking

$$x = \begin{pmatrix} z \\ w \end{pmatrix} \text{ and } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

we have independent systems

$$z' = C_1 z + f_1(t) \text{ or } (e^{-C_1 t} z)' = e^{-C_1 t} f_1(t)$$

and

$$w' = C_2 z + f_2(t) \text{ or } (e^{-C_2 t} w)' = e^{-C_2 t} f_2(t).$$

Now we will obtain (1A) or (1B) depending on the assumptions made at this point. If we seek a solution bounded on R we integrate the first partitioned system from t to ∞ and the second from $-\infty$ to t. This yields

$$z(t) = -\int_t^\infty e^{C_1(t-s)} f_1(s)ds$$

and

$$w(t) = \int_{-\infty}^t e^{C_2(t-s)} f_2(s)ds.$$

If we now take $x = \binom{z(t)}{w(t)}$ and $y = Px$, then we obtain (1B). In this very simple case which we have just described we obtain a compact formula for the periodic solution.

Parallel work for (1A) is found prominently in the literature. In the study of instability, Coppel [4; pp. 74–75] considers

$$y' = C(t)y + f(t, y)$$

and denotes by $Y(t)$ a matrix solution of the linear part. Using projections P_1 and P_2 he gives a variation of parameters argument to get

$$\begin{aligned} x(t) = Y(t)P_1 x(t_0) &+ \int_{t_0}^t Y(t)P_1 Y^{-1}(s) f(s, x(s))ds \\ &- \int_t^\infty Y(t)P_2 Y^{-1}(s) f(s, x(s))ds. \end{aligned}$$

Independent assumptions and arguments are given to ensure that the last integral converges. The lower limit of t_0 is taken because no assumptions concerning a bounded solution are made. From the form of $x(t)$, Coppel obtains an instability result. This extends similar earlier work of Coddington and Levinson [3; pp. 330–332] for an unstable manifold when C is constant.

These are, of course, elementary examples of (1A) and (1B). Parallel considerations for

$$x' = C(t)x + \int_0^t D(t, s)x(s)ds + f(t, x(\cdot))$$

are also of interest when f is a general functional since the full linear part has a variation of parameters formula.

Just as differential equations are special cases of integral equations, so are these special cases of (1A) and (1B). Differential equations need only initial conditions $x(t_0) \in R^n$. But (1A) and (1B) will have initial functions on $[0, t_0)$, on $(-\infty, t_0)$, or on $[0, t_0) \cup (k, \infty)$, for example. For differential equations initial functions can cause discontinuities in the derivative of a solution. But for integral equations, discontinuities may occur in the solution itself. We now illustrate how that can happen.

Given a scalar functional differential equation

$$x' = f(t, x, x(t-1))$$

and an initial function $\phi : [-1, 0] \to R$, we seek a solution $x(t, \phi)$ satisfying the equation for $t \geq 0$ which is continuous on $[-1, \infty)$; in particular, $x(0, \phi) = \phi(0)$. With this experience, investigators are usually surprised to learn that the solution of an integral equation often lacks continuity at the point where it joins the initial function. And this lack of continuity is something to be noticed throughout the sequel.

We will give two elementary examples which we believe will shed considerable light on the abstract work which follows. Sometimes initial conditions are a natural part of (1B) and at other times we need a solution on all of R; in the latter case we often introduce artificial initial functions, obtain a solution, and parlay that into a solution on all of R. In any event, initial functions are an essential part of the problem and these examples let the reader see what is happening.

First, consider the scalar equation

$$x(t) = a(t) + \int_{-\infty}^{t} e^{-(t-s)} x(s) ds + \int_{t}^{\infty} e^{(t-s)} x(s) ds$$

with initial function $x(t) = 1$ for $-\infty < t < -\pi/2$ and $x(t) = 1$ for $\pi/2 < t < \infty$ so that when we substitute those functions into the equation we get

$$x(t) = a(t) + e^{-\pi/2}(e^t + e^{-t}) + \int_{-\pi/2}^{t} e^{-(t-s)}x(s)ds + \int_{t}^{\pi/2} e^{(t-s)}x(s)ds.$$

In order to solve this equation and illustrate properties, we take

$$a(t) = 1 - e^{-\pi/2}(e^t + e^{-t})$$

and are left with the equation

$$x(t) = 1 + \int_{-\pi/2}^{t} e^{-(t-s)}x(s)ds + \int_{t}^{\pi/2} e^{(t-s)}x(s)ds, -\pi/2 \le t \le \pi/2.$$

Notice that if there is a continuous solution on this interval, then it is also differentiable and

$$x'(t) = -\int_{-\pi/2}^{t} e^{-(t-s)}x(s)ds + \int_{t}^{\pi/2} e^{(t-s)}x(s)ds.$$

A further calculation yields

$$x'' + x = -1$$

with general solution

$$x = c_1 \cos t + c_2 \sin t - 1.$$

Clearly, $c_1 = x(0) + 1$ and $c_2 = x'(0)$.

To determine the constants, substitute the solution into the equation for $x(0)$ and $x'(0)$ to obtain:

$$x(0) = 1 + \int_{-\pi/2}^{0} e^{s}[(x(0)+1)\cos s + x'(0)\sin s - 1]ds$$

$$+ \int_{0}^{\pi/2} e^{-s}[(x(0)+1)\cos s + x'(0)\sin s - 1]ds$$

and

$$x'(0) = -\int_{-\pi/2}^{0} e^{s}[(x(0)+1)\cos s + x'(0)\sin s - 1]ds$$

$$+\int_0^{\pi/2} e^{-s}[(x(0)+1)\cos s + x'(0)\sin s - 1]ds.$$

These represent two equations in two unknowns, $x(0)$ and $x'(0)$. The integrals can be simplified and evaluated. In this case we do get a unique solution

$$x(0) = -3,$$

$$x'(0) = 0,$$

and

$$x(t) = -2\cos t - 1.$$

Thus, $x(-\pi/2)$ is not 1, so the solution does not agree with the initial function. It is discontinuous.

We now give an example in which the solution does not match up with its initial function. Moreover, for certain choices of the functions there is no solution.

Given initial functions for (1B) on $(-\infty, a)$ and (b, ∞) (with $a < b$), we obtain an equation which we will greatly simplify as

$$x(t) = 1 + \int_a^t r(s)x(s)ds + \int_t^b h(s)x(s)ds, a \le t \le b.$$

At this point it is impossible to tell what $x(a)$ or $x(b)$ is, regardless of the initial function. It will be very instructive to solve this simple equation. We have

$$x' = [r(t) - h(t)]x$$

so that

$$x(t) = x(a)\exp\int_a^t [r(s) - h(s)]ds.$$

But what is $x(a)$? Substitute the solution into the original equation at $t = a$ and obtain

$$x(a) = 1 + \int_a^b h(s)x(a)(\exp\int_a^s [r(u) - h(u)]du)ds)$$

or

$$x(a) = 1/[1 - \int_a^b h(s)(\exp \int_a^s [r(u) - h(u)]du)ds].$$

We now know three things:

i. $x(a)$ is uniquely determined.

ii. A solution exists if and only if

$$\int_a^b h(s)(\exp \int_a^s [r(u) - h(u)]du)ds \neq 1.$$

iii. For a general equation like (1B) with specified initial functions, we will never know any value of the solution.

We will expect to prove existence, uniqueness, asymptotic behavior, and qualitative properties such as periodicity.

One more item of interest concerns standard inequalities. When we write the standard variation of parameters formula, we commonly take norms and use Gronwall's inequality to get bounds on solutions. Under fairly severe conditions it is also possible to get Gronwall-type inequalities for neutral equations.

1. Existence of solutions of (1A)

For any $t_0 \in R^+$ and any bounded continuous initial function $\phi : [0, t_0) \to R^n$ let $x(t, t_0, \phi)$ denote a solution of Equation (1A) which agrees with ϕ on $[0, t_0)$ and satisfies (1A) on $[t_0, \infty)$. Here, we understand that for $t_0 = 0$ a continuous function $x(t)$ is a solution of Equation (1A) if $x(t)$ satisfies (1A) on R^+. For $t_0 > 0$, we are prepared to accept a discontinuity in $x(t, t_0, \phi)$ at t_0.

1a. Existence by a contraction mapping

Suppose that for any $J > 0$ there are continuous functions $L_J^- : \Delta^- \to R^+$ and

$L_J^+ : \Delta^+ \to R^+$ such that

$$|D(t,s,x) - D(t,s,y)| \le L_J^-(t,s)|x-y| \quad \text{if } (t,s) \in \Delta^-,\ |x|,\ |y| \le J \tag{5}$$

and

$$|E(t,s,x) - E(t,s,y)| \le L_J^+(t,s)|x-y| \quad \text{if } (t,s) \in \Delta^+,\ |x|,\ |y| \le J. \tag{6}$$

Then we have the following result.

THEOREM 1. In addition to (2A), (3A), (4A), (5) and (6), suppose that

$$\lambda := \sup\{\lambda_J : J > 0\} < 1 \tag{7}$$

holds, where

$$\lambda_J := \sup\{\int_0^t L_J^-(t,s)ds + \int_t^\infty L_J^+(t,s)ds : t \in R^+\}.$$

Then for any $t_0 \in R^+$ and any bounded continuous function $\phi : [0,t_0) \to R^n$, Equation (1A) has a unique R^+-bounded solution which agrees with ϕ on $[0,t_0)$ and satisfies (1A) on $[t_0,\infty)$.

PROOF. For any $t_0 \in R^+$, let $C(t_0)$ be the set of bounded continuous functions $\xi : [t_0,\infty) \to R^n$. For any $\xi \in C(t_0)$, define $\|\xi\|_{t_0}$ by

$$\|\xi\|_{t_0} := \sup\{|\xi(t)| : t \ge t_0\}.$$

Then clearly $\|\cdot\|_{t_0}$ is a norm on $C(t_0)$ and $(C(t_0), \|\cdot\|_{t_0})$ is a Banach space. For any $\xi \in C(t_0)$ define a map H on $C(t_0)$ by

$$(H\xi)(t) := a(t) + \int_0^{t_0} D(t,s,\phi(s))ds + \int_{t_0}^t D(t,s,\xi(s))ds + \int_t^\infty E(t,s,\xi(s))ds,\ t \ge t_0.$$

Then, from (2A), (3A) and (4A), it is easy to see that H maps $C(t_0)$ into $C(t_0)$. Moreover, for any $\xi_i \in C(t_0)$ with $\|\xi_i\|_{t_0} \le J$ $(i = 1, 2)$ for some $J > 0$ we have

$$
\begin{aligned}
&|(H\xi_1)(t) - (H\xi_2)(t)| \\
&\quad \le \int_{t_0}^{t} L_J^-(t,s)|\xi_1(s) - \xi_2(s)|ds + \int_t^\infty L_J^+(t,s)|\xi_1(s) - \xi_2(s)|ds \\
&\quad \le \lambda_J \|\xi_1 - \xi_2\|_{t_0}, \quad t \ge t_0,
\end{aligned}
$$

which, together with (7), yields $\|H\xi_1 - H\xi_2\|_{t_0} \le \lambda \|\xi_1 - \xi_2\|_{t_0}$. Thus, $H : C(t_0) \to C(t_0)$ is a contraction mapping and so it has a unique fixed point $\eta \in C(t_0)$. Clearly, the function x defined by

$$
x(t) := \begin{cases} \phi(t), & 0 \le t < t_0 \\ \eta(t), & t \ge t_0 \end{cases}
$$

is a unique R^+-bounded solution of Equation (1A) which agrees with ϕ on $[0, t_0)$ and satisfies (1A) on $[t_0, \infty)$.

EXAMPLE 1. Consider the scalar linear equation

$$
x(t) = a(t) + \alpha \int_0^t e^{s-t} \cos s\, x(s) ds + \int_t^\infty e^{t-s} \sin s\, x(s) ds, \; t \in R^+, \tag{8}
$$

where $a : R^+ \to R$ is a bounded continuous function, and α and β are constants with $|\alpha| + |\beta| < 1$. Equation (8) is obtained from Equation (1A) taking $n = 1$, $D(t, s, x) = \alpha\, x\, e^{s-t} \cos s$, and $E(t, s, x) = \beta\, x\, e^{t-s} \sin s$. For any $J > 0$ we can take the following functions as D_J, E_J, L_J^- and L_J^+:

$$
D_J(t,s) := |\alpha| J\, e^{s-t}, \; (t,s) \in \Delta^-,
$$

$$
E_J(t,s) := |\beta| J\, e^{t-s}, \; (t,s) \in \Delta^+,
$$

$$
L_J^-(t,s) := |\alpha| e^{s-t}, \; (t,s) \in \Delta^-,
$$

and

$$
L_J^+(t,s) := |\beta| e^{t-s}, \; (t,s) \in \Delta^+.
$$

It is easy to see that $a(t)$ and these functions satisfy (2A), (3A), (4A), (5) and (6). Moreover, (7) holds with $\lambda = |\alpha| + |\beta|$. Thus, by Theorem 1, for any $t_0 \in R^+$ and any bounded continuous function $\phi : [0, t_0) \to R$, Equation (8) has a unique R^+-bounded solution $x(t, t_0, \phi)$.

1b. Existence by Schauder's second theorem

Conditions (5)–(7) of Theorem 1 are very strong. We now discuss the existence of R^+-bounded solutions without those assumptions.

For a given $t_0 > 0$, let $\phi : [0, t_0) \to R^n$ be a given bounded continuous function. Let K be the set of positive integers. For any $k \in K$ with $k > t_0$, let $(C_k^+, \|\cdot\|)$ be the Banach space of continuous functions $\xi : [t_0, k] \to R^n$ with the supremum norm $\|\cdot\|$. For any $J > 0$ with $J \geq \sup\{|\phi(t)| : 0 \leq t < t_0\}$, let $C_k^+(J) := \{\xi \in C_k^+ : \|\xi\| \leq J\}$, and let $Y_k^+(J)$ be the space of functions $\eta : R^+ \to R^n$ such that $\eta(t) = \phi(t)$ for $0 \leq t < t_0$, $\eta(t)$ is continuous on $[t_0, \infty)$ except at some m, and $\|\eta\|_0 \leq J$, where $m \in K$, $m \geq k$, and $\|\eta\|_0 := \sup\{|\eta(t)| : t \in R^+\}$. Then $C_k^+(J)$ is convex, and any $\xi \in C_k^+(J)$ can be extended on R^+ by taking an $\eta \in Y_k^+(J)$ such that $\eta(t) = \xi(t)$ for $t_0 \leq t \leq k$. For any $\xi \in C_k^+(J)$, let $\eta \in Y_k^+(J)$ be an extension of ξ. We define a map $H = H_{k,\eta}$ on $C_k^+(J)$ by $\xi \in C_k^+(J)$ implies that

$$(H\xi)(t) := a(t) + \int_0^t D(t, s, \eta(s))ds + \int_t^\infty E(t, s, \eta(s))ds, \ \ t_0 \leq t \leq k. \tag{9}$$

then we have the following result.

LEMMA 1. Under assumption (4A), for any $k \in K$ and $J > 0$ with $k > t_0$ and $J \geq \sup\{|\phi(t)| : 0 \leq t < t_0\}$, there is a continuous increasing function $\delta = \delta_{k,J}(\epsilon) : (0, \infty) \to (0, \infty)$ such that

$$|(H\xi)(t_1) - (H\xi)(t_2)| < \epsilon \text{ if } \xi \in C_k^+(J), \quad \eta \in Y_k^+(J), \tag{10}$$

$$t_0 \leq t_1 < t_2 \leq k \text{ and } t_2 < t_1 + \delta,$$

where η is any extension of ξ.

PROOF. First, it is clear that there is a continuous increasing function $\delta_1 = \delta_1(\epsilon) : (0, \infty) \to (0, \infty)$ with

$$|a(t_1) - a(t_2)| < \frac{\epsilon}{3} \text{ if } t_0 \le t_1 < t_2 \le k \text{ and } t_2 < t_1 + \delta_1. \tag{11}$$

Next we prove that there is a continuous increasing function $\delta^- = \delta^-(\epsilon) : (0, \infty) \to (0, \infty)$ such that

$$\begin{gathered} |(H^-\xi)(t_1) - (H^-\xi)(t_2)| < \frac{\epsilon}{3} \text{ if } \xi \in C_k^+(J), \\ \eta \in Y_k^+(J), \quad t_0 \le t_1 < t_2 \le k \text{ and } t_2 < t_1 + \delta^- \end{gathered} \tag{12}$$

holds, where η is any extension of ξ and H^- on $C_k^+(J)$ is defined by

$$(H^-\xi)(t) := \int_0^t D(t, s, \eta(s))ds, \qquad t_0 \le t \le k.$$

From (4A), for any $\epsilon > 0$ there is a $\tau \ge k$ such that

$$\int_0^{\max(0, t-\tau)} D_J(t, s)ds + \int_{t+\tau}^{\infty} E_J(t, s)ds < \frac{\epsilon}{12} \text{ if } t \in R^+. \tag{13}$$

For any $\xi \in C_k^+(J)$, any extension $\eta \in Y_k^+(J)$ of ξ, and for t_1 and t_2 with $t_0 \le t_1 < t_2 \le k$, we have

$$\begin{aligned} &|(H^-\xi)(t_1) - (H^-\xi)(t_2)| \\ &\qquad = \left| \int_0^{t_1} D(t_1, s, \eta(s))ds - \int_0^{t_2} D(t_2, s, \eta(s))ds \right| \\ &\qquad \le \int_0^{t_1} |D(t_1, s, \eta(s)) - D(t_2, s, \eta(s))|ds + \int_{t_1}^{t_2} D_J(t_2, s)ds. \end{aligned} \tag{14}$$

Since $D(t, s, x)$ is uniformly continuous on $U_1 := \{(t, s, x) : t_0 \le t \le k, \ 0 \le s \le t$ and $|x| \le J\}$, for the ϵ there is a $\delta_2 > 0$ such that $|D(t_1, s, x) - D(t_2, s, x)| < \epsilon/6\tau$ if

$(t_1,s,x),(t_2,s,x)\in U_1$ and $|t_1-t_2|<\delta_2$. From this, if $t_0\le t_1<t_2\le k$ and $t_2<t_1+\delta_2$, then we obtain

$$\int_0^{t_1}|D(t_1,s,\eta(s))-D(t_2,s,\eta(s))|ds<\frac{\epsilon}{6}\,. \tag{15}$$

Moreover, for the ϵ there is a $\delta_3>0$ such that

$$\int_{t_1}^{t_2}D_J(t_2,s)ds<\frac{\epsilon}{6}\ \text{if}\ t_0\le t_1<t_2\le k\ \text{and}\ t_2<t_1+\delta_3. \tag{16}$$

Thus, from (14)–(16) we have (12) with $\delta^-=\delta_4:=\min(\delta_2,\delta_3)$, since we may assume that $\delta_2(\epsilon)$ and $\delta_3(\epsilon)$ are continuous and increasing.

Finally, we prove that there is a continuous increasing function $\delta^+=\delta^+(\epsilon):(0,\infty)\to(0,\infty)$ such that

$$\begin{aligned}&|(H^+\xi)(t_1)-(H^+\xi)(t_2)|<\frac{\epsilon}{3}\ \text{if}\ \xi\in C_k^+(J),\\ &\eta\in Y_k^+(J),\quad t_0\le t_1<t_2\le k\ \text{and}\ t_2<t_1+\delta^+,\end{aligned} \tag{17}$$

where η is any extension of ξ, and H^+ on $C_k^+(J)$ is defined by

$$(H^+\xi)(t):=\int_t^\infty E(t,s,\eta(s))ds,\qquad t_0\le t\le k.$$

Let $\tau\ge k$ be a number in (13). For any $\xi\in C_k^+(J)$, any extension $\eta\in D_k^+(J)$ of ξ, t_1 and t_2 with $t_0\le t_1<t_2\le k$ we have

$$\begin{aligned}&|(H^+\xi)(t_1)-(H^+\xi)(t_2)|\\ &\qquad=\left|\int_{t_1}^\infty E(t_1,s,\eta(s))ds-\int_{t_2}^\infty E(t_2,s,\eta(s))ds\right|\\ &\qquad\le\int_{t_2}^{t_2+\tau}|E(t_1,s,\eta(s))-E(t_2,s,\eta(s))|ds\\ &\qquad\quad+\int_{t_2+\tau}^\infty E_J(t_1,s)ds+\int_{t_2+\tau}^\infty E_J(t_2,s)ds+\int_{t_1}^{t_2}E_J(t_1,s)ds\\ &\qquad<\int_{t_2}^{t_2+\tau}|E(t_1,s,\eta(s))-E(t_2,s,\eta(s))|ds+\int_{t_1}^{t_2}E_J(t_1,s)ds+\frac{\epsilon}{6}\,.\end{aligned} \tag{18}$$

Since $E(t,s,x)$ is uniformly continuous on $U_2 := \{(t,s,x) : t_0 \le t \le k,\ t \le s \le t+\tau+1, |x| \le J\}$, for the ϵ there is a δ_5 such that $0 < \delta_5 < 1$ and $|E(t_1,s,x) - E(t_2,s,x)| < \epsilon/12\tau$ if $(t_1,s,x),(t_2,s,x) \in U_2$ and $|t_1 - t_2| < \delta_5$. From this, if $t_0 \le t_1 < t_2 \le k$ and $t_2 < t_1 + \delta_5$, then we obtain

$$\int_{t_2}^{t_2+\tau} |E(t_1,s,\eta(s)) - E(t_2,s,\eta(s))| ds < \frac{\epsilon}{12}. \tag{19}$$

Moreover for the ϵ there is a $\delta_6 > 0$ such that

$$\int_{t_1}^{t_2} E_J(t_1,s) ds < \frac{\epsilon}{12} \text{ if } t_0 \le t_1 < t_2 \le k \text{ and } t_2 < t_1 + \delta_6. \tag{20}$$

Thus, from (18)–(20), we have (17) with $\delta^+ = \delta_7 := \min(\delta_5,\delta_6)$, since we may assume that $\delta_5(\epsilon)$ and $\delta_6(\epsilon)$ are continuous and increasing.

Hence, from (11), (12), and (17) we can easily conclude that (10) holds for $\delta := \min(\delta_1,\delta^-,\delta^+)$.

This lemma enables us to prove the following result.

THEOREM 2. In addition to (4A), let

$$|a(t)| + \int_0^t D_J(t,s) ds + \int_t^\infty E_J(t,s) ds \le J, \qquad t_0 \le t \le k \tag{21}$$

hold for some $k \in K$ and $J > 0$ with $k > t_0$ and $J \ge \sup\{|\phi(t)| : 0 \le t < t_0\}$. Then for any $\xi_0 \in C(t_0)$ with $\|\xi_0\|_{t_0} \le J$ the equation

$$x(t) = a(t) + \int_0^t D(t,s,x(s)) ds + \int_t^\infty E(t,s,x(s)) ds, \quad t_0 \le t \le k \tag{22}$$

has a solution $x(t)$ on $[t_0,k]$ such that $\|x\|_0 \le J$, $x(t) = \phi(t)$ for $0 \le t < t_0$, and $x(t) = \xi_0(t)$ for $t > k$.

PROOF. Let $\xi_0 \in C(t_0)$ be any fixed function with $\|\xi_0\|_{t_0} \le J$. Then any $\xi \in C_k^+(J)$ can be extended on R^+ by defining $\xi(t) = \phi(t)$ if $0 \le t < t_0$, and $\xi(t) = \xi_0(t)$ if $t > k$. Then the extended function of ξ is an element of $Y_k^+(J)$, and we denote it by η. Let H be a map on $C_k^+(J)$ defined by (9). Then from Lemma 1, (4A), and (21), the set $\{H\xi : \xi \in C_k^+(J)\}$ is equicontinuous, and it is easy to see that H maps $C_k^+(J)$ into $C_k^+(J)$ continuously. Thus, by Schauder's second theorem (cf. Smart [6; p. 25]), H has a fixed point, which yields a solution $x(t)$ of Equation (22) such that $\|x\|_0 \leq J$, $x(t) = \phi(t)$ for $0 \leq t < t_0$, and $x(t) = \xi_0(t)$ for $t > k$.

By using Lemma 1 and Theorem 2, we obtain the following result.

THEOREM 3. In addition to (4A), let

$$|a(t)| + \int_0^t D_J(t,s)ds + \int_t^\infty E_J(t,s)ds \leq J, \qquad t \geq t_0 \tag{23}$$

hold for some $J > 0$ with $J \geq \sup\{|\phi(t)| : 0 \leq t < t_0\}$. Then Equation (1A) has a solution $y(t)$ such that $\|y\|_0 \leq J$, $y(t) = \phi(t)$ for $0 \leq t < t_0$, and $y(t)$ satisfies (1A) on $[t_0, \infty)$.

PROOF. Let $k > t_0$ be any integer, and let ξ_0 be a function with $\xi_0(t) \equiv 0$ on R^+. Then Theorem 2 implies that Equation (22) has a solution $x_k(t)$ such that $\|x_k\|_0 \leq J$, $x_k(t) = \phi(t)$ for $0 \leq t < t_0$, and $x_k(t) = 0$ for $t > k$. Thus, we have a sequence of functions $\{x_k(t)\}$. Let $m > t_0$ be any fixed integer, and for any integer $k > t_0$ let $\xi_k(t)$ be the restriction of $x_k(t)$ on $[t_0, m]$. Then Lemma 1 implies that the set $\{\xi_k(t) : k \geq m\}$ is equicontinuous on $[t_0, m]$. Hence, the sequence $\{x_k(t)\}$ contains a subsequence, say $\{x_k^1(t)\}$, which converges uniformly on $[t_0, t_0 + 1]$. The sequence $\{x_k^2(t)\}$ also has a subsequence converging uniformly on $[t_0, t_0 + 2]$. If we continue in this way we obtain a sequence of sequences $\{x_k^m(t)\}$, $m = 1, 2, \ldots$, each of which is a subsequence of all the preceding ones, such that for each integer $m > t_0$, $x_k^m(t)$ converges uniformly on $[t_0, t_0 + m]$. Consider the sequence of functions $y_k(t) = x_k^k(t)$, where $k > t_0$ is an integer. Then the sequence $\{y_k(t)\}$ is a subsequence of $\{x_k(t)\}$ and is, in fact, a subsequence of each of the sequences $\{x_k^m(t)\}$, for k large. If we define a function $y(t)$ on R^+ by $y(t) = \phi(t)$ for $0 \leq t < t_0$, and $y(t) = \lim_{k\to\infty} y_k(t)$ for $t \geq t_0$, then for any integer $m > t_0$, $\{y_k(t)\}$ converges to $y(t)$ uniformly on $[t_0, m]$ as $k \to \infty$.

Now we show that $y(t)$ satisfies (1A) on $[t_0, \infty)$. For any $t \geq t_0$, let m be an integer with $|t| \leq m$. From the definition of $y_k(t)$, we have $y_k(t) = x_\kappa(t)$ for some integer $\kappa \geq k$. Thus for any integer $k \geq m$ we obtain

$$y_k(t) = a(t) + \int_0^t D(t, s, y_k(s))ds + \int_t^\infty E(t, s, y_k(s))ds, \qquad t_0 \leq t \leq \kappa, \tag{24}$$

where $\kappa = \kappa(k) \in K$ and $y_k(s) = 0$ for $s > \kappa$. Clearly we have

$$\lim_{k\to\infty} \int_0^t D(t, s, y_k(s))ds = \int_0^t D(t, s, y(s))ds.$$

Next we prove that

$$\lim_{k\to\infty} \int_t^\infty E(t, s, y_k(s))ds = \int_t^\infty E(t, s, y(s))ds.$$

From (4A), for any $\epsilon > 0$ there is a $\tau > 0$ with $\int\limits_{t+\tau}^{\infty} E_J(t, s)ds < \epsilon$ if $t \in R^+$. From this we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \left| \int_t^\infty (E(t, s, y_k(s)) - E(t, s, y(s))ds \right| \\
&\qquad \leq \limsup_{k\to\infty} \left| \int_t^{t+\tau} (E(t, s, y_k(s)) - E(t, s, y(s))ds \right| + 2 \int_{t+\tau}^\infty E_J(t, s)ds < 2\epsilon,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \int_t^\infty E(t, s, y_k(s))ds = \int_t^\infty E(t, s, y(s))ds.$$

Thus, letting $k \to \infty$ in (24), we have

$$y(t) = a(t) + \int_0^t D(t, s, y(s))ds + \int_t^\infty E(t, s, y(s))ds. \tag{25}$$

Since $t \geq t_0$ is arbitrary, (25) shows that $y(t)$ is a solution of Equation (1A) such that $\|y\|_0 \leq J$, $y(t) = \phi(t)$ for $0 < t < t_0$, and $y(t)$ satisfies (1A) on $[t_0, \infty)$.

The following example illustrates the result. In work we cited earlier by Coppel and by Coddington and Levinson, a Lipschitz condition was assumed on the functions in the differential equations. Our result does not require that.

EXAMPLE 2. Consider the scalar nonlinear equation

$$x(t) = \arctan t + \int_0^t e^{s-t}\sqrt{|x(s)|}ds + \int_t^\infty e^{t-s}\sqrt{|x(s)|}ds, \quad t \in R^+. \tag{26}$$

This equation is obtained from (1A) taking $n = 1$, $a(t) = \arctan t$, $D(t, s, x) = e^{s-t}\sqrt{|x|}$, and $E(t, s, x) = e^{t-s}\sqrt{|x|}$. For any $J > 0$ we can take

$$D_J(t, s) = e^{s-t}\sqrt{J}, \qquad (t, s) \in \Delta^-$$

and

$$E_J(t, s) = e^{t-s}\sqrt{J}, \qquad (t, s) \in \Delta^+.$$

For any $t_0 \in R^+$, and any bounded continuous function $\phi : [0, t_0) \to R$ with $t_0 > 0$, let

$$J \geq \max\left[\sup\{|\phi(t)| : 0 \leq t < t_0\}, \quad 2\sqrt{J} + \frac{\pi}{2}\right].$$

Then it is easy to see that the conditions (4A) and (23) are satisfied for that J. Thus, by Theorem 3, Equation (26) has a solution $x(t)$ such that $\|x\|_0 \leq J$, $x(t) = \phi(t)$ for $0 \leq t < t_0$, and $x(t)$ satisfies (1A) on $[t_0, \infty)$.

2. Existence of solutions of (1B)

We now show that existence of solutions of (1B) can be obtained by methods similar to those in Section 1. Thus, for any $t_0 \in R$ and any bounded continuous initial function $\phi : (-\infty, t_0) \to R^n$, let $x(t, t_0, \phi)$ again denote a solution of Equation (1B) which agrees with ϕ on $(-\infty, t_0)$ and satisfies (1B) on $[t_0, \infty)$.

2a. Existence by a contraction mapping

Let

$$\lambda_J := \sup\left\{\int_{-\infty}^{t} L_J^-(t,s)ds + \int_t^\infty L_J^+(t,s)ds : t \in R\right\}.$$

Then we have the following result.

THEOREM 4. Suppose that (2A), (3B), (4B), (5)–(7) hold. Then for any $t_0 \in R$ and any bounded continuous function $\phi : (-\infty, t_0) \to R^n$, Equation (1B) has a unique R-bounded solution which agrees with ϕ on $(-\infty, t_0)$ and satisfies (1B) on $[t_0, \infty)$. Moreover, if (2B) holds, then Equation (1B) has a unique R-bounded solution which satisfies (1B) on R.

The first part of the theorem can be proved by a proof similar to that given for Theorem 1 by defining a map H on $C(t_0)$ by

$$\begin{aligned}(H\xi)(t) := a(t) + \int_{-\infty}^{t_0} D(t,s,\phi(s))ds + \int_{t_0}^{t} D(t,s,\xi(s))ds \\ + \int_t^\infty E(t,s,\xi(s))ds, \qquad t \geq t_0.\end{aligned}$$

The second part can be proved by defining H on C by

$$(H\xi)(t) := a(t) + \int_{-\infty}^{t} D(t,s,\xi(s))ds + \int_t^\infty E(t,s,\xi(s))ds, \qquad t \in R,$$

where $(C, \|\cdot\|)$ is the Banach space of bounded continuous functions $\xi : R \to R^n$ with the supremum norm $\|\cdot\|$. The details will not be given.

EXAMPLE 3. Corresponding to Equation (8) in Example 1, consider the scalar linear equation

$$x(t) = a(t) + \alpha \int_{-\infty}^{t} e^{s-t} \cos s\, x(s)ds + \beta \int_t^\infty e^{t-s} \sin s\, x(s)ds, \quad t \in R, \tag{27}$$

where $a : R \to R$ is a bounded continuous function, and α and β are constants with $|\alpha| + |\beta| < 1$. It is easy to see that the assumptions of Theorem 4 are satisfied. Thus, for any $t_0 \in R$ and any bounded continuous function $\phi : (-\infty, t_0) \to R^n$, Equation (27) has a unique R-bounded solution which agrees with ϕ on $(-\infty, t_0)$ and satisfies (1B) on $[t_0, \infty)$. Moreover, Equation (27) has a unique R-bounded solution which satisfies (27) on R.

REMARK. In Section 3 we will consider periodic solutions. For both differential and integral equations it is possible to give periodicity conditions on the functions involved so that whenever $x(t)$ is a solution, so is $x(t+T)$, where T is the period. Thus, when such conditions are in place, then $x(t)$ and $x(t+T)$ are both bounded solutions; if we know that there is a unique bounded solution on R, then we conclude that $x(t) = x(t+T)$. Thus, in Section 3 we parlay Theorem 3 into the existence of a periodic solution.

2b. Existence by Schauder's second theorem

For a given $t_0 \in R$, let $\phi : (-\infty, t_0) \to R^n$ be a given bounded continuous function. For any $k \in K$ with $k > t_0$, let $(C_k^+, \|\cdot\|)$ be the Banach space of continuous functions $\xi : [t_0, k] \to R^n$ with the supremum norm $\|\cdot\|$. For any $J > 0$ with $J \geq \sup\{|\phi(t)| : t < t_0\}$, let $C_k^+(J) := \{\xi \in C_k^+ : \|\xi\| \leq J\}$, and let $Y_k^+(J)$ be the space of functions $\eta : R \to R^n$ satisfying $\eta(t) = \phi(t)$ for $t < t_0$, $\eta(t)$ is continuous on $[t_0, \infty)$ except at some m, and $|\eta(t)| \leq J$ for $t \in R$, where $m \in K$ and $m \geq k$. Then $C_k^+(J)$ is convex, and any $\xi \in C_k^+(J)$ can be extended on R by taking an $\eta \in Y_k^+(J)$ such that $\eta(t) = \xi(t)$ for $t_0 \leq t \leq k$. For any $\xi \in C_k^+(J)$, let $\eta \in Y_k^+(J)$ be an extension of ξ. We define a map $H = H_{k,\eta}$ on $C_k^+(J)$ by $\xi \in C_k^+(J)$ implies that

$$(H\xi)(t) := a(t) + \int_{-\infty}^{t} D(t, s, \eta(s))ds + \int_{t}^{\infty} E(t, s, \eta(s))ds, \quad t_0 \leq t \leq k.$$

Corresponding to Lemma 1, we have the following result.

LEMMA 2. Under assumption (4B), for any $k \in K$ and $J > 0$ with $k > t_0$ and $J \geq \sup\{|\phi(t)| : t < t_0\}$, there is a continuous increasing function $\delta = \delta_{k,J}(\epsilon) : (0, \infty) \to (0, \infty)$ such that

$$|(H\xi)(t_1) - (H\xi)(t_2)| < \epsilon \text{ if } \xi \in C_k^+(J), \quad \eta \in Y_k^+(J),$$

$$t_0 \leq t_1 < t_2 \leq k \text{ and } t_2 < t_1 + \delta,$$

where η is any extension of ξ.

Since we can prove this lemma by a similar method to the one of Lemma 1, we omit the proof.

The next two results are parallel to Theorems 2 and 3.

THEOREM 5. In addition to (4B), let

$$|a(t)| + \int_{-\infty}^{t} D_J(t,s)ds + \int_{t}^{\infty} E_J(t,s)ds \le J, \quad t_0 \le t \le k$$

hold for some $k \in K$ and $J > 0$ with $k > t_0$ and $J \ge \sup\{|\phi(t)| : t < t_0\}$. Then for any $\xi_0 \in C(t_0)$ with $\|\xi_0\|_{t_0} \le J$ the equation

$$x(t) = a(t) + \int_{-\infty}^{t} D(t,s,x(s))ds + \int_{t}^{\infty} E(t,s,x(s))ds, \quad t_0 \le t \le k$$

has a solution $x(t)$ satisfying the equation on $[t_0, k]$ such that $|x(t)| \le J$ for $t \in R$, $x(t) = \phi(t)$ for $t < t_0$, and $x(t) = \xi_0(t)$ for $t > k$.

THEOREM 6. In addition to (4B), let the inequality

$$|a(t)| + \int_{-\infty}^{t} D_J(t,s)ds + \int_{t}^{\infty} E_J(t,s)ds \le J, \quad t \ge t_0$$

hold for some $J > 0$ with $J \ge \sup\{\phi(t)| : t < t_0\}$. Then Equation (1B) has a solution $x(t)$ such that $|x(t)| \le J$ for $t \in R$, $x(t) = \phi(t)$ for $t < t_0$, and $x(t)$ satisfies (1B) on $[t_0, \infty)$.

Since the proofs of these two theorems are similar to the proofs of Theorems 2 and 3, respectively, we omit them.

Next, we discuss the existence of R-bounded solutions of Equation (1B) which satisfy (1B) on R by employing Schauder's second theorem without assumptions (5) and (6).

For any $k \in K$, let $(C_k, \|\cdot\|)$ be the Banach space of continuous functions $\xi : [-k, k] \to R^n$ with the supremum norm $\|\cdot\|$. For any $J > 0$, let $C_k(J) := \{\xi \in C_k : \|\xi\| \le J\}$, and let $Y_k(J)$ be the space of functions $\eta : R \to R^n$ such that $\eta(t)$ is continuous on R except at some $\pm m$, and $|\eta(t)| \le J$ for $t \in R$, where $m \in K$ and $m \ge k$. Then $C_k(J)$ is convex and any $\xi \in C_k(J)$ can be extended on R by taking an $\eta \in Y_k(J)$ with $\eta(t) = \xi(t)$ on $[-k, k]$. We define a map $H = H_{k,\eta}$ on $C_k(J)$ by $\xi \in C_k(J)$ implies that

$$(H\xi)(t) := a(t) + \int_{-\infty}^{t} D(t, s, \eta(s))ds + \int_{t}^{\infty} E(t, s, \eta(s))ds, \quad -k \leq t \leq k, \tag{28}$$

where $\eta \in Y_k(J)$ is an extension of ξ. Then, corresponding to Lemma 1, we have the following result which we state without proof.

LEMMA 3. Under assumption (4B), for any $k \in K$ and $J > 0$, there is a continuous increasing function $\delta = \delta_{k,J} : (0, \infty) \to (0, \infty)$ such that

$$|(H\xi)(t_1) - (H\xi)(t_2)| < \epsilon \text{ if } \xi \in C_k(J), \quad \eta \in Y_k(J),$$

$$-k \leq t_1 < t_2 \leq k \text{ and } t_2 < t_1 + \delta,$$

where η is any extension of ξ.

THEOREM 7. In addition to (4B), let the inequality

$$|a(t)| + \int_{-\infty}^{t} D_J(t, s)ds + \int_{t}^{\infty} E_J(t, s)ds \leq J, \quad -k \leq t \leq k, \tag{29}$$

hold for some $k \in N$ and $J > 0$. Then for any continuous function $\xi_0 : R \to R^n$ with $|\xi_0(t)| \leq J$ for $t \in R$ the equation

$$x(t) = a(t) + \int_{-\infty}^{t} D(t, s, x(s))ds + \int_{t}^{\infty} E(t, s, x(s))ds, \quad -k \leq t \leq k, \tag{30}$$

has a solution $x(t)$ on $[-k, k]$ such that $|x(t)| \leq J$ for $t \in R$, and $x(t) = \xi_0(t)$ if $|t| > k$.

PROOF. Let $\xi_0 : R \to R^n$ be any fixed function with $|\xi_0(t)| \leq J$ for $t \in R$. Then any $\xi \in C_k(J)$ can be extended on R by defining $\xi(t) = \xi_0(t)$ if $|t| > k$. Then the extended function of ξ is an element of $Y_k(J)$, and we denote it by ξ again for simplicity. Let H be a map on $C_k(J)$ defined by (28). Then, from Lemma 3 and (29), H maps $C_k(J)$ into $C_k(J)$ continuously, and the set $\{H\xi : \xi \in C_k(J)\}$ is equicontinuous. Thus, by Schauder's second theorem, H has a fixed point, which yields a solution $x(t)$ of Equation (30) such that $|x(t)| \le J$ for $t \in R$, and $x(t) = \xi_0(t)$ if $|t| > k$.

By using Lemma 3 and Theorem 7, we obtain the following result.

THEOREM 8. In addition to (4B), let

$$|a(t)| + \int_{-\infty}^{t} D_J(t,s)ds + \int_{t}^{\infty} E_J(t,s)ds \le J, \quad t \in R, \tag{31}$$

hold for some $J > 0$. Then Equation (1B) has a solution $y(t)$ which satisfies (1B) on R, and $|y(t)| \le J$ for $t \in R$.

PROOF. Let $k \in K$ be any integer, and let $\phi : R \to R^n$ be a function with $\phi(t) = 0$ on R. Then Theorem 7 implies that Equation (30) has a solution $x_k(t)$ such that $\|x_k\| \le J$, and $x_k(t) = 0$ if $|t| > k$. Thus we have a sequence of functions $\{x_k(t)\}$. Let $m \in K$ be any fixed integer, and for any $k \in K$ let $\xi_k(t)$ be the restriction of $x_k(t)$ on $[-m, m]$. Then, Lemma 3 implies that the set $\{\xi_k(t) : k \ge m\}$ is equicontinuous on $[-m, m]$. By an argument similar to the one in the proof of Theorem 3, it is easily seen that the sequence $\{x_k(t)\}$ contains a subsequence $\{y_j(t)\}$ such that for any integer m, $\{y_j(t)\}$ converges uniformly on $[-m, m]$ as $j \to \infty$. We define a function $y(t)$ on R by $y(t) = \lim_{j\to\infty} y_j(t)$.

Now we show that $y(t)$ is a solution of Equation (1B) and it satisfies (1B) on R. For any $t \in R$, let $m \in K$ be an integer with $|t| \le m$. Since $\{y_j(t)\}$ is a subsequence of $\{x_k(t)\}$, we have $y_j(t) = x_k(t)$ for some $k \in K$ with $k \ge j$. Thus, for any $j \in K$ with $j \ge m$ we obtain

$$y_j(t) = a(t) + \int_{-\infty}^{t} D(t,s,y_j(s))ds + \int_{t}^{\infty} E(t,s,y_j(s))ds, \quad -k \le t \le k, \tag{32}$$

where $k = k(j) \in K$ and $y_j(s) = 0$ if $|s| > k$. By an argument similar to the one in the proof of Theorem 3, it is easily seen that

$$\begin{aligned}\lim_{j\to\infty} &\left(\int_{-\infty}^{t} D(t,s,y_j(s))ds + \int_{t}^{\infty} E(t,s,y_j(s))ds \right) \\ &= \int_{-\infty}^{t} D(t,s,y(s))ds + \int_{t}^{\infty} E(t,s,y(s))ds.\end{aligned}$$

Thus, letting $j \to \infty$ in (32) we have

$$y(t) = a(t) + \int_{-\infty}^{t} D(t,s,y(s))ds + \int_{t}^{\infty} E(t,s,y(s))ds. \tag{33}$$

Since $t \in R$ is arbitrary, (33) shows that $y(t)$ is a solution of Equation (1B) which satisfies (1B) on R and $|y(t)| \le J$ for $t \in R$.

EXAMPLE 4. Corresponding to Equation (26) in Example 2, consider the scalar nonlinear equation

$$x(t) = \arctan t + \int_{-\infty}^{t} e^{t-s}\sqrt{|x(s)|}ds + \int_{t}^{\infty} e^{s-t}\sqrt{|x(s)|}ds, \quad t \in R. \tag{34}$$

Let J be a number with $J \ge 2\sqrt{J} + \pi/2$. Then it is easy to see that (4B) and (31) are satisfied for the same $D_J(t,s)$ and $E_J(t,s)$ in Example 2. Thus, by Theorem 8, Equation (34) has a solution $x(t)$ which satisfies (34) on R, and $|x(t)| \le J$ for $t \in R$.

3. Existence of periodic solutions

In this section we discuss the existence of periodic and asymptotically periodic solutions of neutral integral equations. Thus, we consider the systems (1A) and

$$x(t) = p(t) + \int_{-\infty}^{t} P(t,s,x(s))ds + \int_{t}^{\infty} Q(t,s,x(s))ds, \quad t \in R \tag{35}$$

where $p : R \to R^n$, $P : \Delta^- \times R^n \to R^n$ and $Q : \Delta^+ \times R^n \to R^n$ are continuous. It will be assumed that the functions $a(t)$, $D(t,s,x)$, and $E(t,s,x)$ in (1A) converge to $p(t)$, $P(t,s,x)$, and $Q(t,s,x)$, respectively, in the following sense:

$$q(t) := a(t) - p(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{36}$$

and $p(t)$ is T-periodic, where $T > 0$ is a constant;

$$F(t,s,x) := D(t,s,x) - P(t,s,x), \quad \text{and} \quad P(t+T,s+T,x) = P(t,s,x); \tag{37}$$

$$G(t,s,x) := E(t,s,x) - Q(t,s,x), \quad \text{and} \quad Q(t+T,s+T,x) = Q(t,s,x). \tag{38}$$

In addition, we assume that for any $J > 0$ there are continuous functions P_J and $F_J : \Delta^- \to R^+$, and Q_J and $G_J : \Delta^+ \to R^+$ such that:

$$P_J(t+T, s+T) = P_J(t,s) \text{ if } s \le t;$$

$$Q_J(t+T, s+T) = Q_J(t,s) \text{ if } s \ge t;$$

$$|P(t,s,x)| \le P_J(t,s) \text{ if } s \le t \text{ and } |x| \le J;$$

$$|Q(t,s,x)| \le Q_J(t,s) \text{ if } s \ge t \text{ and } |x| \le J;$$

$$|F(t,s,x)| \le F_J(t,s) \text{ if } s \le t \text{ and } |x| \le J;$$

$$|G(t,s,x)| \le G_J(t,s) \text{ if } s \ge t \text{ and } |x| \le J;$$

$$\int_{-\infty}^{t-\tau} P_J(t,s)ds + \int_{t+\tau}^{\infty} (Q_J(t,s) + G_J(t,s))ds \to 0 \tag{39}$$

$$\text{uniformly for } t \in R \text{ as } \tau \to \infty;$$

$$\int_0^t F_J(t,s)ds + \int_t^{\infty} G_J(t,s)ds \to 0 \text{ as } t \to \infty. \tag{40}$$

First, we have the following lemma.

LEMMA 4. In addition to assumptions (36)–(40), let

$$|a(t)| + \int_0^t (P_J(t,s) + F_J(t,s))ds + \int_t^{\infty} (Q_J(t,s) + G_J(t,s))ds \le J, \quad t \ge t_0,$$

hold for some $J > 0$ with $J \ge \sup\{|\phi(t)| : 0 \le t < t_0\}$. Then Equation (1A) has an R^+-bounded solution $x(t) = x(t, t_0, \phi)$ which satisfies (1A) for $t > t_0$. Moreover, for any sequence $\{s_k\}$ of nonnegative numbers with $s_k \to \infty$ as $k \to \infty$, the sequence of functions $\{x_k(t)\}$ contains a subsequence which converges to an R-bounded solution $x(t)$ of the equation

$$\begin{aligned} x(t) = p(t+\sigma) &+ \int_{-\infty}^{t} P(t+\sigma, s+\sigma, x(s))ds \\ &+ \int_t^{\infty} Q(t+\sigma, s+\sigma, x(s))ds, \quad t \in R, \end{aligned} \tag{35_σ}$$

uniformly on any compact subset of R, where $x_k(t)$ is defined by

$$x_k(t) := \begin{cases} x(0), & t < -s_k \\ x(t+s_k), & t \geq -s_k \end{cases} \quad t \in R,$$

and where σ is a number with $0 \leq \sigma < T$ and $z(t)$ satisfies Equation $(35)_\sigma$ on R.

Since this lemma can be easily proved by Theorem 3 and a standard argument, we omit the proof.

DEFINITION. A function $\xi : R^+ \to R^n$ is said to be asymptotically T-periodic if $\xi = \psi + \mu$ where $\psi : R^+ \to R^n$ is continuous and T-periodic, $\mu : R^+ \to R^n$ is bounded and is continuous on R except at some $t_0 \in R^+$,

$$\mu(t_0) = \mu(t_0+), \quad \text{and} \quad \mu(t) \to 0 \quad \text{as} \quad t \to t_0+ .$$

Concerning the existence of a T-periodic solution of Equation (35) and its attractivity, we have the following result.

THEOREM 9. If (36)–(38) hold, and if Equation (35) has a unique R-bounded solution $x_0(t)$ which satisfies (35) on R, then the following hold:

(i) The solution $x_0(t)$ is T-periodic.

(ii) If the assumptions of Lemma 4 hold, then Equation (1A) has an R^+-bounded solution $x(t)$ such that $x(t) = \phi(t)$ for $0 \leq t < t_0$. Moreover, $x(t)$ is asymptotically T-periodic and approaches $x_0(t)$ as $t \to \infty$.

PROOF. (i) Let $x_1(t)$ be a function obtained by the T-translation of $x_0(t)$ to the left. Then clearly $x_1(t)$ is also an R-bounded solution of Equation (35) which satisfies (35) on R. Thus, from the uniqueness of R-bounded solutions of Equation (35) which satisfy (35) on R, $x_0(t)$ and $x_1(t)$ must be identical on R; that is, $x_0(t)$ is T-periodic.

(ii) Lemma 4 implies that Equation (1A) has an R^+-bounded solution $x(t)$ such that $x(t) = \phi(t)$ for $0 \leq t < t_0$. Let $\{x_k(t)\}$ be the sequence of functions as in Lemma 4 with $s_k = kT$. Then, from Lemma 4 and the uniqueness of R-bounded solutions of Equation

(35) which satisfy (35) on R, $x_k(t)$ converges to $x_0(t)$ uniformly on $[0, T]$. This implies that $x(t)$ is asymptotically T-periodic and its T-periodic part is given by $x_0(t)$.

REMARK. By imposing the Lipschitz conditions (5)–(6) and Condition (7) on $P(t, s, x)$ and $Q(t, s, x)$, we can obtain the uniqueness of R-bounded solutions of Equation (35) which satisfy (35) on R.

We now assume that $P(t, s, x)$ and $Q(t, s, x)$ are linear in x so that we can use the theory of minimal solutions to prove the existence of T-periodic solutions of Equation (35). Consider the equations (1A) and

$$x(t) = p(t) + \int_{-\infty}^{t} P(t, s)x(s)ds + \int_{t}^{\infty} Q(t, s)x(s)ds, \quad t \in R, \tag{41}$$

where $a(t)$, $D(t, s, x) := P(t, s)x + F(t, s, x)$ and $E(t, s, x) := Q(t, s)x + G(t, s, x)$ satisfy (36)–(40) with $P_J(t, s) := J|P(t, s)|$ and $Q_J(t, s) := J|Q(t, s)|$.

Let $h : R \to R^+$ be a continuous positive function with $\int_{-\infty}^{\infty} h(s)ds < \infty$. For any bounded continuous function $x : R \to R^n$, define a function $\lambda(x)$ by

$$\lambda(x) := \sup\left\{ \int_{-\infty}^{\infty} |x(s + t)|^2 h(s)ds : t \in R \right\},$$

and define a number Λ by the infimum of the set of numbers $\lambda(x)$, where x is an R-bounded solution of Equation (41) such that x solves (41) on R and $\|x\| \leq J$, and where $J > 0$ is a constant. Then from the theory of minimal solutions (see [5]), we have the following lemmas which we state without proofs.

LEMMA 5. Let (36)–(40) hold with $D(t, s, x) := P(t, s)x + F(t, s, x)$, $E(t, s, x) := Q(t, s)x + G(t, s, x)$, $P_J(t, s) := J|P(t, s)|$ and $Q_J(t, s) := J|Q(t, s)|$, and

$$|a(t)| + \int_{0}^{t} (P_J(t, s) + F_J(t, s))ds + \int_{t}^{\infty} (Q_J(t, s) + G_J(t, s))ds \leq J, \quad t \geq t_0,$$

hold for some $J > 0$ with $J \geq \sup\{|\phi(t)| : 0 \leq t < t_0\}$. Then Equation (41) has a minimal solution; that is, (41) has an R-bounded solution which attains the value Λ.

LEMMA 6. In addition to the assumptions of Lemma 5, if $C_k(t)$ $(k = 1, 2)$ are minimal solutions of Equation (41), then there is a sequence $\{t_k\}$ with $C_1(t+t_k) - C_2(t+t_k) \to 0$ uniformly on any compact subset of R as $k \to \infty$.

Now we have the following result.

THEOREM 10. Under the assumptions of Lemma 5, Equation (41) has a T-periodic solution.

PROOF. Let $C(t)$ be a minimal solution of Equation (41) which is assured in Lemma 5. Clearly, $C(t+T)$ is also a minimal solution. Thus, from Lemma 6, there is a subsequence $\{t_k\}$ with $C(t+t_k) - C(t+T+t_k) \to 0$ as $k \to \infty$ uniformly on any compact subset of R. For each positive integer k, let ν_k be an integer with $\nu_k T \leq t_k < (\nu_k + 1)T$, and let $\sigma_k := t_k - \nu_k T$. Taking a subsequence if necessary, we may assume that $\sigma_k \to \sigma$ as $k \to \infty$ for some σ with $0 \leq \sigma \leq T$, and that for some bounded continuous function $\gamma(t)$ on R, $C(t+t_k) \to \gamma(t)$ uniformly on any compact subset of R as $k \to \infty$, since the set $\{C(t+t_k)\}$ is uniformly bounded and equicontinuous on R. Clearly, $\gamma(t)$ is T-periodic. Moreover, since (39) holds with $P_J(t,s) := J|P(t,s)|$ and $Q_J(t,s) := J|Q(t,s)|$, from Lemma 4, $\gamma(t)$ is an R-bounded solution of the equation

$$\begin{aligned}\gamma(t) = p(t+\sigma) + &\int_{-\infty}^{t} P(t+\sigma, s+\sigma)\gamma(s)ds \\ &+ \int_{t}^{\infty} Q(t+\sigma, s+\sigma)\gamma(s)ds, \quad t \in R.\end{aligned}$$

For $\delta(t) := \gamma(t-\sigma)$, this equation can be rewritten as

$$\delta(t) = p(t) + \int_{-\infty}^{t} P(t,s)\delta(s)ds + \int_{t}^{\infty} Q(t,s)\delta(s)ds, \quad t \in R,$$

and hence, $\delta(t)$ is a T-periodic solution of Equation (41).

REFERENCES

1. T.A. Burton, Volterra Integral and Differential Equations, Academic Press, Orlando, 1983.
2. T.A. Burton and Tetsuo Furumochi, Periodic solutions of Volterra equations and attractivity, *Dynamic Systems and Applications* **3**(1994), 583–598.
3. E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955, pp. 330–332.
4. W.A. Coppel, Stability and Asymptotic Behavior of Differential Equations, D.C. Heath and Company, Boston, 1965, pp. 74–75.
5. Y. Hino and S. Murakami, Periodic solutions of a linear Volterra system, Proc. Equadiff Conference, ed. by C.M. Dafermos, G. Ladas, and G. Papanicolaou, Dekker, New York, 1989, pp. 319–326.
6. D.R. Smart, Fixed Point Theorems, Cambridge Univ. Press, Cambridge, 1980.