

## The Lurie Control Satisfies a Liénard Equation

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**Abstract.** We transform the Lurie indirect control equations into Liénard type integro-differential equations. Depending on the number of zero eigenvalues of the matrix in the Lurie equations, we obtain a pure Liénard equation, a Liénard equation with memory and an exponentially decaying forcing function, and a Liénard-Volterra-Levin equation. Using the good Liapunov functions known for these type of equations, we prove stability results for the Liénard equations and consequently for the indirect control Lurie problem. The results are then extended to the delay form of the Lurie problem.

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**AMS (MOS) subject classification:** 34A34, 34D05, 34K20, 93D05.

## 1 Introduction

The problem of Lurie began in 1951 [10] and has attracted much interest to the present time, ever changing to encompass more sophisticated systems, but remaining basically the same. A search of the topic "Lurie" in the online Mathematical Reviews will net 85 papers, which are but a fraction of the total literature on the problem, as can be seen by then searching the publications of the authors listed in that first 85. The last 55 of that 85 paper set have appeared since 1995; thus, there is a great resurgence of interest in the subject.

The basic problem is how to ensure the stability of a linearized plant equation using a scalar control that is a non-linear function of the error.

We assume that the plant equation is given by

$$x' = Ax$$

where  $A$  is a  $d \times d$  real constant matrix having  $m$  zero characteristic roots and  $n$  roots with negative real parts.

A control is added to the system and a transformation is made resulting in the  $(d + 1)$ -dimensional system,

$$\begin{aligned}x' &= Ax + bf(\sigma) \\ \sigma' &= c^T x - rf(\sigma)\end{aligned}\tag{1.1}$$

known as the **Lurie's indirect control problem**.

Here  $x$  is the transformed state vector,  $\sigma$  is the scalar control,  $b$  and  $c$  are constant  $d$ -vectors,  $r$  is a positive constant, and  $f$  is an admissible function, defined below.

The task is to give conditions on  $b$  and  $c$  to ensure that the zero solution of (1.1) is globally asymptotically stable for every admissible  $f$ .

We say that  $f : R \rightarrow R$  is **admissible** if:

$$f \text{ is continuous;}\tag{1.2}$$

$$\sigma f(\sigma) > 0 \text{ if } \sigma \neq 0;\tag{1.3}$$

$$\int_0^{\pm\infty} f(s)ds = \infty.\tag{1.4}$$

The books by LaSalle and Lefschetz [5] and by Lefschetz [6] give fine summaries of the problem. The entire book by Lefschetz is devoted to the problem, but Section 19 of [6] is more closely related to our work here. The main efforts by investigators have involved the construction of Liapunov functions (Lurie-Postnikov and Popov are other designations) yielding stability results.

Our thesis here is that the control function,  $\sigma$ , satisfies a Liénard-Volterra-Levin equation for which there are already very effective Liapunov functions.

The Liénard equation in the Liénard plane may be written as

$$\begin{aligned}\sigma' &= z - F(\sigma) \\ z' &= -g(\sigma)\end{aligned}$$

where  $\sigma F(\sigma) > 0$  and  $\sigma g(\sigma) > 0$  for  $\sigma \neq 0$ . It has been widely studied in many contexts since about 1928 when Liénard [9] introduced the Liénard plane.

The equation has also been widely studied when there is a delay. (See Krasovskii [4] [ p. 173] and Zhang [14], for example.) In the Zhang paper a Liapunov functional is given, together with necessary and sufficient conditions for global asymptotic stability.

Again in 1928 Volterra [13] began a study of a biological problem described by a truncated form of

$$\sigma' = - \int_0^t a(t-s)f(\sigma(s))ds$$

with  $a(t) \geq 0$ ,  $a'(t) \leq 0$ , and  $a''(t) \geq 0$ . Volterra suggested that a Liapunov function could be constructed and Levin [7] carried out the details creating

$$V(t, \sigma) = \int_0^\sigma f(s) ds + (1/2)a(t) \left[ \int_0^t f(\sigma(s)) ds \right]^2 - (1/2) \int_0^t a'(t-s) \left[ \int_s^t f(\sigma(u)) du \right]^2 ds.$$

A long line of papers followed Levin's work and some important summaries are found in Krasovskii [4] [pp. 158-160]. It is now understood, but not simple to see, that the Volterra-Levin equation is an integrated form of the Liénard equation and that their Liapunov functions are related.

Let us rewrite now the Lurie control equation in the form of a Liénard equation.

Suppose that the matrix  $A$  can be decomposed and written as

$$A = \begin{pmatrix} L & 0 \\ 0 & J \end{pmatrix}$$

where  $L$  is the  $m \times m$  zero matrix, while  $J$  is an  $n \times n$  real matrix all of whose characteristic roots have negative real parts. Thus, we have the system

$$\begin{aligned} x' &= b_1 f(\sigma) \\ y' &= Jy + b_2 f(\sigma) \\ \sigma' &= c_1^T x + c_2^T y - r f(\sigma) \end{aligned} \tag{1.5}$$

where  $x, b_1, c_1 \in R^m$ ,  $y, b_2, c_2 \in R^n$ ,  $J$  is an  $n \times n$  constant real matrix all of whose characteristic roots have negative real parts,  $r$  is a positive constant, and  $f$  is an admissible control satisfying (1.2), (1.3), and (1.4).

Let  $(x_0, y_0, \sigma_0)$  be any given initial condition. We can write an integral equation for the first equation in (1.5) as

$$x(t) = x_0 + b_1 \int_0^t f(\sigma(s)) ds \tag{1.6}$$

and for the second equation as

$$y(t) = e^{Jt} y_0 + \int_0^t e^{J(t-s)} b_2 f(\sigma(s)) ds. \tag{1.6b}$$

Substituting these into the third equation yields

$$\begin{aligned} \sigma' &= c_1^T \left[ x_0 + b_1 \int_0^t f(\sigma(s)) ds \right] \\ &+ c_2^T \left[ e^{Jt} y_0 + \int_0^t e^{J(t-s)} b_2 f(\sigma(s)) ds \right] - r f(\sigma). \end{aligned} \tag{1.7}$$

Depending on the characteristic roots we have three types of equations:

1. When  $n = 0$ , then the equation in  $\sigma$  becomes a pure Liénard equation; Liapunov theory is known giving necessary and sufficient conditions for global asymptotic stability.

2. When neither  $n$  nor  $m$  is zero, then the equation in  $\sigma$  becomes a classical Liénard equation with a delay. A Liapunov functional for essentially this equation was constructed by Krasovskii in the 1950's.

3. When  $m = 0$ , then the equation in  $\sigma$  becomes a Liénard-Volterra-Levin equation. Levin used a suggestion of Volterra to construct a very good Liapunov functional which will yield the desired stability.

One can also introduce a delay in the control, in the state equation, or in both to obtain delay equations. We will study them in the last section.

Once we know that the  $\sigma$  equation is asymptotically stable we can then make corresponding conclusions about the state variables.

## 2 A Pure Liénard equation.(n=0)

At this point we can note a simple and interesting relation in case  $c_2 = b_2 = 0$ . This is interpreted as a decision that  $y$  need not be controlled or that  $y$  is not even present. In that case our equation becomes

$$\sigma' = c_1^T [x_0 + b_1 \int_0^t f(\sigma(s)) ds] - r f(\sigma) \quad (2.1)$$

which can be written as the system

$$\begin{aligned} \sigma' &= z - r f(\sigma) \\ z' &= c_1^T b_1 f(\sigma). \end{aligned} \quad (2.2)$$

When  $z(0) = c_1^T x_0$  then a solution of (2.2) is a solution of (2.1). When  $\sigma(t)$  is a solution of (2.1), then

$$(\sigma(t), c_1^T [x_0 + b_1 \int_0^t f(\sigma(s)) ds])$$

is a solution of (2.2). If we show that all solutions of (2.2) tend to zero, then all solutions of (2.1) tend to zero.

**REMARK 2.1.** Notice that (2.2) is a differential equation involving the control only.

The sign of  $c_1^T b_1$  will determine the stability or instability of the zero solution of (2.2).

Consider the Liapunov function

$$V(\sigma, z) = z^2 - 2c_1^T b_1 \int_0^\sigma f(s) ds \quad (2.3)$$

having a derivative along a solution  $(\sigma(t), z(t))$  of (2.2) given by

$$V'(\sigma(t), z(t)) = 2c_1^T b_1 r f^2(\sigma). \quad (2.4)$$

**THEOREM 2.1.** Let (1.2), (1.3), and (1.4) hold.

(a) If  $c_1^T b_1 < 0$  then the zero solution of (2.2) is globally asymptotically stable. Moreover, each solution of (2.1) satisfies  $c_1^T x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . In particular, if  $m = 1$  then  $x(t) \rightarrow 0$ .

(b) If  $c_1^T b_1 > 0$  then the zero solution of (2.2) is unstable, and so is the solution of (2.1).

**Proof**

(a) First, notice that  $V$  is positive definite so the zero solution of (2.2) is stable. Moreover,  $V'$  is negative semi-definite, with the set on which  $V' = 0$  containing no positively invariant sets except the origin. (See Krasovskii [4] [p. 67]. Uniqueness is not needed. We see that  $V(\sigma(t), z(t))$  tends to a constant and  $\sigma(t) \rightarrow 0$  so  $\sigma' \rightarrow z_1$ . If  $z_1$  is not zero, then from the first equation in (2.2) we see that  $\sigma'$  approaches a nonzero constant, a contradiction to  $\sigma(t) \rightarrow 0$ .) Hence, the zero solution is asymptotically stable and every bounded solution approaches the origin. Condition  $c_1^T b_1 < 0$  implies that  $V$  is radially unbounded so every solution is bounded.

To prove the last part of the theorem, notice that when the zero solution of (2.2) is asymptotically stable then from (2.2) we see that both  $\sigma(t)$  and  $\sigma'(t)$  tend to zero. Using this in the third equation of (1.5) with  $c_2 = 0$  proves that  $c_1^T x(t) \rightarrow 0$ . This completes the proof of (a).

(b) Consider the curves given by the equation  $V(x, \sigma) = 0$ . In the open domain defined by  $\{(z, \sigma) \mid z > \sqrt{2c_1^T b_1 \int_0^\sigma f(s) ds}\}$  we have  $V > 0$  and in the boundary  $V = 0$ . The origin belongs to the boundary. The derivative along the solutions of (2.2) satisfies  $V' \geq 0$  thus the domain is invariant. We would like to apply Chetaev's Theorem (See Burton [2] Theorem 4.1.25, pg 243), but in that domain the derivative  $V' = 0$  when  $\sigma = 0$ , so the Theorem does not apply directly.

Let us define an invariant subdomain  $D$  in which  $V' > 0$ .

$$D : \left\{ (z, \sigma) \mid \sigma > 0, \quad z > \sqrt{2c_1^T b_1 \int_0^\sigma f(s) ds} \right\}$$

On the part of the boundary of  $D$  where  $\sigma = 0$  we have  $\sigma' > 0$  except at the origin where  $\sigma' = 0$ , thus  $D$  is invariant. Observe also that for any  $(\sigma_0, z_0)$  in  $D$  the region  $z > z_0$  is also invariant, since on the line  $z = z_0$ , we have  $z' = c_1^T b_1 f(\sigma) \geq 0$ . Moreover, if  $(\sigma_0, z_0)$  is any point in  $D$  with  $z_0 > r f(\sigma_0)$ , then no solution in  $D$  crosses the vertical line through  $\sigma_0$  from right to left.

Suppose, by way of contradiction, that the zero solution is stable. Then for  $\epsilon = 1$  there is a  $\delta > 0$  such that  $|(\sigma_0, z_0)| < \delta$  and  $t \geq 0$  imply that the solution through that given point satisfies  $|(\sigma(t), z(t))| < 1$ . Find a point in  $D$  with  $z_0 > r f(\sigma_0)$ ,  $\sigma_0 > 0$ , and  $|(\sigma_0, z_0)| < \delta$ . Then  $V' = 2c_1^T b_1 r f^2(\sigma)$  and

$\sigma_0 \leq \sigma(t) \leq 1$  implies that there is a  $\gamma > 0$  with  $V' \geq \gamma$  for  $t \geq 0$ . Integrate that expression for a contradiction.

### 3 Liénard equation with delay. ( $n > 0, m > 0$ ).

Let us work now with the full Liénard control equation

$$\begin{aligned} \sigma' &= c_1^T [x_0 + b_1 \int_0^t f(\sigma(s)) ds] \\ &+ c_2^T [e^{Jt} y_0 + \int_0^t e^{J(t-s)} b_2 f(\sigma(s)) ds] - r f(\sigma). \end{aligned} \quad (3.1)$$

Calling

$$z(t) = c_1^T [x_0 + b_1 \int_0^t f(\sigma(s)) ds] + c_2^T e^{Jt} y_0$$

the equation (3.1) can be written as

$$\begin{aligned} \sigma' &= z - r f(\sigma) + c_2^T \int_0^t e^{J(t-s)} b_2 f(\sigma(s)) ds \\ z' &= c_1^T b_1 f(\sigma) + c_2^T J e^{Jt} y_0, \end{aligned} \quad (3.2)$$

which is a Liénard equation with memory and with an exponentially decaying forcing function.

Note again that if  $(\sigma(t), z(t))$  is a solution of (3.2) with  $z(0) = c_1^T x_0 + c_2^T y_0$ , then the equation

$$\begin{aligned} z' &= c_1^T b_1 f(\sigma) + c_2^T J e^{Jt} y_0. \\ z(0) &= c_1^T x_0 + c_2^T y_0 \end{aligned}$$

has a solution  $z = c_1^T [x_0 + b_1 \int_0^t f(\sigma(s)) ds] + c_2^T e^{Jt} y_0$ , so  $\sigma(t)$  satisfies (3.1), and conversely.

Define

$$C(t-s) = -(1/2) c_1^T b_1 |c_2^T e^{J(t-s)} b_2| \quad (3.3)$$

and then define the classical Liapunov functional

$$V(t, \sigma, z) = (1/2) z^2 - c_1^T b_1 \int_0^\sigma f(s) ds + \int_0^t \int_{t-s}^\infty C(u) du f^2(\sigma(s)) ds. \quad (3.4)$$

**THEOREM 3.1.** Let  $c_1^T b_1 < 0$  and suppose that there is an  $\alpha > 0$  with

$$c_1^T b_1 (r - \int_0^\infty |c_2^T e^{Ju} b_2| du) \leq -\alpha < 0 \quad (3.5)$$

Then the zero solution of (3.2) is globally asymptotically stable for each admissible  $f$ . Moreover, when  $\sigma(t) \rightarrow 0$ , so does  $y(t)$ , while  $c_1^T x(t) \rightarrow 0$ .

Proof. Taking the derivative of  $V$  along a solution of (3.2) yields

$$\begin{aligned}
V' &= -c_1^T b_1 f(\sigma) [z - r f(\sigma) + c_2^T \int_0^t e^{J(t-s)} b_2 f(\sigma(s)) ds] \\
&\quad + z [c_1^T b_1 f(\sigma) + c_2^T J e^{Jt} y_0] + f^2(\sigma(t)) \int_0^\infty C(u) du \\
&\quad - \int_0^t C(t-s) f^2(\sigma(s)) ds \\
&\leq c_1^T b_1 r f^2(\sigma) - c_1^T b_1 \int_0^t |c_2^T e^{J(t-s)} b_2| |f(\sigma(t)) f(\sigma(s))| ds \\
&\quad + \int_0^\infty C(u) du f^2(\sigma) - \int_0^t C(t-s) f^2(\sigma(s)) ds + z c_2^T J e^{Jt} y_0.
\end{aligned}$$

Then

$$\begin{aligned}
V' &\leq +c_1^T b_1 r f^2(\sigma) \\
&\quad - c_1^T b_1 \frac{1}{2} \left\{ \int_0^t |c_2^T e^{J(t-s)} b_2| |f(\sigma(t))|^2 ds + \int_0^t |c_2^T e^{J(t-s)} b_2| |f(\sigma(s))|^2 ds \right\} \\
&\quad + \int_0^\infty C(u) du f^2(\sigma) - \int_0^t C(t-s) f^2(\sigma(s)) ds + z c_2^T J e^{Jt} y_0 \\
&\leq c_1^T b_1 r f^2(\sigma) \left\{ r - \int_0^t \frac{1}{2} |c_2^T e^{J(t-s)} b_2| |f(\sigma(t))|^2 ds \right\} + \int_0^\infty C(u) du f^2(\sigma) \\
&\quad + z c_2^T J e^{Jt} y_0
\end{aligned}$$

where we used the definition of  $C(t-s)$ , (3.3). Substituting again for  $C(u)$

$$\begin{aligned}
V' &\leq c_1^T b_1 f^2(\sigma) \left\{ r - \int_0^\infty \frac{1}{2} |c_2^T e^{J(t-s)} b_2| ds - (1/2) \int_0^\infty |c_2^T e^{J(t-s)} b_2| du \right\} \\
&\quad + z c_2^T J e^{Jt} y_0 \\
&= c_1^T b_1 f^2(\sigma) \left\{ r - \int_0^\infty |c_2^T e^{J(t-s)} b_2| ds \right\} + z c_2^T J e^{Jt} y_0 \\
&\leq -\alpha f^2(\sigma) + M(V+1)e^{-\beta t}.
\end{aligned}$$

for positive constants  $\beta$  and  $M$ . Here we used (3.5), the fact that all the eigenvalues of  $J$  have negative real parts, and  $|z| < (V+1)$ .

Define a new function of the form

$$W(t, \sigma, z) = [V(t, \sigma, z) + 1] e^{-\int_0^t M e^{-\beta s} ds}$$

and obtain

$$\begin{aligned} W'(t, \sigma, z) &= V' e^{-\int_0^t M e^{-\beta s} ds} + (V + 1)(-M e^{-\int_0^t M e^{-\beta s} ds} e^{-\beta t}) \\ &\leq -\alpha f^2(\sigma) e^{-\int_0^t M e^{-\beta s} ds}. \end{aligned}$$

From this, we argue (in the same way Krasovskii does in [4] [p. 67]) that  $f^2(\sigma) \rightarrow 0$ . Then we see that  $V(t, \sigma, z) \rightarrow z^2/2$  which tends to a constant. If  $z^2$  does not tend to zero, then  $\sigma' \rightarrow z$  implies that  $\sigma$  does not tend to zero.

Looking now at (1.6b) we see that the integral is the convolution of an  $L^1$  function with a function tending to zero so the convolution tends to zero. Hence,  $y$  tends to zero. Then looking at the last equation in (1.5) we see that  $\sigma', f(\sigma), y$  all tend to zero. This means that  $c_1^T x \rightarrow 0$ ; if  $m = 1$ , then  $x \rightarrow 0$ . This completes the proof.

**COROLLARY 1.** If  $b_1^T c_1 < 0$ ,  $c_2^T e^{J(t)} b_2 > 0$  and  $r + c_2^T J^{-1} b_2 > 0$  then the zero solution of (3.2) is globally asymptotically stable for every admissible  $f$ .

Proof. Consider

$$\begin{aligned} r - \int_0^\infty |c_2^T e^{J(s)} b_2| ds &= r - \int_0^\infty c_2^T e^{J(s)} b_2 ds \\ &= r - c_2^T J^{-1} e^{Ju} \Big|_0^\infty b_2 \\ &= r + c_2^T J^{-1} b_2. \end{aligned}$$

Thus the condition (3.5)

$$c_1^T b_1 (r - \int_0^\infty |c_2^T e^{Ju} b_2| du) = c_1^T b_1 (r + c_2^T J^{-1} b_2) := -\alpha < 0$$

obtains, and we can apply the theorem.

NOTE: We encounter here the condition  $r + c_2^T J^{-1} b_2 > 0$  with a long tradition in the literature. See the books by LaSalle and Lefschetz [5], [6] and the papers by Burton [2], Moser [11] and Somolinos [12].

**COROLLARY 2.** Suppose that  $c_1^T b_1 < 0$  and that  $J$  has a real characteristic root  $-\lambda < 0$ . Let us assume that  $b_2$  or  $c_2$  can be selected as a characteristic vector belonging to  $-\lambda$

$$Jb_2 = -\lambda b_2, \quad \text{or} \quad c_2 J = -\lambda c_2.$$

Then, if  $c_2^T b_2 > 0$ , the zero solution of (3.2) is globally asymptotically stable for every admissible  $f$ .



Proof. We have

$$\begin{aligned} r - \int_0^\infty |c_2^T e^{J(s)} b_2| \, ds &= r - \int_0^\infty |c_2^T b_2 e^{-\lambda u}| du \\ &= r + \frac{e^{-\lambda}}{\lambda} \Big|_0^\infty c_2^T b_2 \\ &= r + \frac{c_2^T b_2}{\lambda} \end{aligned}$$

and the condition 3.5 is satisfied.

The proof for  $c_2$  is similar.

**COROLLARY 3.** Suppose that  $c_1^T b_1 < 0$  and that  $J$  has a real characteristic root  $-\lambda < 0$ . Let us assume that  $b_2$  and  $c_2$  can be selected as a characteristic vector  $b$  belonging to  $-\lambda$ ,  $Jb = -\lambda b$

Then the zero solution of (3.2) is globally asymptotically stable for every admissible  $f$ .

Proof. As in the previous Corollary 2. Observe that in the last line we have now  $b^T b > 0$ .

## 4 The Liénard-Volterra-Levin equation

Something needs to be said about the case in which  $m = 0$  relative to the use of the Liénard equation for stability analysis. In that case, the Lurie plant equation is

$$y(t) = e^{Jt} y_0 + \int_0^t e^{J(t-s)} b_2 f(\sigma(s)) ds \quad (4.0)$$

and the Liénard equation (1.7), becomes

$$\sigma' = -r f(\sigma) + \int_0^t c_2^T e^{J(t-s)} b_2 f(\sigma(s)) ds + c_2^T e^{Jt} y_0. \quad (4.1)$$

That equation has the form of one studied by Levin in [7] and for which he constructed a very exact Liapunov function using a suggestion of Volterra [13]. But it still falls in the Liénard category because it can be shown that Levin's equation is an integrated Liénard equation.

Levin requires that

$$a(t) := c_2^T e^{Jt} b_2 \leq 0, \quad a'(t) \geq 0, \quad a''(t) \leq 0. \quad (4.2)$$

This condition is satisfied, for example, if  $J$  has a real root  $-\lambda < 0$ . Picking  $c_2 = -b_2 = -b$ , where  $b$  is a characteristic vector belonging to  $-\lambda$ , we have

$$a(t) = -b^T e^{Jt} b = -b^T b e^{-\lambda t}$$

and (4.2) is satisfied. It will then be the case that every solution of (4.1) tends to zero, that  $\sigma' \rightarrow 0$ , and then from (3.6) that  $y(t) \rightarrow 0$  for any  $f(\sigma)$  which is admissible.

**THEOREM 4.1.** If (4.2) is satisfied then every solution of (4.1) tends to zero as  $t \rightarrow \infty$  for every admissible  $f$  and for  $r > 0$ , and the same is true of  $y$  in equation (4.0)

Proof. With  $a(t)$  defined in (4.2) we define a Liapunov functional by

$$\begin{aligned} V(t, \sigma) &= \int_0^\sigma f(s)ds - (1/2)a(t) \left[ \int_0^t f(\sigma(u))du \right]^2 \\ &\quad + (1/2) \int_0^t a'(t-s) \left[ \int_s^t f(\sigma(u))du \right]^2 ds. \end{aligned}$$

Then

$$\begin{aligned} V' &= f(\sigma) \left[ -rf(\sigma) + \int_0^t a(t-s)f(\sigma(s))ds + c_2^T e^{Jt}y_0 \right] \\ &\quad - a(t) \int_0^t f(\sigma(u))du f(\sigma) - (1/2)a'(t) \left[ \int_0^t f(\sigma(u))du \right]^2 \\ &\quad + (1/2) \int_0^t a''(t-s) \left[ \int_s^t f(\sigma(u))du \right]^2 ds \\ &\quad + \int_0^t a'(t-s) \int_s^t f(\sigma(u))duds f(\sigma). \end{aligned}$$

We integrate the last term by parts and obtain

$$a(t) \int_0^t f(\sigma(u))du f(\sigma) - \int_0^t a(t-s)f(\sigma(s))ds f(\sigma).$$

This yields

$$\begin{aligned} V' &= -rf^2(\sigma) + f(\sigma)c_2^T e^{Jt}y_0 - (1/2)a'(t) \left[ \int_0^t f(\sigma(u))du \right]^2 \\ &\quad + (1/2) \int_0^t a''(t-s) \left[ \int_s^t f(\sigma(u))du \right]^2 ds. \end{aligned}$$

Now

$$|f(\sigma)c_2^T e^{Jt}y_0| \leq (1/2)(rf^2(\sigma) + Me^{-\beta t})$$

for positive constants  $M, \beta$ . Thus, define

$$W(t, \sigma) = [V(t, \sigma) + 1]e^{-\int_0^t Me^{-\beta s} ds}$$

and obtain

$$\begin{aligned} W'(t, \sigma) &\leq [V'(t, \sigma) - Me^{-\beta t}]e^{-\int_0^t Me^{-\beta s} ds} \\ &\leq [-rf^2(\sigma) + (1/2)(rf^2(\sigma) + Me^{-\beta t}) - Me^{-\beta t}]e^{-\int_0^t Me^{-\beta s} ds} \\ &\leq -(1/2)rf^2(\sigma)e^{-\int_0^\infty Me^{-\beta s} ds}. \end{aligned}$$

Hence, we can argue with Krasovskii that  $f(\sigma) \rightarrow 0$ . Looking now at (4.1) we see that the integral is the convolution of an  $L^1$  function with a function tending to zero. Hence,  $\sigma'(t) \rightarrow 0$ . Next, looking back at (4.0) we give the same argument to show that  $y(t) \rightarrow 0$ . This completes the proof.

## 5 Equations with delay.

It has long been recognized that there may be a time delay involved in the control. We first consider the system in which the roots of  $A$  are all zero,  $n = 0$ .

Thus, we look at

$$\begin{aligned} x' &= bf(\sigma(t)) \\ \sigma' &= c^T x(t-L) - rf(\sigma) \end{aligned} \tag{5.1}$$

where  $L$  is a positive constant. When  $L = 0$  we recover equation (2.2).

As before, we obtain

$$x(t) = x_0 + b \int_0^t f(\sigma(s)) ds$$

and then

$$\sigma' = c^T [x_0 + b \int_0^{t-L} f(\sigma(s)) ds] - rf(\sigma). \tag{5.2}$$

Let us rewrite it as a Lienard equation

$$\begin{aligned} \sigma' &= z - rf(\sigma) - c^T b \int_{t-L}^t f(\sigma(s)) ds \\ z' &= c^T b f(\sigma). \end{aligned} \tag{5.3}$$

where  $z = c^T [x_0 + b \int_0^t f(\sigma(s)) ds]$

One can show that, for a given initial condition, a solution of equation (5.3) is also a solution of (5.2).

Equation (5.3) is the delayed Liénard equation which is extensively discussed in the literature. See Zhang [14], for example. For Lurie problems with delay see Cao-Li-Ho [3].

**THEOREM 5.1.** Suppose that  $c^T b < 0$  and

$$r + Lc^T b > 0. \tag{5.4}$$

Then the zero solution of (5.3) is asymptotically stable and  $c^T x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Proof. Define

$$V(t, \sigma, z) = z^2 - 2 \int_0^\sigma c^T b f(s) ds + \int_{-L}^0 \int_{t+v}^t (c^T b)^2 f^2(\sigma(u)) du dv.$$

The derivative along the solutions satisfies

$$\begin{aligned}
V' &= +2zc^T b f(\sigma) \\
&\quad - 2c^T b f(\sigma)[z - r f(\sigma) - c^T b \int_{t-L}^t f(\sigma(s)) ds] \\
&\quad + \int_{-L}^0 (c^T b)^2 f^2(\sigma(t)) dv - \int_{-L}^0 (c^T b)^2 f^2(\sigma(t+v)) dv \\
&= 2rc^T b f^2(\sigma) + 2(c^T b)^2 \int_{t-L}^t f(\sigma(t)) f(\sigma(s)) ds \\
&\quad + L(c^T b)^2 f^2(\sigma(t)) - \int_{t-L}^t (c^T b)^2 f^2(\sigma(v)) dv \\
V' &\leq 2rc^T b f^2(\sigma) + (c^T b)^2 \int_{t-L}^t (f^2(\sigma(t)) + f^2(\sigma(s))) ds \\
&\quad + L(c^T b)^2 f^2(\sigma(t)) - \int_{t-L}^t (c^T b)^2 f^2(\sigma(v)) dv \\
&\leq 2rc^T b f^2(\sigma) + (c^T b)^2 L f^2(\sigma(t)) + (c^T b)^2 \int_{t-L}^t f^2(\sigma(s)) ds \\
&\quad + L(c^T b)^2 f^2(\sigma(t)) - \int_{t-L}^t (c^T b)^2 f^2(\sigma(v)) dv \\
&\leq (2rc^T b + 2(c^T b)^2 L) f^2(\sigma(t)) \leq 2c^T b(r + Lc^T b) f^2(\sigma(t)) \leq 0.
\end{aligned}$$

We now argue (as did Krasovskii) that  $f(\sigma(t)) \rightarrow 0$ . Then  $V \rightarrow z^2$  and so  $z \rightarrow 0$ . Hence  $\sigma' \rightarrow 0$ . Finally,  $c^T x(t) \rightarrow 0$ . This completes the proof.

Note that when  $L = 0$  the condition  $c^T b < 0$  is enough to ensure absolute stability as we expected from Theorem 2.1.

We now consider

$$\begin{aligned}
x' &= b_1 f(\sigma) \\
y' &= Jy + b_2 f(\sigma) \\
\sigma' &= c_1^T x(t-L) + c_2^T y(t-L) - r f(\sigma)
\end{aligned} \tag{5.5}$$

where the characteristic roots of  $J$  have negative real parts.

**THEOREM 5.2.** Let us select

$$c_1^T b_1 < 0 \quad \text{and} \quad L(|c_1^T b_1| + |c_2^T b_2|) < r. \tag{5.6}$$

If

$$a(t) := -c_2^T e^{J(t-L)} b_2 \geq 0 \quad \text{and} \quad a'(t) \leq 0, \quad a''(t) \geq 0 \tag{5.7}$$

then each solution of (5.5) satisfies  $\sigma(t)$ ,  $\sigma'(t)$ ,  $c^T x(t)$ ,  $y(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Proof. As we have done before, write

$$\begin{aligned} x(t) &= x_0 + b_1 \int_0^t f(\sigma(s)) ds \\ y(t) &= e^{Jt} y_0 + \int_0^t e^{J(t-s)} b_2 f(\sigma(s)) ds \end{aligned}$$

so that

$$\begin{aligned} \sigma' &= c_1^T [x_0 + b_1 \int_0^t f(\sigma(s)) ds - b_1 \int_{t-L}^t f(\sigma(s)) ds] - r f(\sigma) \\ &+ c_2^T [e^{J(t-L)} y_0 + \int_0^t e^{J(t-L-s)} b_2 f(\sigma(s)) ds - \int_{t-L}^t e^{J(t-L-s)} b_2 (f(\sigma(s)) ds)]. \end{aligned} \quad (5.8)$$

Let  $z = c_1^T [x_0 + b_1 \int_0^t f(\sigma(s)) ds] + c_2^T e^{J(t-L)} y_0$ . We will now use our definition of  $a(t)$  and write

$$\begin{aligned} \sigma' &= z - c_1^T b_1 \int_{t-L}^t f(\sigma(s)) ds - \int_0^t a(t-s) f(\sigma(s)) ds \\ &+ \int_{t-L}^t a(t-s) f(\sigma(s)) ds - r f(\sigma) \\ z' &= c_1^T b_1 f(\sigma) + c_2^T J e^{J(t-L)} y_0. \end{aligned} \quad (5.9)$$

Next, define a Liapunov functional by

$$\begin{aligned} V(t, \sigma, z) &= -c_1^T b_1 \int_0^\sigma f(s) ds + (1/2) z^2 - (1/2) c_1^T b_1 a(t) [\int_0^t f(\sigma(s)) ds]^2 \\ &+ (1/2) c_1^T b_1 \int_0^t a'(t-s) [\int_s^t f(\sigma(u)) du]^2 ds \\ &+ (1/2) \int_{-L}^0 \int_{t+s}^t [(c_1^T b_1)^2 + |c_1^T b_1 c_2^T b_2|] f^2(\sigma(u)) du ds. \end{aligned}$$

The derivative of  $V$  along a solution satisfies

$$\begin{aligned}
V' &= -c_1^T b_1 f(\sigma) [z - c_1^T b_1 \int_{t-L}^t f(\sigma(s)) ds - \int_0^t a(t-s) f(\sigma(s)) ds \\
&\quad + \int_{t-L}^t a(t-s) f(\sigma(s)) ds - r f(\sigma)] + z c_1^T b_1 f(\sigma) + z c_2^T J e^{J(t-L)} y_0 \\
&\quad - (1/2) c_1^T b_1 a'(t) [\int_0^t f(\sigma(s)) ds]^2 - a(t) f(\sigma(t)) \int_0^t c_1^T b_1 f(\sigma(s)) ds \\
&\quad + (1/2) c_1^T b_1 \int_0^t a''(t-s) [\int_s^t f(\sigma(u)) du]^2 ds \\
&\quad + c_1^T b_1 \int_0^t a'(t-s) \int_s^t f(\sigma(u)) du ds f(\sigma) \\
&\quad + (1/2) \int_{-L}^0 [(c_1^T b_1)^2 + |c_1^T b_1 c_2^T b_2|] [f^2(\sigma(t)) - f^2(\sigma(t+s))] ds.
\end{aligned}$$

Integrating by parts the expression  $c_1^T b_1 \int_0^t a'(t-s) \int_s^t f(\sigma(u)) du ds f(\sigma(t))$  we obtain

$$\begin{aligned}
&- [a(t-s) \int_s^t c_1^T b_1 f(\sigma(u)) du] \Big|_0^t - \int_0^t a(t-s) c_1^T b_1 f(\sigma(s)) ds f(\sigma) \\
&= +a(t) \int_0^t c_1^T b_1 f(\sigma(u)) du f(\sigma) - \int_0^t a(t-s) c_1^T b_1 f(\sigma(s)) ds f(\sigma).
\end{aligned}$$

Taking into account that  $a'(t) \leq 0$ ,  $a(t-s) := -c_2^T e^{J(t-s-L)} b_2 \geq 0$ , is decreasing in  $[t-L, t]$ , we have  $a(t-s) \leq -c_2^T b_2 = |c_2^T b_2|$  and thus

$$\begin{aligned}
V' &\leq (1/2) \int_{t-L}^t [(c_1^T b_1)^2 - c_1^T b_1 a(t-s)] (f^2(\sigma(s)) + f^2(\sigma(t))) ds \\
&\quad + c_1^T b_1 r f^2(\sigma) + z c_2^T J e^{J(t-L)} y_0 \\
&\quad + (1/2) \int_{-L}^0 [(c_1^T b_1)^2 + |c_1^T b_1 c_2^T b_2|] [f^2(\sigma(t)) - f^2(\sigma(t+s))] ds \\
&\leq \{L[(c_1^T b_1)^2 + |c_1^T b_1 c_2^T b_2|] + c_1^T b_1 r\} f^2(\sigma) + z c_2^T J e^{J(t-L)} y_0.
\end{aligned}$$

The first term on that last line is negative, while the last term can be handled as before by defining  $W$  in terms of  $V$ . The conclusion will now follow as before.

Now we look briefly at the case in which  $\sigma$  is delayed in all the connections. It actually can be handled the same way as the problem just solved, except that  $L$  is replaced by  $2L$ .

Consider the system

$$\begin{aligned}x' &= b_1 f(\sigma(t-L)) \\y' &= Jy + b_2 f(\sigma(t-L)) \\ \sigma' &= c_1^T x(t-L) + c_2 y(t-L) - r f(\sigma)\end{aligned}$$

where the characteristic roots of  $J$  have negative real parts.

As before, we write

$$\begin{aligned}x(t) &= x_0 + b_1 \int_0^t f(\sigma(s-L)) ds \\y(t) &= e^{Jt} y_0 + \int_0^t e^{J(t-s)} b_2 f(\sigma(s-L)) ds \\ \sigma'(t) &= c_1^T [x_0 + b_1 \int_0^{t-L} f(\sigma(s-L)) ds] - r f(\sigma) \\ &\quad + c_2^T [e^{J(t-L)} y_0 + \int_0^{t-L} e^{J(t-s-L)} b_2 f(\sigma(s-L)) ds].\end{aligned}$$

Rewriting the last equation

$$\begin{aligned}\sigma'(t) &= c_1^T [x_0 + b_1 \int_{-L}^0 f(\sigma(s)) ds + b_1 \int_0^{t-2L} f(\sigma(s)) ds] - r f(\sigma) \\ &\quad + c_2^T [e^{J(t-L)} y_0 + \int_{-L}^0 e^{J(t-s-2L)} b_2 f(\sigma(s)) ds \\ &\quad + \int_0^{t-2L} e^{J(t-s-2L)} b_2^2 f(\sigma(s)) ds].\end{aligned}$$

We need initial conditions of the form  $x_0, y_0$ , and  $\psi : [-L, 0] \rightarrow R$  which is an initial function for  $\sigma(t)$ . In the above integrals from  $-L$  to  $0$  we will write  $\sigma(s) = \psi(s)$ . Then this is just our old problem with

$$x_0 \text{ replaced by } x_0 + b_1 \int_{-L}^0 f(\psi(s)) ds$$

$$c_2^T e^{Jt} y_0 \text{ replaced by } c_2^T [e^{J(t-L)} y_0 + e^{J(t-2L)} \int_{-L}^0 e^{-Js} b_2 f(\psi(s)) ds].$$

We readily prove the following result.

**THEOREM 5.3.** Let us select

$$c_1^T b_1 < 0 \quad \text{and} \quad 2L(|c_1^T b_1| + |c_2^T b_2|) < r$$

and suppose that

$$a(t) = -c_2^T e^{J(t-2L)} b_2 \geq 0, \quad a' \leq 0, \quad \text{and} \quad a'' \geq 0. \quad (5.10)$$

Then each solution of (5.8) satisfies

$$\sigma(t), \sigma'(t), c^T x(t), y(t) \rightarrow 0$$

as  $t \rightarrow \infty$ .

## References

- [1] T. A. Burton, The generalized Liénard equation, *SIAM J. Control*, **3**, (1965), 223-230.
- [2] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, New York, (1985).
- [3] Jinde Cao, H. X. Li, and Daniel W. C. Ho, Synchronization criteria of Luré systems with time-delay feedback control, *Chaos, Solitons & Fractals*, **23:4**, (2005), 1285-1298.
- [4] N. N. Krasovskii, *Stability of Motion*, Stanford, (1963).
- [5] J. P. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, (1961).
- [6] S. Lefschetz, *Stability of Nonlinear Control Systems*, Academic Press, New York, (1965).
- [7] J. J. Levin, The asymptotic behavior of the solution of a Volterra equation, *Proc. Amer. Math. Soc.*, **14**, (1963), 534-541.
- [8] J. J. Levin and J. A. Nohel, On a system of integrodifferential equations occurring in reactor dynamics, *J. Math. Mech.*, **9**, (1960), 347-368.
- [9] A. Liénard, Étude des oscillations entretenues., *Rev. Gen de l'Electricit'e*, **23**, (1928), 901-946.
- [10] A. I. Lurie, *On some Nonlinear Problems in the Theory of Automatic Control*, *H. M. Stationery Office*, London, (1951).
- [11] J. Moser, On nonoscillating networks, *Quart. Appl. Math.*, **25**, (1967), 1-9.
- [12] A. S. Somolinos, On the condition  $c^T A^{-1}b + r > 0$  in the Lurie Problem, *Proc. Amer. Math. Soc.*, **47**, (1975) 432-441.
- [13] V. Volterra, Sur la théorie mathématique des phénomènes héréditaires, *J. Math. Pures Appl.*, **7**, (1928), 249-298.
- [14] B. Zhang, On the retarded Liénard equation, *Proc. Amer. Math. Soc.*, **115**, (1992), 779-785.

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