FUNCTIONAL DIFFERENTIAL EQUATIONS AND JENSEN'S INEQUALITY

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1 Introduction

This paper is concerned with stability and boundededness properties of the functional differential equation

$$x'(t) = F(t, x_t) \tag{1}$$

where $x_t(s) = x(t+s)$ for $-h \le s \le 0$ and h is a fixed positive constant. The equation is investigated by means of Liapunov's direct method.

In this discussion, $(C, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-h, 0] \to R^n$, $\|\phi\| = \sup_{-h \le s \le 0} |\phi(s)|$, and $|\cdot|$ is any convenient norm in R^n. The symbol $|||\cdot|||$ is used to

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denote the L^2-norm. For a positive constant H, by C_H we denote the subset of C for which $\|\phi\| < H$.

It is supposed that $F : [0,\infty) \times C_H \to R^n$, that F is continuous, and that F takes bounded sets into bounded sets. It is then known that if $t_0 \geq 0$ and $\phi \in C_H$ then there is a solution $x(t_0, \phi)$ satisfying (1) on an interval $[t_0, t_0 + \alpha)$ with $x_{t_0}(t_0, \phi) = \phi$, and with value at t denoted by $x(t, t_0, \phi)$. Moreover, if there is an $H_1 < H$ and if $|x(t, t_0, \phi)| \leq H_1$ for all $t \geq t_0$ for which $x(t_0, \phi)$ can be defined, then $\alpha = \infty$.

Throughout this paper we work with wedges, denoted by W_i, which are continuous functions from $[0,\infty) \to [0,\infty)$, which are strictly increasing, and which satisfy $W_i(0) = 0$. These wedges are related to properties of continuous scalar functionals (called Liapunov functionals) $V : [0,\infty) \times C_H \to [0,\infty)$ which are differentiated along solutions of (1) by the relation

$$V'_{(1)}(t, \phi) = \limsup_{\delta \to 0^+} [V(t + \delta, x_{t+\delta}(t, \phi)) - V(t, \phi)]/\delta\,.$$

Detailed consequences of this derivative are discussed in ([2],[6],[7],[11]). Those consequences are concerned with the following properties of (1).

DEFINITION 1. Let $F(t, 0) = 0$ so that $x = 0$ is a solution of (1).

(a) The zero solution of (1) is *stable* if for each $\epsilon > 0$ and $t_0 \geq 0$ there exists $\delta > 0$ such that $[\phi \in C_\delta,\ t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \epsilon$.

(b) The zero solution of (1) is *uniformly stable* (U.S.) if it is stable and if δ is independent of t_0.

(c) The zero solution of (1) is *asymptotically stable* (A.S.) if it is stable and if for each $t_0 \geq 0$ there is a $\gamma > 0$ such that $\phi \in C_\gamma$ implies that $|x(t, t_0, \phi)| \to 0$ as $t \to \infty$.

(d) The zero solution of (1) is *uniformly asymptotically stable* (U.A.S.) if it is U.S. and if there is an $\gamma > 0$ and for each $\mu > 0$ there is a $T > 0$ such that $[t_0 \geq 0,\ \phi \in C_\gamma,\ t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \mu$.

The following result is the standard theorem for (1).

THEOREM 0. Let $V : [0, \infty) \times C_H \to [0, \infty)$ be continuous.

(a) If $W_1(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$, and $V'_{(1)}(t, \phi) \leq 0$, then $x = 0$ is stable.

(b) If $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ and $V'_{(1)}(t, \phi) \leq 0$, then $x = 0$ is U.S.

(c) If $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$, $V'_{(1)}(t, \phi) \leq -W_3(|\phi(0)|)$, and if $|F(t, \phi)|$ is bounded for ϕ bounded, then $x = 0$ is U.A.S.

(d) If $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)|) + W_3(|||\phi|||)$ and $V'_{(1)}(t, \phi) \leq -W_4(|\phi(0)|)$, then $x = 0$ is U.A.S.

So frequently in applications a functional V is constructed with numerous properties similar (but different from) those listed in Theorem 0. It is then of interest to find alternate properties which will imply some type of stability. In this paper we show some effective ways of doing that using Jensen's inequality. The discussion here closely follows that of Natanson [9; pp. 36–46].

DEFINITION 2. Let $G : [a, b] \to (-\infty, \infty)$ with

$$G\left([t_1 + t_2]/2\right) \leq [G(t_1) + G(t_2)]/2$$

for any $t_1, t_2 \in [a,b]$, then G is *convex downward.*

LEMMA 1. If $f : [a,b] \to (-\infty, \infty)$ is increasing, then

$$F(t) = \int_a^t f(u)du$$

is convex downward.

We note, in particular, that if $W(r)$ is a wedge then $W_1(r) = \int\limits_0^r W(s)ds$ is a wedge and that on $[0,1]$ then $W_1(r) \le W(r)$. This means that for any local result which we state with

$$V'_{(1)}(t,\phi) \le -W(|\phi(0)|)$$

it is no loss of generality to assume that W is convex downward.

THEOREM 2 (JENSEN). Let $\Phi : (-\infty,\infty) \to (-\infty,\infty)$ be continuous and convex downward. If f and p are continuous on $[a,b]$ with $p(t) \ge 0$ and $\int\limits_a^b p(t)dt > 0$, then

$$\Phi\left[\int\limits_a^b f(t)p(t)dt \Big/ \int\limits_a^b p(t)dt\right] \le \int\limits_a^b \Phi(f(t))p(t)dt \Big/ \int_a^b p(t)dt\,.$$

Throughout this paper we will apply this inequality to wedges; thus it suffices to regard $\Phi : [0,\infty) \to [0,\infty)$.

The following type of function plays a central role with Jensen's inequality and, hence, is called a J-function.

DEFINITION 3. A continuous function $\eta : [0,\infty) \to [0,\infty)$ is said to be a J-function if η is non-increasing, $\eta \notin L^1[0,\infty)$, and for each $h > 0$ there is an $M > 0$ with $\int\limits_{t-h}^t \eta(s)ds \le M\eta(t)$ for $h \le t < \infty$.

The function defined by $\eta(t) = 1/(t+1)$ is a J-function.

In the way of notation we remark that when a function is written without its argument, then that argument is t.

Parts (a), (b), and (c) of Theorem 0 constitute the classical result for (1) which stood from Krasovskii's formulation in 1956 (cf. [8; pp. 152–157]) until 1978 when part (d) was proved by Burton [1]. While the upper bound on V in part (d) is more stringent than the one used in part (c), the requirement in (c) that $|F(t,\phi)|$ be bounded is considered by most investigators to be entirely unacceptable and one of the main thrusts of investigators has been to eliminate that type of condition. A counterpart for part (d) of Theorem 0 has been obtained by Wen [10] using a Razumikhin technique. For a summary, see [4].

2 Asymptotic Stability

Our first results focus on relations which are *variants* of $V'_{(1)}(t,\phi) \le -\delta|F(t,\phi)|$, $\delta > 0$. This means that a solution of (1) satisfies

$$V(t,x_t) \le V(t_0,x_{t_0}) - \delta(\text{Arc length } x(t)).$$

While this appears to be a strong condition, with the aid of Jensen's inequality we show that the net result can frequently be realized. This leads us to the scalar equation

$$x' = a(t)x(t) + b(t)x(t-h)$$

in which we show that if (among other conditions) we have $a(t) + b(t+h) \le -\beta < 0$ for all t, then $x(t) \to 0$ as $t \to \infty$; in fact, $a(t)$ and $b(t)$ can change sign.

The results frequently require U.S., which follows from Theorem 0(b), but examples show that it is sometimes prudent to give a separate set of conditions for the U.S.

THEOREM 1. Let $V : R_+ \times C_H \to [0,\infty)$ and $\eta : [0,\infty) \to [0,\infty)$ both be continuous with

(i) $W_1(|x(t)|) \le V(t,x_t)$,

(ii) $V'_{(1)}(t, x_t) \leq -\eta(t)[W_2(|x|) + W_3(|x'|)]$,

(iii) W_3 convex downward,

(iv) η a J-function, and

(v) $x = 0$ U.S.

Then $x = 0$ is A.S.

PROOF. By (v), there is a $\gamma > 0$ such that $[t_0 \geq 0,\ \phi \in C_\gamma,\ t \geq t_0]$ imply that $|x(t, t_0, \phi)| < H$. Suppose that for some such (t_0, ϕ), the solution $x(t) = x(t, t_0, \phi) \not\to 0$ as $t \to \infty$. By the uniform stability there is an $\epsilon > 0$ and an r_i in each interval $I_i = [t_0 + ih, t_0 + (i+1)h]$ with $|x(r_i)| \geq \epsilon$. If $t \in I_{n+1}$, then

$$\begin{aligned} V(t, x_t) \leq V(t_0, \phi) &- \sum_{i=0}^{n} \int_{I_i} \eta(s) W_2(|x(s)|) ds \\ &- \sum_{i=0}^{n} \int_{I_i} \eta(s) W_3(|x'(s)|) ds. \end{aligned}$$

On each I_i either $|x(t)| \geq \epsilon/2$ for every t in I_i or there is an s_i with $|x(s_i)| \leq \epsilon/2$ and, in the latter case, we have

$$\int_{I_i} |x'(s)| ds \geq \left| \int_{r_i}^{s_i} |x'(s)| ds \right| \geq |x(s_i) - x(r_i)| \geq \epsilon/2.$$

In the first case we have

$$\int_{I_i} \eta(s) W_2(|x(s)|) ds \geq W_2(\epsilon/2) \int_{I_i} \eta(s) ds,$$

whereas the second case yields

$$
\begin{aligned}
&\int_{I_i} \eta(s) W_3(|x'(s)|)ds \\
&\qquad \geq \left(\int_{I_i} \eta(s)ds\right) W_3\left(\left[\int_{I_i} |\eta(s)x'(s)|ds\right] \Big/ \left[\int_{I_i} \eta(s)ds\right]\right) \\
&\qquad \geq \left(\int_{I_i} \eta(s)ds\right) W_3\Big(\eta(t_0 + (i+1)h)\Big) \left[\int_{I_i} |x'(s)|ds\right] \Big/ \left[\int_{I_i} \eta(s)ds\right] \\
&\qquad \geq \left(\int_{I_i} \eta(s)ds\right) W_3\left[(1/M) \int_{I_i} |x'(s)|ds\right] \\
&\qquad \geq W_3(\epsilon/2M) \int_{I_i} \eta(s)ds.
\end{aligned}
$$

If $J = \min[W_2(\epsilon/2), W_3(\epsilon/2M)]$, then

$$0 \leq V(t, x_t) \leq V(t_0, \phi) - J \sum_{i=0}^{n} \int_{I_i} \eta(s)ds \to -\infty,$$

a contradiction. This completes the proof.

EXAMPLE A. Busenberg and Cooke [5] consider the scalar equation

(A1) $$x' = b(t)x(t-h) - c(t)x(t)$$

with $b, c : [0, \infty) \to (-\infty, \infty)$ continuous. They assume that for each $\eta > 0$ there exists $\tau > 0$ such that

(A2) $$\int_t^{t+\tau} |b(s)|ds < \eta \text{ for all } t \geq 0$$

so that

(A3) $$\int_{-h}^{0} |b(t+h+\theta)|d\theta \leq B \text{ for some } B \text{ and all } t \geq 0$$

and that for some $a > 0$ and $q > 0$ then

(A4) $$2c(t) - a|b(t)| - |b(t+h)|/a \geq q \text{ for } t \geq 0.$$

They conclude U.A.S.

Condition (A4) is not transparent. It seems to ask (very roughly) that $c(t) \geq q/2$, that $c(t) > |b(t)|$, and that $c(t) > |b(t+h)|$.

We ask instead that there exist a number $a > 1$ with

$$\text{(A5)} \qquad c(t) > a|b(t+h)|$$

and that there exist an $\eta \leq 1$ satisfying Theorem 1 with

$$\text{(A6)} \qquad c(t) \geq \eta(t).$$

Our conclusion then is only A.S., but we note that $c(t)$ may tend to 0 as $t \to \infty$.

To this end we define $\overline{a} = (a+1)/2$ and

$$V(t, x_t) = |x| + \overline{a} \int_{t-h}^{t} |b(u+h)|\,|x(u)|du$$

so that

$$\begin{aligned} V'(t, x_t) &\leq |b(t)x(t-h)| - c(t)|x(t)| \\ &\qquad + \overline{a}|b(t+h)|\,|x| - \overline{a}|b(t)|\,|x(t-h)| \\ &= (-\overline{a}+1)|b(t)|\,|x(t-h)| + \big[-c(t) + \overline{a}|b(t+h)|\big]|x| \\ &\leq c(t)\left[-1 + \overline{a}\big\{|b(t+h)|/c(t)\big\}\right]|x| \\ &\leq c(t)[-1 + (\overline{a}/a)]|x| \stackrel{\text{def}}{=} -\delta c(t)|x|. \end{aligned}$$

Next, note that

$$V'(t-h, x_{t-h}) \leq -\delta c(t-h)|x(t-h)|$$

so that if we define

$$H(t, x_t, x_{t-h}) = V(t, x_t) + V(t-h, x_{t-h})$$

we have

$$\begin{aligned} H' &\le -\delta c(t)|x(t)| - \delta c(t-h)|x(t-h)| \\ &\le -(\delta/2)c(t)|x(t)| - (\delta/2)c(t)|x(t)| - \delta|b(t)x(t-h)| \\ &\le -(\delta/2)\eta(t)\big[|x(t)| + |x'(t)|\big] \end{aligned}$$

because $\eta(t) \le 1$.

REMARK. If (A2) holds we have U.S. If, in addition, $c(t) \ge c_0 > 0$, then we have

$$W_1(|x|) \le H(t, x_t, x_{t-h}) \le W_2(|x|) + W_3(\|x_t\|) + W_4(\|x_{t-h}\|)$$

and

$$H' \le -\delta\big[|x| + |x'|\big].$$

It is then trivial to show U.A.S.

The following concept was introduced in [3].

DEFINITION 4. A measurable function $\eta : R_+ \to R_+$ is said to be uniformly integrally positive with parameter h (UIP(h)) if there exists $\delta > 0$ with $\int\limits_{t-h}^{t} \eta(s)ds \ge \delta$ for $t \ge h$.

THEOREM 2. Let $V : R_+ \times C_H \to [0, \infty)$ and let $\eta_1, \eta_2 : R_+ \to R_+$ where $\int\limits_{0}^{\infty} \eta_1(s)ds = \infty$ and η_2 is UIP(h). If

(i) $x = 0$ is U.S. and

(ii) $$V'_{(1)}(t, x_t) \le -\eta_1(t)\left\{W_1\left(\int_{t-h}^{t} |F(s, x_s)|ds\right) + W_2\left(\int_{t-h}^{t} \eta_2(s)W_3(|x(s)|)ds\right)\right\},$$

then $x = 0$ is A.S.

PROOF. Let $x(t)$ be a solution of (1) on $[t_0, \infty)$, $|x(t)| < H$, and suppose that $|x(t)| \nrightarrow 0$ as $t \to \infty$. Then there is an $\epsilon > 0$ and $\{t_n\} \uparrow \infty$ such that $|x(t_n)| \geq \epsilon$. For the $\epsilon > 0$ there exists $\delta > 0$ such that $[\phi \in C_\delta,\ t \geq t_1]$ imply that $|x(t, t_1, \phi)| < \epsilon$. Thus, on each interval $[t-h, t]$ there is a t_* with $|x(t_*)| \geq \delta$. There are two possibilities:

(a) $|x(s)| \geq \delta/2$ for all $s \in [t-h, t]$, or

(b) $|x(s_1)| < \delta/2$ at some $s_1 \in [t-h, t]$.

If (a) holds, then there exists $\beta > 0$ with

$$\int_{t-h}^{t} \eta_2(s) W_3(\delta/2) ds \geq \beta.$$

If (b) holds, then $\int\limits_{t-h}^{t} |F(s, x_s)| ds \geq \delta/2$.

In any case, for every t we have

$$V'_{(1)}(t, x_t) \leq -\eta_1(t) \min\left[W_1(\delta/2), W_2(\beta)\right]$$

so that $V(t, x_t) \to -\infty$ as $t \to \infty$, a contradiction. This completes the proof.

REMARK. The next example seems significant. Using standard theory it is some chore to show that solutions of $x' = -ax + bx(t-h)$ tend to zero even when a and b are constants with $-a + b < 0$. Using Theorem 2 we allow $a(t)$ and $b(t)$ to both change sign so long as $-a(t) + b(t+h) \leq -\beta < 0$, plus other conditions.

EXAMPLE B. Consider the scalar equation

$$\text{(B1)} \qquad x'(t) = -a(t)x(t) + b(t)x(t-h)$$

with $a, b : [-h, \infty) \to R$ being continuous. We wish to use $b(t)$ to help stabilize the equation. It is assumed that there is an $\alpha > 0$ such that

$$\text{(B2)} \qquad 2[b(t+h) - a(t)] + |b(t+h) - a(t)| \int_{t-h}^{t} |b(u+h)| du$$
$$+\alpha h \lambda(t) \stackrel{\text{def}}{=} \Gamma(t) \le 0$$

where $\lambda(t) = \max\big[|a(t)|,\ |b(t+h)|\big]$ is UIP(h),

$$\text{(B3)} \qquad \alpha - |b(t+h) - a(t)| \stackrel{\text{def}}{=} \eta_1(t) \ge 0,$$

$$\text{(B4)} \qquad \overline{\eta}(t) = \min[\eta_1(t),\ \eta_1(t-h)] \notin L^1[0,\infty),$$

$$\text{(B5)} \qquad 0 < \int_{t-h}^{t} |b(s+h)| ds \le K,\ 0 < \int_{t-h}^{t} |a(s)| ds \le K,$$

some $K > 0$.

Then U.S. implies A.S. If, in addition, $\Gamma(t) \le -\Gamma_0 < 0$, and if $-2a(t) + |b(t)| + |b(t+h)|$ is bounded above, then $x = 0$ is U.S.

PROOF. Write (B1) as

$$\text{(B1)}' \qquad x' = [-a(t) + b(t+h)]x - (d/dt) \int_{t-h}^{t} b(u+h)x(u)du$$

and define

$$V(t, x_t) = \left(x + \int_{t-h}^{t} b(u+h)x(u)du \right)^2$$
$$+ \alpha \int_{-h}^{0} \int_{t+s}^{t} \lambda(u)x^2(u)du\, ds$$

so that

$$
\begin{aligned}
V'(t,x_t) &= 2\left(x+\int_{t-h}^{t} b(u+h)x(u)du\right)[-a(t)+b(t+h)]x \\
&\quad +\alpha\int_{-h}^{0}\lambda(t)x^2(t)ds-\alpha\int_{-h}^{0}\lambda(t+s)x^2(t+s)ds \\
&\le 2[b(t+h)-a(t)]x^2+|b(t+h)-a(t)|\int_{t-h}^{t}|b(u+h)|du\,x^2 \\
&\quad +|b(t+h)-a(t)|\int_{t-h}^{t}|b(u+h)|x^2(u)du+\alpha h\lambda(t)x^2 \\
&\quad -\alpha\int_{t-h}^{t}\lambda(s)x^2(s)ds \\
&= \Gamma(t)x^2+|b(t+h)-a(t)|\int_{t-h}^{t}|b(u+h)|x^2(u)du \\
&\quad -\alpha\int_{t-h}^{t}\lambda(s)x^2(s)ds.
\end{aligned}
$$

First, we note that

$$
\text{(B6)} \qquad V'(t,x_t)\le -\eta_1(t)\int_{t-h}^{t}\lambda(s)x^2(s)ds
$$

and λ is UIP(h), so this is the term

$$
-\eta_1(t)W_2\left(\int_{t-h}^{t}\eta_2(s)W_3(|x(s)|)ds\right)
$$

of Theorem 2. Next, we see that

$$
V'(t,x_t)\le -\eta_1(t)\int_{t-h}^{t}|b(s+h)|x^2(s)ds
$$

so that by Jensen's inequality we have

$$
V'(t,x_t)\le -\left[\eta_1(t)/\int_{t-h}^{t}|b(s+h)|ds\right]\left[\int_{t-h}^{t}|b(s+h)x(s)|ds\right]^2.
$$

This means that

$$
\text{(B7)} \qquad V'(t-h,x_{t-h})\le -[\eta_1(t-h)/K]\left[\int_{t-h}^{t}|b(s)x(s-h)|ds\right]^2.
$$

Finally, $V'(t,x_t) \le -\eta_1(t) \int\limits_{t-h}^{t} |a(s)|x^2(s)ds$ so that by Jensen's inequality

$$V'(t,x_t) \le -[\eta_1(t)/K]\left[\int_{t-h}^{t} |a(s)x(s)|ds\right]^2.$$

If we define

$$\Omega(t,x_t,x_{t-h}) = V(t,x_t)(1+K) + KV(t-h,x_{t-h}),$$

then for

$$\overline{\eta}(t) = \min[\eta_1(t), \eta_1(t-h)] \notin L^1[0,\infty)$$

the conditions of Theorem 2 are satisfied with $W_1(u) = \frac{1}{2}u^2$.

Next, we show U.S. Define

$$H(t,x_t) = x^2 + \int_{t-h}^{t} |b(u+h)|x^2(u)du$$

so that

$$\begin{aligned} H'(t,x_t) &= -2a(t)x^2 + 2|b(t)|\,|x(t-h)| \\ &\qquad + |b(t+h)|x^2 - |b(t)|x^2(t-h) \\ &\le \big[-2a(t) + |b(t)| + |b(t+h)|\big]x^2 \le Jx^2 \end{aligned}$$

for some $J > 0$. Since $V'(t,x_t) \le -\Gamma_0 x^2$, then for

$$U(t,x_t) = V(t,x_t) + (\Gamma_0/2J)H(t,x_t)$$

we have

$$U'(t,x_t) \le -\Gamma_0 x^2 + (\Gamma_0/2)x^2 \le 0.$$

Evidently there are W_i with

$$W_4(|x(t)|) \le U(t,x_t) \le W_5(\|x_t\|)$$

and this implies U.S.

EXAMPLE OF EXAMPLE B. Let

$$(\mathrm{B1})' \qquad x' = b(t)x(t-h)$$

with $b(t) < 0$ and continuous. Suppose there is an $\alpha > 0$ with

$$(\mathrm{B2})' \qquad |b(t+h)|\left[-2+\alpha h+\int_{t-h}^{t}|b(u+h)|du\right] \le -\Gamma_0 < 0,$$

$$(\mathrm{B3})' \qquad 0 \le \alpha - |b(t+h)| - \eta_1(t),$$

$$(\mathrm{B4})' \qquad \overline{\eta}(t) = \min[\eta_1(t), \eta_1(t-h)] \notin L^1[0,\infty),$$

and

$$(\mathrm{B5})' \qquad -b \in \mathrm{UIP}(h),\ \int_{t-h}^{t}|b(s+h)|ds \le K,\ K > 0.$$

Then $x = 0$ is U.S. and A.S.

The conditions (B1)–(B4) are readily verified. Moreover, it is not hard to see that when $a(t) \equiv 0$ then the requirement in (B5) of

$$0 < \int_{t-h}^{t}|a(s)|ds$$

is not needed.

In Example B the size of h plays a significant role. In the next example, the condition labelled (B2) is simplified. As a result, it is easier to see that when functions a and b are bounded and satisfy the condition $-a(t) + b(t+h) \le -\beta < 0$, solutions may tend to zero for sufficiently small h even when each function is allowed to change its sign.

EXAMPLE C. Consider again the scalar equation

$$(\mathrm{C1}) \qquad x'(t) = -a(t)x(t) + b(t)x(t-h),$$

where $a, b : [-h,\infty) \to R$ are continuous and λ denotes the UIP(h) function that was defined in Example B. Assume α is a positive constant such that

(C2) $$-a(t)+b(t+h)+\alpha h\lambda(t) \overset{\text{def}}{=} Q(t)\le 0$$

and

(C3) $$\alpha - K|b(t+h)-a(t)| \overset{\text{def}}{=} \eta_1(t)\ge 0,$$

where K again represents the upper bound on the two integrals in Example B. Assume $\overline{\eta}(t)=\min[\eta_1(t),\eta_1(t-h)]\notin L^1[0,\infty)$. Then U.S. implies A.S. Furthermore, if there is a positive constant Q_0 such that $Q(t)\le -Q_0$ and if $-2a(t)+|b(t)|+|b(t+h)|$ is bounded above, then $x=0$ is U.S.

PROOF. Define the functional $V(t,x_t)$ exactly as in the proof of Example B. Then differentiation yields

$$\begin{aligned}V'(t,x_t)&\le 2[b(t+h)-a(t)]x^2(t)\\ &\qquad + |b(t+h)-a(t)|\left\{x^2(t)+\left(\int_{t-h}^{t}|b(u+h)|\,|x(u)|du\right)^2\right\}\\ &\qquad + \alpha h\lambda(t)x^2(t)-\alpha\int_{t-h}^{t}\lambda(s)x^2(s)ds\\ &=\Gamma(t)x^2(t)+|b(t+h)-a(t)|\left[\int_{t-h}^{t}|b(u+h)|\,|x(u)|du\right]^2\\ &\qquad -\alpha\int_{t-h}^{t}\lambda(s)x^2(s)ds\end{aligned}$$

where $\Gamma(t)=2[b(t+h)-a(t)]+|b(t+h)-a(t)|+\alpha h\lambda(t)$. By Jensen's inequality,

$$\begin{aligned}V'(t,x_t)&\le \Gamma(t)x^2(t)\\ &\qquad + |b(t+h)-a(t)|\int_{t-h}^{t}|b(u+h)|du\int_{t-h}^{t}|b(u+h)|x^2(u)du\\ &\qquad -\alpha\int_{t-h}^{t}\lambda(s)x^2(s)ds.\end{aligned}$$

We note that $\Gamma(t)\le-\delta$ if and only if $Q(t)\le-\delta$, for $\delta\ge 0$. Using $\Gamma(t)\le 0$, the integral bounds, and (C3), we find

$$\text{(C4)} \qquad V'(t, x_t) \le -\eta_1(t) \int_{t-h}^{t} \lambda(s) x^2(s) ds.$$

Next, we see that

$$\begin{aligned} V'(t, x_t) &\le |b(t+h) - a(t)| \left[\int_{t-h}^{t} |b(u+h)x(u)| du \right]^2 \\ &\quad - \alpha \int_{t-h}^{t} |b(s+h)| x^2(s) ds \\ &\le -[\eta_1(t)/K] \left[\int_{t-h}^{t} |b(s+h)x(s)| ds \right]^2, \end{aligned}$$

as

$$\int_{t-h}^{t} |b(u+h)| x^2(u) du \ge [1/K] \left[\int_{t-h}^{t} |b(u+h)x(u)| du \right]^2$$

by the integral bounds and Jensen's inequality. This implies

$$\text{(C5)} \qquad V'(t-h, x_{t-h}) \le -[\eta_1(t-h)/K] \left[\int_{t-h}^{t} |b(s)x(s-h)| ds \right]^2.$$

By (C4),

$$V'(t, x_t) \le -\eta_1(t) \int_{t-h}^{t} |a(s)| x^2(s) ds$$

which, upon applying Jensen's inequality again, yields

$$\text{(C6)} \qquad V'(t, x_t) \le -[\eta_1(t)/K] \left[\int_{t-h}^{t} |a(s)x(s)| ds \right]^2.$$

Since $Q(t) \le -Q_0$ implies that $\Gamma(t) \le -Q_0$, it follows from the inequalities (C4), (C5), (C6) that the rest of this proof proceeds just like Example B's, the only notable change being that the constant Q_0 replaces Γ_0 in the definition of the functional $U(t, x_t)$.

EXAMPLE OF EXAMPLE C. Let $b = -4$, $a(t) = -1 + 2\sin t$, $\lambda(t) = 4$, $K = 4h$, $\alpha = 20h$, and $h \le 1/9$. Then $x = 0$ is U.S. and A.S.

PROOF. Since $B(t+h) - a(t) = -3 - 2\sin t$, $\alpha - K|b(t+h) - a(t)| = 8h(1 - \sin t) = \eta_1(t) \geq 0$, $\overline{\eta}(t) = \min[\eta_1(t), \eta_1(t-h)] \notin L^1[0,\infty)$. Then (C2) is $-3 - 2\sin t + 80h^2 = Q(t)$. If $h \leq 1/9$, then $Q(t) \leq -Q_0$, where $Q_0 = 1/81$. All the conditions in Example C are satisfied.

The ideas in Theorem 2 are very useful in locating limit sets, as we now illustrate.

THEOREM 3. Let $V : R_+ \times C_H \to [0,\infty)$ be continuous and satisfy

$$V'_{(1)}(t, x_t) \leq -\int_{t-h}^{t} W(|x'_i(s)|)ds$$

(where $x = (x_1, \ldots, x_n)$) for some i. Then any solution $x(t)$ satisfying $|x(t)| < H$ on $[t_0, \infty)$ also satisfies

$$\sup_{0\leq\theta\leq h} |x_i(t) - x_i(t-\theta)| \to 0 \ \text{ as } \ t \to \infty$$

and $\int\limits_{t-h}^{t} |x'_i(s)|ds \to 0$ as $t \to \infty$. Here, W is convex downward.

PROOF. If the theorem is false then there is a solution $x(t)$, there is an $\epsilon > 0$, and there is a sequence $\{t_n\} \uparrow +\infty$ with

$$\int_{t_n-h}^{t_n} |x'_i(s)|ds \geq \epsilon$$

for some i. Moreover, it is shown in [3] that there is a sequence $\{\overline{t}_n\} \uparrow \infty$, a $\delta > 0$, and an $h_1 > 0$ with

$$\int_{t-h}^{t} |x'_i(u)|du \geq \delta$$

for $\overline{t}_n \leq t \leq t_n + h_1$. This means that

$$\begin{aligned} V'(t, x_t) &\leq -hW\left([1/h]\int_{t-h}^{t} |x'_i(u)|du\right) \\ &\leq -hW([1/h]\delta) \end{aligned}$$

on $[\,\overline{t}_n, \overline{t}_n + h_1]$. Thus, $V(t, x_t) \to -\infty$ as $t \to \infty$, a contradiction. Hence

$$\sup_{0\leq\theta\leq h} |x_i(t) - x_i(t-\theta)| \to 0 \text{ as } t \to \infty$$

as required.

EXAMPLE D. Krasovskii [8; p. 173] considers a system

$$\text{(D1)} \qquad \begin{cases} x'(t) = y(t) \\ y'(t) = -\phi(y(t),t) - f(x(t)) + \int_{-h(t)}^{0} f^*(x(t+s))y(t+s)ds \end{cases}$$

where

$$\text{(D2)} \qquad [\phi(y,t)/y] \geq b > 0 \text{ for } y \neq 0,$$

$$\text{(D3)} \qquad 0 \leq h(t) \leq h,\ [f(x)/x] > a > 0 \text{ for } x \neq 0,$$

and

$$\text{(D4)} \qquad f^*(x) = (d/dx)f(x) \text{ satisfies } |f^*(x)| < N.$$

Consider the functional

$$\text{(D5)} \qquad V(x_t, y_t) = 2\int_0^x f(s)ds + y^2(t) + [b/h]\int_{-h}^0\int_u^0 y^2(t+s)ds\,du.$$

Then

$$\text{(D6)} \qquad V'(x_t, y_t) \leq -\gamma\left[hy^2(t) + \int_{t-h}^t y^2(u)du\right],$$

where $\gamma > 0$ for $h < b/N$. Note that with $x_i = x$, (D6) satisfies the conditions of Theorem 3. This means that:

$$\text{(i)} \quad \int_{-h(t)}^0 f^*(x(t+s))y(t+s)ds \to 0 \text{ as } t \to \infty,$$

$$\text{(ii)} \quad \int_{-h}^0\int_u^0 y^2(t+s)ds\,du \leq h\int_{-h}^0 y^2(t+s)ds \to 0 \text{ as } t \to \infty,$$

and for any $L > 0$ then

$$\text{(iii)} \quad \sup_{-L\leq\theta\leq 0} |x(t) - x(t+\theta)| \to 0 \text{ as } t \to \infty.$$

Since $V' \leq 0$ we see that

$$\text{(iv)} \quad 2\int_0^{x(t)} f(s)ds + y^2(t) \to \text{constant as } t \to \infty.$$

If $\phi(t,y)$ is bounded for y bounded, then it would follow readily that $x(t)$ approaches a constant and $y(t)$ approaches zero as $t \to \infty$. It seems unclear that this might be derived from (iii) and (iv).

The proof of Lemma 2 is a simple exercise.

LEMMA 2. Let η be UIP(δ) for some $\delta > 0$. Then the zero solution of the ordinary differential equation

$$(*) \quad |x(t)|' = -\eta(t) W\big(|x(t)|/2\big)$$

is U.A.S.

THEOREM 4. Let $D, V : [0,\infty) \times C_H \to [0,\infty)$ be continuous with V locally Lipschitz in ϕ such that

$$\text{(i)} \quad W_0(|x|) \le V(t, x_t) \le W_1(|x|) + W_2(D(t, x_t)),$$

$$\text{(ii)} \quad V'_{(1)}(t, x_t) \le -\eta(t)\big[W_3(|x|) + W_4(D(t, x_t))\big],$$

$$\text{(iii)} \quad D(t, x_t) \le W_5(\|x_t\|),$$

and

$$\text{(iv)} \quad \eta \text{ is UIP}(\delta), \text{ some } \delta > 0.$$

Then $x = 0$ is U.A.S.

PROOF. The U.S. follows from Theorem 0(b).

Let $x(t)$ be a solution of (1) on $[t_0, \infty)$ with $|x(t)| < H$. If

$$W(r) = \min\big[W_3(W_1^{-1}(r)), W_4(W_2^{-1}(r))\big],$$

then

$$V'_{(1)}(t, x_t) \le -\eta(t) W(V(t, x_t)/2),$$

where, by renaming, we assume that W is convex downward. By Lemma 2 and a comparison theorem, the zero solution of (1) is U.A.S.

In the same way, the following result may be proved.

THEOREM 5. Let $D, V : [0,\infty) \times C_H \to [0,\infty)$ and $\eta : [0,\infty) \to [0,\infty)$ be continuous with

$$\text{(i)} \quad W_0(|x|) \le V(t, x_t) \le W_1(|x|) + W_2(D(t, x_t)),$$

$$\text{(ii)} \quad V'_{(1)}(t, x_t) \le -\eta(t)\big[W_3(|x|) + W_4(D(t, x_t))\big],$$

and

$$\text{(iii)} \quad \int_0^\infty \eta(t)dt = \infty.$$

If $x(t)$ is a solution of (1) on $[t_0, \infty)$ with $|x(t)| < H$, then $|x(t)| \to 0$ as $t \to \infty$.

THEOREM 6. Let $D, V : [0,\infty) \times C_H \to [0,\infty)$ be continuous and satisfy

$$\text{(i)} \quad 0 \le V(t, x_t) \le W_2(|x|) + W_3\bigg(\int_{t-h}^t D(s, x_s)ds\bigg)$$

and

$$\text{(ii)} \quad V'_{(1)}(t, x_t) \le W_4(|x|) - W_5(D(t, x_t)) \text{ where } W_5 \text{ is convex downward.}$$

If $x(t)$ is a solution of (1) on $[t_0, \infty)$ with $|x(t)| < H$, then $V(t, x_t) \to 0$ as $t \to \infty$.

PROOF. If $V(t, x_t) \nrightarrow 0$ as $t \to \infty$, then there exists $C > 0$ with $V(t, x_t) \ge C$ for $t \ge t_0$. Hence,

$$W_2(|x(t)|) + W_3\bigg(\int_{t-h}^t D(s, x_s)ds\bigg) \ge C \text{ for } t \ge t_0.$$

This implies that either:

$$\text{(a)} \quad |x(t)| \ge W_2^{-1}(C/2)$$

or

$$\text{(b)} \quad \int_{t-h}^{t} D(s,x_s)ds \geq W_3^{-1}(C/2)$$

for each $t \geq t_0$.

By Jensen's inequality

$$hW_5\left([1/h]\int_{t-h}^{t} D(s,x_s)ds\right) \leq \int_{t-h}^{t} W_5(D(s,x_s))ds$$

and so

$$\int_{t-h}^{t} W_5(D(s,x_s))ds \geq hW_5\big([1/h]W_3^{-1}(C/2)\big) \stackrel{\text{def}}{=} L > 0$$

in case (b) holds.

Let $E_1 = \{t \geq t_0 :\ \text{(a) holds}\}$ and

$$E_2 = [t_0,\infty) - E_1 \subset \{t \geq t_0 :\ \text{(b) holds}\}.$$

Suppose N is the positive integer such that

$$NL > V(t_0, x_{t_0}) \geq (N-1)L$$

and $\mu > 0$ is a number such that

$$\mu W_4(W_2^{-1}(C/2)) > V(t_0, x_{t_0}).$$

Let $T = Nh + \mu$ and consider the interval $I = [t_0, t_0 + T]$. Then one of the following cases must hold.

$$\text{(A)} \quad \text{measure } (E_1 \cap I) \geq \mu$$

or

$$\text{(B)} \quad \text{measure } (E_2 \cap I) \geq Nh.$$

If (A) is true, then

$$\begin{aligned}V(t_0+T, x_{t_0+T}) &\le V(t_0, x_{t_0}) - \int_{t_0}^{t_0+T} W_4(|x(s)|)ds \\ &\le V(t_0, x_{t_0}) - \int_{E_1\cap I} W_4(W_2^{-1}(C/2))ds \\ &\le V(t_0, x_{t_0}) - \mu W_4(W_2^{-1}(C/2)) < 0.\end{aligned}$$

If (B) is true then in $E_2 \cap I$ there must exist N points $t_1 < t_2 < \ldots < t_N$ with $t_1 \ge t_0$ and $t_j \ge t_{j-1} + h$ for $j = 2, 3, \ldots, N$. Hence

$$\begin{aligned}V(t_0+T, x_{t_0+T}) &\le V(t_0, x_{t_0}) - \int_{t_0}^{t_0+T} W_5(D(s, x_s))ds \\ &\le V(t_0, x_{t_0}) - \int_{E_2\cap I} W_5(D(s, x_s))ds \\ &\le V(t_0, x_{t_0}) - \sum_{j=1}^{N} \int_{t_j-h}^{t_j} W_5(D(s, x_s))ds \\ &\le V(t_0, x_{t_0}) - NL < 0.\end{aligned}$$

Thus, both (A) and (B) yield contradictions and so $V(t, x_t) \to 0$ as $t \to \infty$.

THEOREM 7. Let $V, D : [0, \infty) \times C_H \to [0, \infty)$ be continuous with

(i) $0 \le V(t, x_t) \le W_2(|x|) + W_3\left[\int_{t-h}^{t} D(s, x_s)ds\right]$

and

(ii) $V'_{(1)}(t, x_t) \le -\gamma(t)W_4(|x|)$,

where $\gamma : [0, \infty) \to [0, \infty)$ is a measurable function with the property that $\liminf_{t\to\infty} \int_t^{t+\xi} \gamma(s)ds > 0$ for each $\xi > 0$.

If $x(t)$ is a solution of (1) on $[t_0, \infty)$ with $|x(t)| < H$, then either

(a) $V(t, x_t) \to 0$ as $t \to \infty$

or for any $\delta > 0$

$$\text{(b)} \quad \int_{t-h-\delta}^{t} D(s, x_s)ds \geq M \text{ for some } M > 0$$

and all large t.

In particular, if we define

$$H(t, x(\cdot)) = V(t, x_t) + V(t-h, x_{t-h})$$

then

$$\text{(i)}' \quad 0 \leq H(t, x(\cdot)) \leq W_2(|x|) + W_2(|x(t-h)|) + 2W_3\left[\int_{t-2h}^{t} D(s, x_s)ds\right]$$

and

$$\text{(ii)}' \quad H'_{(1)}(t, (\cdot)) \leq -\gamma(t)W_4(|x|) - \gamma(t-h)W_4(|x(t-h)|)$$

so that either

$$\text{(a)}' \quad H(t, x(\cdot)) \to 0 \text{ as } t \to \infty$$

or

$$\text{(b)}' \quad \int_{t-2h}^{t} D(s, x_s)ds \geq M \text{ for all large } t.$$

PROOF. Let $x(t)$ be such a solution and suppose that $V(t, x_t) \nrightarrow 0$ as $t \to \infty$. Then $V(t, x_t) \geq C$ for some $C > 0$. Choose $\epsilon > 0$ so that $W_2(\epsilon) + W_3(\epsilon) = C$. We observe that $\int_{t-h}^{t} D(s, x_s)ds > \epsilon$ whenever $|x(t)| < \epsilon$ and that $\gamma(t)W_4(|x(t)|) \in L^1[0, \infty)$.

We claim that for each $\delta > 0$ there corresponds a $T \geq \delta + h$ such that $t \geq T$ implies the existence of a point $t_* \in [t - \delta, t]$ with $\int_{t_*-h}^{t_*} D(s, x_s)ds > \epsilon$. If this were not the case, then there would be infinitely many mutually disjoint intervals $[t_n - \delta, t_n]$, with $t_1 \geq \delta + h$

and $t_n \to \infty$, such that $|x(t)| \geq \epsilon$ for all $t \in [t_n - \delta, t_n]$. By a result referred to in [3, cf. the definition of integrally positive], $\int_I \gamma(s)ds = \infty$ where $I = \bigcup_{n=1}^{\infty} [t_n - \delta, t_n]$. Hence, $\int_0^\infty \gamma(s)W_4(|x(s)|)ds \geq \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n} \gamma(s)W_4(\epsilon)ds = \infty$, a contradiction.

Let $h > \delta$, say $\delta = h/4$. Then for $t \geq T$, $\int_{t-h-\delta}^{t} D(s, x_s)ds \geq \int_{t_*-h}^{t_*} D(s, x_s) > \epsilon$, which completes the proof.

EXAMPLE E. Consider the scalar equation

$$\text{(E1)} \qquad x' = b(t)x(t-h) - c(t)x(t)$$

with $b, c : [0, \infty) \to R$ continuous and assume

$$\text{(i)} \quad -\gamma(t) \overset{\text{def}}{=} -c(t) + |b(t+h)| < 0,$$

$$\text{(ii)} \quad \liminf_{t\to\infty} \int_t^{t+\xi} \gamma(s)ds > 0 \text{ for every } \xi > 0,$$

and suppose there is a function μ with

$$\text{(iii)} \quad \mu(t) \geq \int_{t-2h}^{t} b^2(u+h)du \text{ for } t \geq h.$$

If, in addition, $\gamma(t) \geq \overline{\gamma}_i \geq 0$ on $[2ih, 2(i+1)h]$ and

$$\text{(iv)} \quad \sum_{i=0}^{\infty} \overline{\gamma}_i / \mu(2(i+1)h) = \infty$$

then $x = 0$ is A.S.

PROOF. Define

$$V(t, x_t) = |x| + \int_{t-h}^{t} |b(u+h)|\,|x(u)|du$$

so that

$$
\begin{aligned}
V'(t,x_t) &\le -c(t)|x| + |b(t)|\,|x(t-h)| \\
&\qquad + |b(t+h)|\,|x| - |b(t)|\,|x(t-h)| = -\gamma(t)|x| \\
&\le -\gamma(t)|x|^2 \text{ if } |x| \le 1.
\end{aligned}
$$

Since $x = 0$ is stable, we may take $H = 1$ and have $|x(t)| < 1$.

Referring to Theorem 7, the conditions labelled (i) and (ii) are fulfilled. If $V(t,x_t) \nrightarrow 0$ as $t \to \infty$, then there is a constant $M > 0$ with

$$
M \le \int_{t-2h}^{t} |b(u+h)|\,|x(u)|du \le \left[\int_{t-2h}^{t} b^2(u+h)du \int_{t-2h}^{t} x^2(u)du\right]^{1/2}
$$

so that

$$
\int_{t-2h}^{t} x^2(u)du \ge M^2 \Big/ \int_{t-2h}^{t} b^2(u+h)du \ge M^2/\mu(t).
$$

If m and n are chosen so that $2mh \ge t_0$ and $n \ge m$, then for $t \ge 2(n+1)h$ we have

$$
\begin{aligned}
V(t,x_t) &\le V(t_0,x_{t_0}) - \sum_{i=m}^{n} \int_{2ih}^{2(i+1)h} \gamma(s)x^2(s)ds \\
&\le V(t_0,x_{t_0}) - \sum_{i=m}^{n} \overline{\gamma}_i \int_{2ih}^{2(i+1)h} x^2(s)ds \\
&\le V(t_0,x_{t_0}) - \sum_{i=m}^{n} \overline{\gamma}_i M^2/\mu(2(i+1)h) \to -\infty
\end{aligned}
$$

as $n \to \infty$. This is a contradiction and so $V(t,x_t)$ (hence, $|x(t)|$) tends to 0 as $t \to \infty$, completing the proof.

REMARK. Let $b(t) = \sqrt{t}$. Then

$$
\begin{aligned}
\int_{t-2h}^{t} b^2(u+h)du &= \int_{t-2h}^{t} (u+h)du \le (t+h)2h \\
&\stackrel{\text{def}}{=} \mu(t).
\end{aligned}
$$

Thus, if $\gamma(t) \geq 1$, then (ii) and (iv) hold. This means that $b(t)$ can be unbounded of order $\sqrt{t}$ and we can still conclude A.S.

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