A STABILITY THEORY FOR INTEGRAL EQUATIONS

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1. Introduction. In large measure, stability theory of differential equations centers around equilibrium points, either those occurring naturally in the equation or constructed by a change of variable. Thus, if we are interested in a stability theory for integral equations, then we need to decide just what will play the role of an equilibrium point. This is particularly important if we wish to employ Liapunov functions because they are constructed so as to be positive definite with respect to an equilibrium point.

In this paper we offer one choice for equilibrium points and we show that it is a good choice by developing a Liapunov theory around it and use it to obtain new results on limit sets for three problems of classical interest.

In particular, we study three forms of the integral equation

$$x(t) = a(t) - \int_{\alpha(t)}^{t} Q(t,s,x(s))\,ds \tag{1}$$

where $\alpha(t) \geq \alpha \geq -\infty$. We focus on functions which are analogous to equilibrium points of ordinary differential equations and obtain results, by way of Liapunov's direct method, concerning the long-time behavior of solutions.

DEF. 1. A pair of functions (ψ, Ψ), each mapping $[\alpha, \infty) \to R^n$ with $\alpha \leq 0$, is said to be a *near equilibrium* for (1) if

$$\Psi(t) := a(t) - \psi(t) - \int_{\alpha(t)}^{t} Q(t,s,\psi(s))\,ds \in L^1[0,\infty). \tag{2}$$

Thus, if (1) is perturbed by the L^1 function Ψ, then ψ is a solution of (1); in other words, ψ fails to be a solution of (1) by an amount of an L^1 function.

We will discuss the relation of a near equilibrium to the concept of an equilibrium point for an ordinary differential equation in a moment.

EXAMPLE 1.

(i) If $a \in L^1[0,\infty)$ and $Q(t,s,0) = 0$, then $\psi(t) = 0$ and $\Psi(t) = a(t)$ so (ψ, Ψ) is a near equilibrium for (1).

(ii) If $x(t) = a(t) + \int_{-\infty}^{t} C(t-s)x(s)\,ds$ where $a \in L^1[0,\infty)$ and $\int_0^\infty C(t)\,dt = 1$, then for every constant x_0, $\psi(t) = x_0$ and $\Psi(t) = a(t)$, so (ψ, Ψ) is a near equilibrium for this equation.

(iii) If $x(t) = a + a_1(t) + \int_{-\infty}^{t} C(t-s)x(s)\,ds$ where a is constant, $a_1 \in L^1[0,\infty)$, $\int_0^\infty C(t)\,dt = c \neq 1$, then for α defined by $\alpha(1-c) = a$, it follows that $\psi(t) = \alpha$ and $\Psi(t) = a_1(t)$ so (ψ, Ψ) is a near equilibrium.

(iv) If ψ is an L^1 solution of $x(t) = a(t) + \int_{-\infty}^{t} D(t,s)g(x(s))\,ds$ and if $E \in L^1$, then (ψ, Ψ) is a near equilibrium for

$$x(t) = a(t) + \int_{-\infty}^{t} D(t,s)g(x(s))\,ds + \int_0^t E(t-s)x(s)\,ds$$

where $\Psi(t) := \int_0^t E(t-s)\psi(s)\,ds$.

REMARK. In the work to follow we frequently ask that not only Ψ, but powers of Ψ be $L^1[0,\infty)$. A number of transformations may be used to achieve this. In the equation

$$x(t) = a(t) + \int_0^t C(t-s)x(s)\,ds$$

with a and c in $L^1[0,\infty)$ and $C(t) \to 0$ as $t \to \infty$, let $y = x - a(t)$ so that

$$y(t) = \int_0^t C(t-s)a(s)\,ds + \int_0^t C(t-s)y(s)\,ds.$$

The first term on the right is $L^1[0,\infty)$ and it tends to zero. Hence, all powers are $L^1[0,\infty)$.

For a given t_0 we require a continuous initial function $\varphi : [\alpha, t_0] \to R^n$ and seek a solution $x(t, t_0, \varphi)$ of (1) with x continuous on $[\alpha, \infty)$, $x(t) = \varphi(t)$ on $[\alpha, t_0]$, and $x(t, t_0, \varphi)$ satisfying (1) for $t \geq t_0$. While existence theory may be given for (1) which allows a discontinuity of x at t_0, in most of our work we perform certain integration by parts which requires continuity; thus, φ must be selected with care.

DEF. 2. A metric space $(\Omega(t_0), \rho)$ of continuous functions $\varphi : [\alpha, t_0] \to R^n$ is said to be *admissible* if for each $\varphi \in \Omega(t_0)$ there is a solution $x(t, t_0, \varphi)$ of (1) with $x(t, t_0, \varphi) = \varphi(t)$ for $\alpha \leq t \leq t_0$, $x(t, t_0, \varphi)$ satisfies (1) for $t \geq t_0$ and $x(t, t_0, \varphi)$ is continuous on $[\alpha, \infty)$.

Thus, given $\varphi \in \Omega(t_0)$, Equation (1) is usually written as

$$x(t) = a(t) - \int_{\alpha(t)}^{t_0} Q(t, s, \varphi(s))\, ds - \int_{t_0}^{t} Q(t, s, x(s))\, ds$$

and the first two terms on the right are taken as the inhomogeneous term. In this form there is much existence theory, as may be seen in Corduneanu [2] or Gripenberg-Londen-Staffans [3], for example.

NOTATION. The symbol $\Omega(t_0)$ will always denote an admissible set. If $\varphi \in \Omega(t_0)$ and $\Psi : [\alpha, \infty) \to R^n$, then $\rho(\varphi, \Psi)$ means Ψ is restricted to $[\alpha, t_0]$.

Clearly, φ must be chosen so that

$$\varphi(t_0) = a(t_0) - \int_{\alpha(t_0)}^{t_0} Q(t_0, s, \varphi(s))\, ds. \tag{3}$$

However, if for large t we have $\alpha(t) > \alpha(t_0)$ then (3) can be avoided, as we will see in the next section.

But what is important here is that any bounded continuous φ on $(-\infty, 0]$ can be approximated arbitrarily well by a function satisfying (3) with $t_0 = 0$.

PROPOSITION. Let $Q : R \times R \times R^n \to R^n$ be continuous and suppose that $\int_{-\infty}^{0} Q(0, s, \varphi(s))\, ds$ converges for each bounded and continuous $\varphi : (-\infty, 0] \to R^n$. Let

$\varphi : (-\infty, 0] \to R^n$ be an arbitrary bounded and continuous function. For each $\varepsilon > 0$ there is a $t_1 < 0$, t_1 near 0, and $\varphi_1 : (-\infty, 0] \to R^n$ which is continuous, which satisfies

$$\begin{aligned} &\varphi_1(0) = a(0) - \int_{-\infty}^{0} Q(0, s, \varphi_1(s))\, ds, \\ &\varphi(t) = \varphi_1(t) \text{ for } -\infty < t \le t_1, \text{ and } |\tilde{\varphi}(0) - \varphi_1(0)| \le \varepsilon \end{aligned} \tag{3*}$$

where

$$\tilde{\varphi}(0) = a(0) - \int_{-\infty}^{0} Q(0, s, \varphi(s))\, ds.$$

Proof. For any $x \in R^n$ and any $t_1 < 0$ define

$$\varphi^x = \begin{cases} \varphi(s), & \text{if } s \le t_1, \\ [(t_1 - s)x + s\varphi(t_1)]/t_1, & \text{if } t_1 < s \le 0. \end{cases}$$

Now, let t_1 be any number such that for any $x \in R^n$ with $|x - \tilde{\varphi}(0)| \le \varepsilon$ we have

$$\left| \int_{-\infty}^{0} Q(0, s, \varphi(s))\, ds - \int_{-\infty}^{0} Q(0, s, \varphi^x(s))\, ds \right| \le \varepsilon. \tag{$*$}$$

By the continuity of Q and the assumed convergence, $(*)$ can be satisfied. Also, t_1 is as near 0 as we please.

Next, let $S = \{x \in R^n : |x - \tilde{\varphi}(0)| \le \varepsilon\}$ and define $P : S \to S$ by $x \in S$ implies that

$$P(x) = a(0) - \int_{-\infty}^{0} Q(0, s, \varphi^x(s))\, ds.$$

Now P is continuous and, by construction, maps S into S. By Brouwer's theorem, there is a fixed point x_1 and φ^{x_1} is the required function.

REMARK. The next definition is a straight-forward generalization of the standard definition of stability from differential equations and, in fact, contains it as a special case, as we later see. We will also point out why the standard definition may be inadequate for integral equations.

DEF. 3. A near equilibrium (ψ, Ψ) is said to be stable relative to Ω if there is a wedge W and continuous functions $\gamma(t)$ and $p(t)$, where $\gamma \in L^1[0, \infty)$, while $p(t) \to 0$ and $W(t) \to \infty$

as $t \to \infty$, and for each $\varepsilon > 0$ and $t_0 \in R$ there is a $\delta > 0$ such that $[\varphi \in \Omega(t_0),\ \rho(\varphi, \Psi) < \delta]$ imply that

$$W(|x(t, t_0, \varphi) - \Psi(t)|) < \varepsilon + p(t_0) + \int_{t_0}^{t} \gamma(s)\, ds.$$

If, in addition, $|x(t, t_0, \varphi) - \Psi(t)| \to 0$ as $t \to \infty$, then (ψ, Ψ) is asymptotically stable relative to Ω.

To relate this to differential equations, first note in (1) that if $a(t) \equiv 0$ and $Q(t, s, 0) \equiv 0$, then ψ and Ψ may be both zero so $(0, 0)$ is a near equilibrium. If we take W as the identity function and $p(t) = \gamma(t) = 0$ then our definition is the usual one for stability of an integrodifferential equation

$$x'(t) = \int_{\alpha(t)}^{t} Q(t, s, x(s))\, ds,\ Q(t, s, 0) \equiv 0,$$

so that the zero function is a solution (equilibrium point). See, for example, Yoshizawa [11; pp. 27–31, 183–190], Burton [1; pp. 12–3, 33–34, 227–237].

Next, if $\varphi(t)$ is a solution of (1) and we wish to study the behavior of solutions starting near it, we can write $x = y + \varphi$ so that

$$(1^*) \qquad y(t) = -\int_{\alpha(t)}^{t} [Q(t, s, y(s) + \varphi(s)) - Q(t, s, \varphi(s))]\, ds$$

has the near equilibrium $(0, 0)$.

The very construction of a differential equation frequently produces an equation with some constant solutions, say $x = 0$ is a solution. And the vast majority of stability considerations surround stability of $x = 0$. By contrast, uncontrived forms of (1) seldom have constant solutions, nor do they have easily recognizable solutions φ so that (1^*) can be treated, except in the linear case. Even in the linear case, when (1^*) is analyzed without the knowledge of φ, stability of the zero solution tells little about ultimate behavior of all solutions.

The natural idea in the study of (1) is to show that $x(t)$ follows $a(t)$ in some sense. For example, consider

$$(1^{**}) \qquad x(t) = a(t) + \int_{-\infty}^{t} D(t - s) x(s)\, ds$$

where $D \in L^1[0,\infty)$. Three facts are derived by elementary considerations which motivated Definitions 1 and 3:

(i) Does (1^{**}) have any constant solutions?

It does if and only if $a(t)$ is constant.

(ii) Does (1^{**}) have a solution in $L^1[0,\infty)$?

It does only if $a(t) \in L^1[0,\infty)$.

(iii) Does (1^{**}) have any solutions tending to zero?

It does only if $a(t) \to 0$ as $t \to \infty$.

Part (ii) is the most interesting. We frequently show that there is not only a solution in L^1, but it converges pointwise to $a(t)$ as $t \to \infty$.

2. A finite delay problem. In our discussions we always consider a pair of equations: one is linear, one nonlinear. The linear equation will be the prototype and will lead us to the results; in effect, it will be an example. But the basic theory is nonlinear and we provide nonlinear examples.

Let h be a positive constant, Q be continuous, and consider the scalar equations

$$x(t) = a(t) - \int_{t-h}^{t} D(t,s)x(s)\,ds \tag{4}$$

and

$$x(t) = a(t) - \int_{t-h}^{t} Q(t,s,x(s))\,ds \tag{4_N}$$

with

$$a : R \to R \text{ being continuous, } a \text{ and } a^2 \in L^1[0,\infty), \tag{5}$$

and suppose there is a $P > 0$ with

$$D(t,t) \le P, D(t,s) \ge 0, D_s(t,s) \ge 0, D_{st}(t,s) \le 0, D(t,t-h) = 0. \tag{6}$$

Condition (6) might be called the Volterra-Levin condition because of work in differential equations of both bounded and unbounded delay found in Levin ([6], [7]), Levin and Nohel ([8], [9]), extended by Hale in ([4], [5]), and summarized in Corduneanu [2] and Gripenberg-Londen-Staffans [3]. But (6) has also been used extensively in circuit theory and statistics for a very long time. There are technical reasons for these assumptions, but elementary considerations strongly suggest them.

For example, let $a(t)$ be bounded and consider the convolution equation

$$x(t) = a(t) - \int_{t-h}^{t} C(t-s)x(s)\, ds.$$

If $C(t) < 0$ and large, for a positive initial function, we readily expect $x(t)$ to grow; thus, we ask $C(t) > 0$. But this is an equation with memory and, although the memory is lost on each interval of length h, we still expect the memory to immediately begin to fade with time; thus, we ask that $C'(t) \le 0$. For technical reasons we ask that $C(h) = 0$, but if $C(h) > 0$ a translation could be made. Hence, there is an uncontrived reason for $D(t,s) \ge 0$, $D_s(t,s) \ge 0$, and $D(t,t-h) = 0$, and investigators traditionally ask $D_{st} \le 0$ out of technical necessity. One of our goals is to reduce $D_{st} \le 0$.

The discussion here is the same for any t_0 so we take $t_0 = 0$ and $\Omega = \Omega(0)$ to be the set of continuous $\varphi : [-h, 0] \to R$ with

(7) $$\varphi(0) = a(0) - \int_{-h}^{0} D(0,s)\varphi(s)\, ds$$

for the stability statements. But (7) will not be needed for the study of limit sets.

The metric ρ on Ω will be the L^2-norm, $|||\cdot|||$. Also, if $q : [-h, A) \to R$, $A > 0$, then $q_t(s) = q(t+s)$ for $-h \le s \le 0$ and

(8) $$|||q_t|||^2 = \int_{-h}^{0} q^2(t+s)\, ds.$$

Clearly, the pair $(0, a(t))$ is a near equilibrium for (4) and we will show that it is asymptotically stable relative to Ω. In addition, it will motivate a general theorem. It is convenient to give them in reverse order and to prove Theorem 1B first.

Theorem 1A. *Suppose that for some continuous function* $\psi : [-h, \infty) \to R$,

$$\Psi(t) := a(t) - \psi(t) - \int_{t-h}^{t} Q(t, s, \psi(s))\, ds \in L^1[0, \infty),$$

and that there exist continuous functions $p, q : [0, \infty) \to [0, \infty)$, $p(t) \to 0$ *as* $t \to \infty$, $q \in L^1[0, \infty)$, *a continuous function* $V(t, x(\cdot))$ *defined for a solution* $x(t) = x(t, 0, \varphi)$ *of* (4_N) *with* $\varphi \in \Omega$, *and wedges* W_i *such that* $W_1(r) \to \infty$ *as* $r \to \infty$,

(i) $W_1(|x(t) - \Psi(t)|) \le V(t, x(\cdot)) \le W_2(|||(x - \Psi)_t|||) + p(t)$, *and*

(ii) $V'(t, x(\cdot)) \le -W_3(|x(t) - \Psi(t)|) + q(t)$.

Then the near equilibrium (ψ, Ψ) *of* (4_N) *is asymptotically stable relative to* Ω. *If* p *and* q *depend on* φ *then* $|x(t) - \Psi(t)| \to 0$ *as* $t \to \infty$.

Theorem 1B. *Let (5), (6), (7) hold. Then there exist continuous functions* $p, q : [0, \infty) \to [0, \infty)$, $p(t) \to 0$ *as* $t \to \infty$, $q \in L^1[0, \infty)$, *a continuous function* $V(t, x(\cdot))$ *defined for a solution* $x(t) = x(t, 0, \varphi)$ *of (4) with* $\varphi \in \Omega$, *and wedges* W_i *such that* $W_1(r) \to \infty$ *as* $r \to \infty$,

(i) $$W_1(|x(t) - a(t)|) \le PV(t, x(\cdot)) \le W_2(|||(x - a)_t|||) + p(t),$$

and

(ii) $$V'(t, x(\cdot)) \le -W_3(|x(t) - a(t)|) + q(t).$$

Thus, the near equilibrium $(0, a(t))$ *of (4) is asymptotically stable relative to* Ω.

Proof. To prove Theorem 1B, let $\varphi \in \Omega$, $x(t) = x(t, 0, \varphi)$, and define

(9) $$V(t) = V(t, x(\cdot)) = \int_{t-h}^{t} D_s(t, s)\left(\int_{s}^{t} x(v)\, dv\right)^2 ds.$$

Then

$$\begin{aligned}V'(t) &= -D_s(t,t-h)\Big(\int_{t-h}^t x(v)\,dv\Big)^2 ds + \int_{t-h}^t D_{st}(t,s)\Big(\int_s^t x(v)\,dv\Big)^2 ds\\ &+ 2x(t)\int_{t-h}^t D_s(t,s)\int_s^t x(v)\,dv\\ &\le 2x(t)\Big[D(t,s)\int_s^t x(v)\,dv\Big|_{t-h}^t + \int_{t-h}^t D(t,s)x(s)\,ds\Big]\\ &= 2x(t)[a(t)-x(t)]\\ &= -x^2(t) - (x(t)-a(t))^2 + a^2(t) \le -(x(t)-a(t))^2 + a^2(t)\\ &=: -W_3(|x(t)-a(t)|) + q(t)\end{aligned}$$

so (ii) holds.

Next, from (4) we have

$$\begin{aligned}&W_1(|x(t)-a(t)|) := (x(t)-a(t))^2 = \Big(-\int_{t-h}^t D(t,s)x(s)\,ds\Big)^2\\ &= \Big\{D(t,s)\int_s^t x(v)\,dv\Big|_{t-h}^t - \int_{t-h}^t D_s(t,s)\int_s^t x(v)\,dv\,ds\Big\}^2\\ &\le \int_{t-h}^t D_s(t,s)\,ds\int_{t-h}^t D_s(t,s)\Big(\int_s^t x(v)\,dv\Big)^2 ds \le PV(t)\\ &\le P\int_{t-h}^t D_s(t,s)2\Big[\Big(\int_s^t |x(v)-a(v)|\,dv\Big)^2 + \Big(\int_s^t |a(v)|\,dv\Big)^2\Big] ds\\ &\le 2P\int_{t-h}^t D_s(t,s)(t-s)\int_s^t |x(v)-a(v)|^2\,dv + 2P\Big(\int_{t-h}^t D_s(t,s)\,ds\Big)\Big(\int_{t-h}^t |a(v)|\,dv\Big)^2\\ &\le 2P^2h\int_{t-h}^t |x(v)-a(v)|^2\,dv + 2P^2\Big(\int_{t-h}^t |a(v)|\,dv\Big)^2\\ &=: W_2(|||(x-a)_t|||) + p(t)\end{aligned}$$

where $p(t) \to 0$ as $t \to \infty$; hence, (i) holds and Theorem 1B will be proved when we have proved Theorem 1A.

To that end, in Theorem 1A we note that an integration of (ii) yields $V(t)$ bounded and, since $W_1(r) \to \infty$ as $r \to \infty$, in (i) we see that $|x(t) - \Psi(t)|$ is bounded. This means that (ii) can be sharpened to

(ii^*) $$V'(t) \le -W_4(|x(t)-\Psi(t)|^2) + q(t)$$

where W_4 is convex downward. (See Natanson [10; pp. 36–46] for a good discussion of convexity and Jensen's inequality. In particular, if W is a wedge, then for $0 \leq r \leq 1$ we have

$$W^*(r) = \int_0^r W(s)\,ds = W(\xi)r \leq W(r)$$

for some ξ in $[0, r]$ and W^* is convex downward.)

From (i) and (ii) we have

$$\begin{aligned} W_1(|x(t) - \Psi(t)|) &\leq V(t, x(\cdot)) \leq V(0) + \int_0^t q(s)\,ds \\ &\leq W_2(|||(\varphi - \Psi)_0|||) + p(0) + \int_0^t q(s)\,ds, \end{aligned}$$

(as we have taken t_0 to be zero for convenience) and this is the required stability.

We now show that $|x(t) - \Psi(t)| \to 0$ as $t \to \infty$. If it does not, then there is an $\varepsilon > 0$ and $\{t_n\} \uparrow \infty$ with $h < t_n$, $t_{n+1} > t_n + h$, and $|x(t_n) - \Psi(t_n)| \geq \varepsilon$. Using (i) and the fact that $p(t) \to 0$, we can say that there is a $\delta > 0$ with $|||(x - \Psi)_{t_n}|||^2 \geq \delta$ for large n, say $n \geq 1$. Using (ii^*) and Jensen's inequality, we take N large, integrate (ii^*) from t_1 to t_N and obtain

$$\begin{aligned} V(t_N) - V(t_1) &\leq -\int_{t_1}^{t_N} W_4(|x(s) - \Psi(s)|^2)\,ds + \int_{t_1}^{t_N} q(s)\,ds \\ &\leq -\sum_{i=2}^{N} \int_{t_i - h}^{t_i} W_4(|x(s) - \Psi(s)|^2)\,ds + \int_{t_1}^{t_N} q(s)\,ds \\ &\leq -\sum_{i=2}^{N} hW_4\left(\frac{1}{h}\int_{t_i - h}^{t_i} |x(s) - \Psi(s)|^2\,ds\right) + \int_{t_1}^{t_N} q(s)\,ds \\ &\leq -\sum_{i=2}^{N} hW_4(\delta/h) + \int_{t_1}^{t_N} q(s)\,ds, \end{aligned}$$

a contradiction for large N since $V(t) \geq 0$ and $q \in L^1[0, \infty)$. This proves Theorem 1A, so 1B is also true.

The only place (7) was used was in the integration by parts when differentiating V. For any continuous φ there is a solution $x(t, 0, \varphi)$ for $t > 0$ of (4) which may have a discontinuity at $t = 0$ but V is differentiable for $t > h$. There is the question of stability, but it can be resolved using continuous dependence of solutions on initial conditions in conjunction with the following result.

COR. 1. *If (5) and (6) hold then there exist continuous functions $p, q : [0, \infty) \to [0, \infty)$, $p(t) \to 0$ as $t \to \infty$, $q \in L^1[0, \infty)$, and wedges W_i such that if $\varphi : [-h, 0] \to R$ is continuous and $x(t) = x(t, 0, \varphi)$ solves (4), then there is a continuous function $V(t, x(\cdot))$ satisfying (i) of Theorem 1B for $t > 0$ and (ii) for $t > h$. In particular, $|x(t) - a(t)| \to 0$ as $t \to \infty$.*

One of our main stated goals is to reduce $D_{st} \le 0$ and the next result gives us one way to do that. But it forces us to write (ii) as an integral inequality, which we do in later results. Here, $f_+ = \max(f(t), 0)$.

COR. 2. *Let the conditions of Theorem 1B hold except*

$$2h^2 \int_{t-h}^{t} (D_{st}(t, s))_+ \, ds \le 1.$$

Then $(0, a(t))$ is asymptotically stable relative to Ω.

Proof. We readily obtain

$$\begin{aligned} V'(t) &\le -x^2(t) - (x(t) - a(t))^2 + a^2(t) + \int_{t-h}^{t} \Big(D_{st}(t, s)\Big)_+ \Big(\int_s^t x(v)\, dv\Big)^2 ds \\ &\le -x^2(t) - (x(t) - a(t))^2 + a^2(t) + \frac{1}{2h} \int_{t-h}^{t} x^2(v)\, dv \end{aligned}$$

Again, take $t_0 = 0$ and let $t = 2Nh$. On each interval $[(n-1)h, nh]$, choose t_n such that

$$\int_{t_n - h}^{t_n} x^2(v)\, dv \ge \int_{t-h}^{t} x^2(v)\, dv \text{ on } [(n-1)h, nh].$$

An integration of V' will yield

$$0 \le V(t) \le V(0) - \int_0^t x^2(s)\,ds - \int_0^t (x(s)-a(s))^2\,ds$$

$$+ \int_0^t a^2(s)\,ds + \frac{1}{2h}\int_0^t \int_{s-h}^s x^2(v)\,dv\,ds$$

$$\le V(0) - \frac{1}{2}\sum_{i=1}^N \int_{2ih-2h}^{2ih} x^2(s)\,ds - \frac{1}{2}\sum_{i=2}^N \int_{(2i-1)h-2h}^{(2i-1)h} x^2(s)\,ds$$

$$+ \frac{1}{2}\sum_{i=1}^N \int_{t_{2i}-h}^{t_{2i}} x^2(s)\,ds + \frac{1}{2}\sum_{i=1}^N \int_{t_{2i-1}-h}^{t_{2i-1}} x^2(s)\,ds$$

$$- \int_0^t (x(s)-a(s))^2\,ds + \int_0^t a^2(s)\,ds$$

(Notice that the lengths of intervals of integration in the first pair of integrals is $2h$, but only h in the second pair.)

$$\le V(0) + \text{ constant } - \int_0^t (x(s)-a(s))^2\,ds + \int_0^t a^2(s)\,ds$$

and this will allow us to prove the result as before.

Theorem 1A emphasizes that linearity is not essential; it merely serves as a convenient example with fewer hypotheses. We now give examples of superlinear and sublinear cases. The wedges in the theorems still arise in a natural way.

For the equation

$$x(t) = a(t) - \int_{t-h}^t D(t,s)g(s,x(s))\,ds \tag{4*}$$

with (5), (6), g bounded for x bounded, and

$$\varphi(0) = a(0) - \int_{-h}^0 D(0,s)g(s,\varphi(s))\,ds \tag{7*}$$

then

$$V(t,x(\cdot)) = \int_{t-h}^t D_s(t,s)\left(\int_s^t g(v,x(v))\,dv\right)^2 ds$$

satisfies

$$(x(t)-a(t))^2 \le D(t,t)V(t,x(\cdot))$$

and

$$V'(t, x(\cdot)) \le -2g(t,x)[x - a(t)].$$

EXAMPLE 2. If (5), (6), and (7^*) hold and if a^3 and $a^4 \in L^1[0,\infty)$, then conditions (i) and (ii) of Theorem 1A hold when $g(t,s) = x^3$ in (4^*) and $|x(t) - a(t)| \to 0$ as $t \to \infty$.

Proof. We have just defined V and we have

$$\begin{aligned} V'(t) &\le -2x^4 + 2a(t)x^3 \\ &= -x^4 - (x - a(t))^4 - 2a(t)x^3 + 6x^2a^2(t) - 4xa^3(t) + a^4(t) \end{aligned}$$

so that

$$\begin{aligned} V(t) \le V(0) &- \int_0^t x^4(s)\,ds - \int_0^t (x(s) - a(s))^4\,ds \\ &+ 2\Big(\int_0^t a^4(s)\,ds\Big)^{1/4}\Big(\int_0^t x^4(s)\,ds\Big)^{3/4} + 6\Big(\int_0^t x^4(s)\,ds\Big)^{1/2}\Big(\int_0^t a^2(s)\,ds\Big)^{1/2} \\ &+ 4\Big(\int_0^t x^4(s)\,ds\Big)^{1/4}\Big(\int_0^t a^4(s)\,ds\Big)^{3/4} + \int_0^t a^4(s)\,ds \end{aligned}$$

and it follows that each term in the expression for V' is $L^1[0,\infty)$. Indeed, if $\int_0^t x^4(s)\,ds \to \infty$, then it dominates all the positive terms, so the right hand side tends to $-\infty$. Thus, all the positive terms have finite integrals and, therefore, so does $(x - a)^4$. We may write

$$V'(t) \le -(x(t) - a(t))^4 + q(t)$$

where $q \in L^1[0,\infty)$, satisfying (ii) of Theorem 1A. Next, if we take

$$r(s) = 3(x(s) - a(s))^2|a(s)| + 3|x(s) - a(s)|a^2(s) + |a(s)|^3,$$

then we have

$$\begin{aligned} \int_{t-h}^t x^2(s)|a(s)|\,ds &\le \Big(\int_{t-h}^t |x(s)|^3\,ds\Big)^{2/3}\Big(\int_{t-h}^t |a(s)|^3\,ds\Big)^{1/3} \\ &\le h^{1/6}\Big(\int_{t-h}^t |x(s)|^4\,ds\Big)^{1/2}\Big(\int_{t-h}^t |a(s)|^3\,ds\Big)^{1/3} \end{aligned}$$

$$\to 0 \text{ as } t \to \infty$$

and

$$\int_{t-h}^{t} |x(s)|a^2(s)\,ds \le \left(\int_{t-h}^{t} |x(s)|^3\,ds\right)^{1/3}\left(\int_{t-h}^{t} |a(s)|^3\,ds\right)^{2/3}$$
$$\to 0 \text{ as } t \to \infty.$$

Clearly,

$$\int_{t-h}^{t} |a(s)|^3\,ds \to 0 \text{ as } t \to \infty.$$

Thus, we have, for r defined above,

$$\begin{aligned}
V(t,x(\cdot)) &= \int_{t-h}^{t} D_s(t,s)\left(\int_s^t x^3(v)\,dv\right)^2 ds \\
&\le \int_{t-h}^{t} D_s(t,s)2\left\{\left(\int_s^t |x(v)-a(v)|^3\,dv\right)^2 + \left(\int_s^t r(v)\,dv\right)^2\right\} ds \\
&\le \int_{t-h}^{t} D_s(t,s)2\left\{h^{1/4}\left(\int_{t-h}^{t} |x(v)-a(v)|^4\,dv\right)^{3/4} + \left(\int_{t-h}^{t} r(v)\,dv\right)^2\right\} ds \\
&\le 2D(t,t)h^{1/4}\left(\int_{t-h}^{t} (x(v)-a(v))^4\,dv\right)^{3/4} + p(t)
\end{aligned}$$

where we have verified that $p(t) \to 0$ as $t \to \infty$, so that (i) of Theorem 1A is satisfied and the conclusion follows.

EXAMPLE 3. If (5), (6), and (7*) hold for (4*) and if $g(t,x) = x^{1/3}$, while $a(t)$ is bounded, then the conditions of Theorem 1A hold and $|x(t)-a(t)| \to 0$ as $t \to \infty$.

Proof. We have

$$V'(t) \le -2x^{4/3} + 2x^{1/3}a(t)$$

so that

$$0 \le V(t) \le V(0) - 2\int_0^t x^{4/3}(s)\,ds + 2\left(\int_0^t x^{4/3}(s)\,ds\right)^{1/4}\left(\int_0^t a^{4/3}(s)\,ds\right)^{3/4}$$

and so the terms in V' are $L^1[0,\infty)$. Moreover, familiar arguments yield (i). Hence, V is bounded so $(x(t)-a(t))^2$ is bounded; but $a(t)$ bounded yields $x(t)$ bounded. Thus, there exists $M > 0$ with

$$\int_0^t x^2(s)\,ds = \int_0^t x^{2/3}(s)x^{4/3}(s)\,ds \le M\int_0^t x^{4/3}(s)\,ds$$

Hence, $(x-a)^2 = x^2 - 2ax + a^2$ is in $L^1[0,\infty)$ and we can write

$$V'(t) \le -(x(t)-a(t))^2 + q(t)$$

so that (ii) of Theorem 1A holds and the proof is complete.

Theorem 1A is predicated on finding a near equilibrium; once that is found, the limit set of all solutions is specified by Cor. 1. To find a near equilibrium is to find a function which fails to solve (4) only by an amount of an L^1-function. If we can find a function which fails to solve (4) only by an amount of a bounded function, then we can locate a bounded set which contains the limit set of all solutions of (4). When the conditions of this theorem hold, then we are assured that all stable near equilibria are a bounded distance from that function.

Theorem 2A. *Let $x(t)$ solve (4_N) with $x(t) = x(t,0,\varphi)$ and $\varphi : [-h,0] \to R$ be continuous. Suppose there is a continuous function $\Psi : [-h,\infty) \to R$, positive constants Q and L, wedges W_i with $W_1(r) \to \infty$ as $r \to \infty$, and a continuous function $V(t,x(\cdot))$ so that for $t > h$*

$$\text{(i)} \qquad W_1(|x(t) - \Psi(t)|) \le V(t,x(\cdot)) \le W_2(|||(x-\Psi)_t|||) + Q$$

and for $t > h$

$$\text{(ii)} \qquad V'(t,x(\cdot)) \le -W_3(|x(t)-\Psi(t)|^2) + L$$

with W_3 convex downward. Then there is a number B independent of φ with $|x(t)| \le B$ for large t.

Proof. Consider the intervals $I_n = [(n-1)h, nh]$ for $n = 2, 3, \ldots$. Either

(a) $V(nh) \ge V((n-1)h) - 1$ so that from (ii)

$$-1 \le V(nh) - V((n-1)h) \le -hW_3\left(\frac{1}{h}|||(x-\Psi)_{nh}|||^2\right) + Lh$$

or

$$|||(x-\Psi)_{nh}|||^2 \le hW_3^{-1}\left(L+\frac{1}{h}\right)$$

so from (i)

$$(A) \qquad V(nh) \le W_2((hW_3^{-1}(L+\frac{1}{h}))^{1/2}) + Q =: C$$

or

(b) $V(nh) \le V((n-1)h) - 1.$

Since (b) can not hold for all n, there is a k with (A) holding for $n = k$:

$$V(kh) \le C.$$

From (ii) we have

$$V(t) \le C + Lh \text{ if } kh \le t \le (k+1)h.$$

But by the arguments in (a) and (b), either

$$V((k+1)h) \le V(kh) - 1 < C \text{ by (b)}$$

or

$$V((k+1)h) \le C \text{ by (A)}.$$

Hence,

$$W_1(|x(t)-\Psi(t)|) \le V(t) \le C + Lh$$

for all large t and we take

$$B = W_1^{-1}(C+Lh).$$

This completes the proof.

Suppose there is an $A > 0$ with

$$(5^*) \qquad a : R \to R \text{ is continuous and } |a(t)| \le A \text{ for } t \ge 0.$$

Theorem 2B. *Let (5*) and (6) hold. Then there are constants Q and L, wedges W_i with $W_1(r) \to \infty$ as $r \to \infty$ and a continuous function V with the following properties. If $\varphi : [-h, 0] \to R$ is continuous and if $x(t) = x(t, 0, \varphi)$ satisfies (4) then*

$$(i) \qquad W_1(|x(t) - a(t)|) \leq PV(t, x(\cdot)) \leq W_2(|||(x - a)_t|||) + Q$$

and for $t > h$

$$(ii) \qquad V'(t, x(\cdot)) \leq -W_3(|x(t) - a(t)|^2) + L$$

where W_3 is convex downward. Thus, there is a $B > 0$ independent of φ with $|x(t)| \leq B$ for large t.

Proof. The proof of (i) proceeds by familiar arguments. We have

$$Q = 2P^2h^2A^2 \text{ and in (ii) } L = A^2.$$

REMARK. When we study the proof of Theorem 2A, part (b), we see that for each $B_1 > 0$ there is a $B_2 > 0$ such that $|||(\varphi - \Psi)_0||| < B_1$ and $t \geq 0$ imply $|x(t, 0, \varphi)| < B_2$. Also, for each $B_3 > 0$ there is a $T > 0$ such that $|||(\varphi - \Psi)_0||| < B_3$ and $t \geq T$ imply $|x(t)| \leq B$. This may be called uniform boundedness and uniform ultimate boundedness.

3. Infinite delay. Consider the equation

$$(10) \qquad x(t) = a(t) - \int_{-\infty}^{t} D(t, s)x(s)\, ds$$

or

$$(10_N) \qquad x(t) = a(t) - \int_{-\infty}^{t} Q(t, s, x(s))\, ds$$

where Q is continuous,

(11) $\qquad a : R \to R$ is bounded and continuous, $a \in L^1[0, \infty)$,

there is a constant $P > 0$ with

$$D(t,s) \geq 0, D_s(t,s) \geq 0, D(t,t) \leq P, \tag{12}$$

$$\int_{-\infty}^{t} [D(t,s) + \{D_s(t,s) + |D_{st}(t,s)|\}(t-s)^2]\, ds \text{ is continuous} \tag{13}$$

$$\lim_{s\to-\infty} (t-s)D(t,s) = 0 \text{ for fixed } t, \tag{14}$$

there is a function

$$g : [0,\infty) \to (0,1], g \text{ decreasing, } g(0) = 1, \int_0^\infty g(s)\, ds =: G < \infty, \tag{15}$$

there are constants $L > 0$ and $M > 0$ with $MG < 1$ and

$$\begin{aligned} &\int_{-\infty}^{t} [D_s(t,s)(t-s)/g(t-s)]\, ds \leq L \, and \\ &2\int_{-\infty}^{0} \Big[(D_{st}(t,s))_+(t-s)/g(t-s)\Big]\, ds + \int_0^t \Big[(D_{st}(t,s))_+(t-s)/g(t-s)\Big]\, ds \leq M, \end{aligned} \tag{16}$$

$$\begin{gathered} d^*(t) := \int_{-\infty}^{0} \Big(D_{st}(t,s)_+\Big) s^2\, ds \in L^1[0,\infty) \text{ and for each} \\ T > 0 \text{ then } \int_{-\infty}^{T} D_s(t,s)(t-s)^2\, ds \to 0 \text{ as } t \to \infty. \end{gathered} \tag{17}$$

Define Ω by

$$\Omega = \{\varphi : (-\infty, 0] \to R, \varphi \in C, \varphi \text{ bounded, } \varphi(0) = a(0) - \int_{-\infty}^{0} D(0,s)\varphi(s)\, ds\} \tag{18}$$

where C denotes a set of continuous functions on $(-\infty, 0]$, and ρ by

$$\varphi \in \Omega \text{ implies that } \rho(\varphi) = \int_{-\infty}^{0} g(-s)|\varphi(s)|\, ds. \tag{19}$$

Solutions are denoted as before.

Theorem 3A. *Suppose that there is a continuous function $\psi : R \to R$ with $\Psi(t) := a(t) - \psi(t) - \int_{-\infty}^{t} Q(t,s,\psi(s))\, ds$ in $L^1[0,\infty)$, and that there are continuous functions p,*

$q : [0,\infty) \to [0,\infty)$, $p(t) \to 0$ as $t \to \infty$, $q \in L^1[0,\infty)$, wedges W_i with $W_1(r) \to \infty$ as $r \to \infty$, and a continuous function $V(t,x(\cdot))$defined for a solution $x(t) = x(t,t_0,\varphi)$ of (10_N) such that

$$(i) \qquad W_1(|x(t) - \Psi(t)|) \le V(t,x(\cdot)) \le W_2(\rho(W_3(|x-\Psi|_t))) + p(t)$$

and for $V(t) = V(t,x(\cdot))$ then

$$(ii) \qquad V(t) \le V(t_0) - \int_{t_0}^t W_3(|x(s) - \Psi(s)|)\,ds + \int_{t_0}^t q(s)\,ds.$$

Then the near equilibrium (ψ,Ψ) of (10_N) is asymptotically stable relative to Ω. If p and q depend on $x(t)$, then $|x(t) - \Psi(t)| \to 0$ as $t \to \infty$.

Theorem 3B. *Let (11) – (18) hold. Then there are continuous functions p, $q : [0,\infty) \to [0,\infty)$, $p(t) \to 0$ as $t \to \infty$, $q \in L^1[0,\infty)$, wedges W_i with $W_1(r) \to \infty$ as $r \to \infty$, and a continuous function $V(t,x(\cdot))$ defined for a solution $x(t) = x(t,t_0,\varphi)$ of (10) with $\varphi \in \Omega$ such that*

$$(i) \qquad W_1(|x(t) - a(t)|) \le PV(t,x(\cdot)) \le W_2(\rho(W_3(|x-a|_t))) + p(t)$$

and for $V(t) = V(t,x(\cdot))$ then

$$(ii) \qquad V(t) \le V(t_0) - \int_{t_0}^t W_3(|x(s) - a(s)|)\,ds + \int_{t_0}^t q(s)\,ds.$$

Moreover, the near equilibrium $(0,a(t))$ of (10) is asymptotically stable relative to Ω.

Proof. We begin the proof of Theorem 3B first. As before, we can obtain

$$\begin{aligned}
(x(t) - a(t))^2 &\le D(t,t)\int_{-\infty}^t D_s(t,s)\left(\int_s^t x(v)\,dv\right)^2 ds =: D(t,t)V(t,x(\cdot)) \\
&\le P\int_{-\infty}^t D_s(t,s)2\left\{\left(\int_s^t |x(v) - a(v)|\,dv\right)^2 + \left(\int_s^t |a(v)|\,dv\right)^2\right\} ds \\
&\le 2P\int_{-\infty}^t D_s(t,s)(t-s)\int_s^t |x(v) - a(v)|^2 dv\,ds + 2P\int_{-\infty}^t D_s(t,s)\left(\int_s^t |a(v)|\,dv\right)^2 ds \\
&\le 2P\int_{-\infty}^t [D_s(t,s)(t-s)/g(t-s)]\int_s^t |x(v) - a(v)|^2 g(t-s)\,ds \\
&\qquad + 2P\int_{-\infty}^t D_s(t,s)\left(\int_s^t |a(v)|dv\right)^2 ds
\end{aligned}$$

(g decreasing implies that $g(t-v) \geq g(t-s)$)

$$
\begin{aligned}
&\leq 2P \int_{-\infty}^{t} [D_s(t,s)(t-s)/g(t-s)] \int_{-\infty}^{t} |x(v)-a(v)|^2 g(t-v)\,dv \\
&\quad + 2P \int_{-\infty}^{t} D_s(t,s)\Big(\int_s^t |a(v)|\,dv\Big)^2 ds \\
&\leq 2PL\rho(|x-a|_t^2) + p(t)
\end{aligned}
$$

where p is the last integral. We later show that $p(t) \to 0$ as $t \to \infty$.

A calculation yields

$$
\begin{aligned}
V'(t,x(\cdot)) &\leq -2x[x-a(t)] + \int_{-\infty}^{t} (D_{st}(t,s))_+ \Big(\int_s^t x(v)\,dv\Big)^2 ds \\
&\leq -x^2 - (x-a(t))^2 + a^2(t) + \int_{-\infty}^{t} (D_{st}(t,s))_+ \Big(\int_s^t x(v)\,dv\Big)^2 ds.
\end{aligned}
$$

Now there is a positive constant H with $|\varphi(t)| \leq H$ on $(-\infty, 0]$ so the last term can be bounded by

$$
\begin{aligned}
&2\int_{-\infty}^{0} (D_{st}(t,s))_+ \Big(\int_s^0 \varphi(v)\,dv\Big)^2 ds \\
&+ 2\int_{-\infty}^{0} \Big[(D_{st}(t,s))_+(t-s)/g(t-s)\Big]\,ds \int_0^t g(t-v)x^2(v)\,dv \\
&+ \int_0^t \Big[(D_{st}(t,s))_+(t-s)/g(t-s)\Big] \int_0^t g(t-v)x^2(v)\,dv \\
&\leq 2d^*(t)H^2 + M\int_0^t g(t-v)x^2(v)\,dv.
\end{aligned}
$$

If we now integrate V' and interchange the order of integration in the last term above, taking $a^2(t) + 2d^*(t)H^2 = q(t)$, then we will have, by taking $t_0 = 0$ for brevity,

$$
\begin{aligned}
V(t) &\leq V(0) - \int_0^t x^2(s)\,ds + \int_0^t q(s)\,ds + M\int_0^t \int_0^u g(u-v)x^2(v)\,dv\,ds \\
&\quad - \int_0^t (x(s)-a(s))^2\,ds \\
&= V(0) - \int_0^t x^2(s)\,ds + \int_0^t M\int_v^t g(u-v)\,du\,x^2(v)\,dv + \int_0^t q(s)\,ds \\
&\quad - \int_0^t (x(s)-a(s))^2\,ds \\
&\leq V(0) - (1-MG)\int_0^t x^2(s)\,ds + \int_0^t q(s)\,ds - \int_0^t (x(s)-a(s))^2\,ds
\end{aligned}
$$

yielding (ii).

We now complete the proof of (i) by noting that $(x(t)-a(t))^2$, $x^2(t)$, and $a^2(t)$ are all in $L^1[0,\infty)$. From the first line of the proof we have

$$\begin{aligned}PV(t) &\le 2P\int_{-\infty}^{t}[D_s(t,s)(t-s)/g(t-s)]\int_s^t |x(v)-a(v)|^2 g(t-v)\,dv\,ds\\ &+2P\int_{-\infty}^{t} D_s(t,s)\Big(\int_s^t |a(v)|\,dv\Big)^2 ds\\ &\le 2PL\int_{-\infty}^{t}|x(v)-a(v)|^2 g(t-v)\,dv\\ &+2P\int_{-\infty}^{T} D_s(t,s)(t-s)^2\,ds\|a\|^2+2P\int_T^t D_s(t,s)\Big(\int_s^t |a(v)|\,dv\Big)^2 ds\end{aligned}$$

where $\|a\|$ is the bound on a and T will be large. The second term tends to zero by assumption; the last term can be made small by taking T large since $a\in L^1[0,\infty)$. This proves (i).

We now prove Theorem 3A which will also complete the proof of Theorem 3B. Using (ii) we have $V(t)\le V(t_0)+\int_{t_0}^t q(s)\,ds$ which in (i) yields

$$\begin{aligned}W_1(|x(t)-\Psi(t)|)\le V(t,x(\cdot))\le W_2(\rho(W_3(|\varphi-\Psi|_{t_0})))+p(t_0)\\ +\int_{t_0}^t q(s)\,ds.\end{aligned}$$

This yields stability. From (ii), $W_3(|x(t)-\Psi(t)|)\in L^1[0,\infty)$ and $g(t)\to 0$ so $\rho(W_3(|x-\Psi|_t))\to 0$ as $t\to\infty$. Since $p(t)\to 0$ as $t\to\infty$, $V(t)\to 0$, completing the proof.

We now give a general boundedness result for (10_N) and for (10) when $a(t)$ is bounded. Let

(11*) $\qquad a: R\to R$ be bounded and continuous,

(12*) $\qquad D(t,s)\ge 0, D_s(t,s)\ge 0, D(t,t)\le P, D_{st}(t,s)\le 0,$

and for g defined in (15) let

$$\int_{-\infty}^{t}[D_s(t,s)(t-s)/g(t-s)]\,ds\le L.$$

Theorem 4A. *Let g satisfy (15), V, f, $p : [0,\infty) \to [0,\infty)$ be continuous, $p(t) \to 0$ as $t \to \infty$, W be a wedge, and let M be a positive constant. Suppose that*

$$(i) \qquad V(t) \le W\left(\int_0^t f(s)g(t-s)\,ds\right) + p(t),$$

$$(ii) \qquad V'(t) \le M - f(t),$$

and that $V(t)$ being bounded implies that $f(t)$ is bounded. Then $V(t) \le B := W(MG+1)+1$ for large t.

Theorem 4B. *Let (11*), (12*), (13) – (15), (16*) hold. Suppose also that there are wedges W_i, $M > 0$, $p : [0,\infty) \to [0,\infty)$ with $p(t) \to 0$ as $t \to \infty$, such that $W_1(r) \to \infty$ as $r \to \infty$. If $\varphi \in \Omega$ and $x(t) = x(t,0,\varphi)$ solves (10), then there is a continuous function $V(t, x(\cdot))$ with*

$$(i) \qquad W_1(|x(t) - a(t)|) \le PV(t, x(\cdot)) \le W_2\left(\int_0^t W_3(|x(s)|)g(t-s)\,ds\right) + p(t)$$

and

$$(ii) \qquad V'(t, x(\cdot)) \le M - W_3(|x(t)|)$$

so that $V(t, x(\cdot)) \le B := W(MG+1)+1$ for large t.

Proof. We first verify the conditions in Theorem 4B. If $a^2(t) \le M$, then the calculations in the proof of Theorem 3B yield (ii) with $V'(t) \le M - x^2$. For the same V we have

$$\begin{aligned}(x(t) - a(t))^2 &\le P\int_{-\infty}^t D_s(t,s)\left(\int_s^t x(v)\,dv\right)^2 ds = PV(t, x(\cdot))\\ &\le P\int_{-\infty}^t [D_s(t,s)(t-s)/g(t-s)]\,ds \int_{-\infty}^t x^2(v)g(t-v)\,dv\\ &\le PL\int_0^t x^2(v)g(t-v)\,dv + PL\int_{-\infty}^0 x^2(v)g(t-v)\,dv\end{aligned}$$

so (i) is satisfied since $g \in L^1[0,\infty)$. Notice that M is independent of φ.

We now consider Theorem 4A and suppose there is a $t > 0$ with $V(t) \geq V(s)$ for $0 \leq s \leq t$. Then

$$g(t-s)V'(s) \leq Mg(t-s) - g(t-s)f(s)$$

so by a mean value theorem, there is an $\xi \in [0,t]$ with

$$\begin{aligned} 0 \leq g(0)[V(t) - V(\xi)] = g(0)\int_{\xi}^{t} V'(s)\,ds = \int_0^t g(t-s)V'(s)\,ds \\ \leq M\int_0^t g(s)\,ds - \int_0^t g(t-s)f(s)\,ds \end{aligned}$$

or

$$\int_0^t g(t-s)f(s)\,ds \leq MG$$

where $G = \int_0^\infty g(s)\,ds$. This means that either $V(0)$ is the maximum of V or $V(t) \leq W(MG) + \|p\|$. In either case, V is bounded and there is a $k > 0$ with $f(t) \leq k$.

Let $\{t_n\} \uparrow \infty$ have the property that $V(t_n) \to \limsup_{t\to\infty} V(t)$, and find m such that

$$(*) \qquad t \geq t_m \text{ implies that } V(t) \leq V(t_j) + 1 \text{ if } j \geq m.$$

Now, let $n > m$ and from (ii) consider

$$g(t_n - s)V'(s) \leq Mg(t_n - s) - f(s)g(t_n - s)$$

so that if $t_m \leq t^* < t_n$ there is some $\xi \in [t^*, t_n]$. We have from $(^*)$ that

$$\begin{aligned} -g(0) \leq g(0)[V(t_n) - V(\xi)] = g(0)\int_{\xi}^{t_n} V'(s)\,ds \\ = \int_{t^*}^{t_n} g(t_n - s)V'(s)\,ds \leq MG - \int_{t^*}^{t_n} g(t_n - s)f(s)\,ds \end{aligned}$$

or

$$\int_{t^*}^{t_n} g(t_n - s)f(s)\,ds \leq MG + 1.$$

Thus, if $t_m \leq t^* < t_n$ then

$$V(t_n) \leq W\Big(\int_0^{t^*} f(s)g(t_n - s)\,ds + MG + 1\Big) + p(t_n)$$

For t^* fixed and $t_n \to \infty$, $\int_0^{t^*} f(s)g(t_n - s)\,ds \to 0$ since f is bounded and $g \in L^1[0,\infty)$. This means that

$$V(t_n) \to \limsup_{t\to\infty} V(t) \le W(MG+1)+1 = B.$$

This completes the proof.

EXAMPLE 4. Let

$$x(t) = a(t) - \int_{-\infty}^{t} D(t,s)r(s,x(s))\,ds \tag{10_{NN}}$$

where a and D satisfy (11^*), (12^*), and

$$\int_{-\infty}^{t} [D_s(t,s)/g^2(t-s)]\,ds \le L,\ g \text{ defined in (15).} \tag{16^{**}}$$

Let r be continuous, bounded for x bounded, and suppose there is an $M > 0$ with $-2r(t,x)[x - a(t)] \le M - |r(t,x)|$. For $\varphi : (-\infty, 0] \to R$ bounded and continuous with $\varphi(0) = a(0) - \int_{-\infty}^{0} D(0,s)r(s,\varphi(s))\,ds$, let $x(t) = x(t,0,\varphi)$ solve (10_{NN}). Then for $f(t) = |r(t,x(t))|$ and

$$V(t) = \int_{-\infty}^{t} D_s(t,s)\left(\int_s^t r(v,x(v))\,dv\right)^2 ds,$$

the conditions of Theorem 4A are satisfied.

Proof. A calculation yields $V'(t) \le M - |r(t,x)|$ and

$$\begin{aligned}
(x(t)-a(t))^2 \le PV(t,x(\cdot)) &\le P\int_{-\infty}^{t} [D_s(t,s)/g^2(t-s)]\left(\int_s^t |r(v,x(v))|g(t-v)\,dv\right)^2 ds \\
&\le 2PL\left(\int_0^t |r(v,x(v))|g(t-v)\,dv\right)^2 \\
&+ 2PL\left(\int_{-\infty}^{0} |r(v,\varphi(v))|g(t-v)\,dv\right)^2 \\
&=: W\left(\int_0^t f(v)g(t-v)\,dv\right) + p(t).
\end{aligned}$$

4. Unbounded delay. While the theory for the following equations is generally quite different from that for (10) and (10_N), they can be treated in much the same way in this context. Let

$$x(t) = a(t) - \int_0^t D(t,s)x(s)\,ds \tag{20}$$

or

$$x(t) = a(t) - \int_0^t Q(t,s,x(s))\,ds \tag{20_N}$$

with Q continuous,

$$a : [0,\infty) \to R \text{ is continuous, } a \text{ and } a^2 \in L^1[0,\infty), \tag{21}$$

$$D(t,s) \geq 0, D(t,t) \leq P, D_t(t,s) \leq 0, D_s(t,s) \geq 0, tD(t,0) \to 0 \text{ as } t \to \infty. \tag{22}$$

Let

$$g : [0,\infty) \to (0,1], g(0) = 1, g \text{ decreasing, } \int_0^t g(s)\,ds \leq G \tag{23}$$

and suppose there are constants $L > 0$, $M \geq 0$ with $GM < 1$ and for $t \geq 0$ then

$$\int_0^t [D_s(t,s)(t-s)/g(t-s)]\,ds \leq L, \int_0^t \Big[(D_{st}(t,s))_+(t-s)/g(t-s)\Big]\,ds \leq M. \tag{24}$$

Now for each $t_0 \geq 0$ and each continuous $\varphi : [0,t_0] \to R$ there is a solution $x(t,t_0,\varphi)$ satisfying (20) if $t > t_0$ and $x(t,t_0,\varphi) = \varphi(t)$ for $0 \leq t \leq t_0$. We then require that

$$\varphi(t_0) = a(t_0) - \int_0^{t_0} D(t_0,s)\varphi(s)\,ds \tag{25}$$

so that $x(t,t_0,\varphi)$ is continuous on $[0,\infty)$ enabling us to integrate by parts when we compute V'.

Theorem 5A. *Suppose that for a continuous function $\psi : [0,\infty) \to R$, $\Psi(t) := a(t) - \psi(t) - \int_0^t Q(t,s,\psi(s))\,ds$ is in $L^1[0,\infty)$, and that there are continuous functions p, q :*

$[0,\infty) \to [0,\infty)$ with $p(t) \to 0$ as $t \to \infty$, $q \in L^1[0,\infty)$, a continuous funciton $V(t,x(\cdot))$ defined for a solution $x(t) = x(t,t_0,\varphi)$ of (20_N) with $\varphi \in \Omega$, and wedges W_i such that

$$(i) \qquad W_1(|x(t)-\Psi(t)|) \le V(t,x(\cdot)) \le W_2\left(\int_0^t W_3(|x(v)-\Psi(v)|)g(t-v)dv\right) + p(t)$$

where $W_1(r) \to \infty$ as $r \to \infty$ and

$$(ii) \qquad V(t,x(\cdot)) \le V(t_0) - \int_{t_0}^t W_3(|x(s)-\Psi(s)|)\,ds + \int_{t_0}^t q(s)\,ds.$$

Then the near equilibrium (ψ,Ψ) of (20_N) is asymptotically stable relative to Ω. If p and q depend on $x(t)$ then $|x(t)-\Psi(t)| \to 0$ as $t \to \infty$.

Theorem 5B. *Let (21) – (25) hold. Then there are wedges W_i such that $W_1(r) \to \infty$ as $r \to \infty$ and for any solution $x(t) = x(t,t_0,\varphi)$ of (20) with $\varphi \in \Omega$, there are continuous functions p, $q : [t_0,\infty) \to [0,\infty)$, $p(t) \to 0$ as $t \to \infty$, $q \in L^1[t_0,\infty)$ and a continuous function $V(t,x(\cdot))$ such that*

(i)

$$W_1(|x(t)-a(t)|) \le 2(D(0,0)+P)V(t,x(\cdot)) \le W_2\left(\int_0^t W_3(|x(v)-a(v)|)g(t-v)\,dv\right) + p(t)$$

and for $V(t) = V(t,x(\cdot))$ then

$$(ii) \qquad V(t) \le V(t_0) - \int_{t_0}^t W_3(|x(s)-a(s)|)\,ds + \int_{t_0}^t q(s)\,ds + G\int_0^{t_0} \varphi^2(s)\,ds$$

and $|x(t)-a(t)| \to 0$ as $t \to \infty$.

Proof. We consider Theorem 5B first. Let

$$V(t,x(\cdot)) = \int_0^t D_s(t,s)\left(\int_s^t x(v)\,dv\right)^2 ds + D(t,0)\left(\int_0^t x(s)\,ds\right)^2$$

so that

$$
\begin{aligned}
&(x(t)-a(t))^2 = \left(-\int_0^t D(t,s)x(s)\,ds\right)^2 = \\
&\left(D(t,s)\int_s^t x(v)\,dv\Big|_0^t - \int_0^t D_s(t,s)\int_s^t x(v)\,dv\,ds\right)^2 \\
&= \left(-D(t,0)\int_0^t x(v)\,dv - \int_0^t D_s(t,s)\int_s^t x(v)\,dv\,ds\right)^2 \\
&\le 2D^2(t,0)\left(\int_0^t x(v)\,dv\right)^2 + 2\int_0^t D_s(t,s)\,ds\int_0^t D_s(t,s)\left(\int_s^t x(v)\,dv\right)^2 ds \\
&\le 2D^2(t,0)\left(\int_0^t x(v)\,dv\right)^2 + 2\left[D(t,t)-D(t,0)\right]\int_0^t D_s(t,s)\left(\int_s^t x(v)\,dv\right)^2 ds \\
&\le 2\left[D(t,0)+D(t,t)\right]V(t,x(\cdot)) \le 2\left[D(0,0)+D(t,t)\right]V(t,x(\cdot))
\end{aligned}
$$

or

$$(x(t)-a(t))^2 \le 2[D(0,0)+P]V(t,x(\cdot))$$

satisfying the left-hand-side of (i).

Next

$$
\begin{aligned}
V'(t,x(\cdot)) &\le 2\int_0^t D_s(t,s)\int_s^t x(v)\,dv\,ds\,x(t) + D_t(t,0)\left(\int_0^t x(s)\,ds\right)^2 \\
&+ 2D(t,0)x(t)\int_0^t x(s)\,ds + \int_0^t D_{st}(t,s)\left(\int_s^t x(v)\,dv\right)^2 ds \\
&= 2x(t)\left[D(t,s)\int_s^t x(v)\,dv\Big|_0^t + \int_0^t D(t,s)x(s)\,ds\right] \\
&+ D_t(t,0)\left(\int_0^t x(s)\,ds\right)^2 + 2x(t)D(t,0)\int_0^t x(s)\,ds \\
&+ \int_0^t D_{st}(t,s)\left(\int_s^t x(v)\,dv\right)^2 ds \\
&= -2x(t)D(t,0)\int_0^t x(v)\,dv + 2x(t)[a(t)-x(t)] \\
&+ D_t(t,0)\left(\int_0^t x(s)\,ds\right)^2 + 2x(t)D(t,0)\int_0^t x(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t D_{st}(t,s)\Big(\int_s^t x(v)\,dv\Big)^2\,ds\\
&\le -x^2-(x-a(t))^2+a^2(t)+\int_0^t (D_{st}(t,s))_+(t-s)\int_s^t x^2(v)\,dv\,ds\\
&\le -x^2-(x-a(t))^2+a^2(t)+\int_0^t \Big[(D_{st}(t,s))_+(t-s)/g(t-s)\Big]\int_s^t x^2(v)g(t-s)\,dv\,ds\\
&\le -x^2-(x-a(t))^2+a^2(t)+M\int_0^t x^2(v)g(t-v)\,dv
\end{aligned}
$$

so that

$$
\begin{aligned}
V(t,x(\cdot)) \le V(t_0,\varphi) &- \int_{t_0}^t x^2(s)\,ds - \int_{t_0}^t (x(s)-a(s))^2\,ds + \int_{t_0}^t a^2(s)\,ds\\
&+ M\int_{t_0}^t\int_0^u x^2(v)g(u-v)\,dv\,du
\end{aligned}
$$

and the last term is

$$
\begin{aligned}
&M\int_{t_0}^t x^2(v)\int_v^t g(u-v)\,du\,dv + M\int_0^{t_0}\int_{t_0}^t x^2(v)g(u-v)\,du\,dv\\
&\le MG\int_{t_0}^t x^2(v)\,dv + M\int_0^{t_0} x^2(v)G\,dv
\end{aligned}
$$

and this verifies (ii). In particular, $a^2 \in L^1[0,\infty)$ and so is $(x(t)-a(t))^2$, as $MG<1$ we can also argue that $x^2 \in L^1$.

To satisfy the right-hand-side of (i), we note that

$$
D(t,0)\Big(\int_0^t x(v)\,dv\Big)^2 \le D(t,0)t\int_0^t x^2(v)\,dv =: p_1(t) \to 0
$$

as $t\to\infty$ by (22) and the fact that $x^2 \in L^1[0,\infty)$. Next,

$$
\begin{aligned}
&\int_0^t D_s(t,s)\Big(\int_s^t x(v)\,dv\Big)^2\,ds\\
&\le \int_0^t D_s(t,s)2\Big\{\Big(\int_s^t |x(v)-a(v)|\,dv\Big)^2+\Big(\int_s^t |a(v)|\,dv\Big)^2\Big\}\,ds\\
&\le 2\int_0^t [D_s(t,s)(t-s)/g(t-s)]\int_s^t (x(v)-a(v))^2 g(t-s)\,dv\,ds\\
&+2\int_0^t D_s(t,s)\Big(\int_s^t |a(v)|\,dv\Big)^2\,ds\\
&\le 2L\int_0^t (x(v)-a(v))^2 g(t-v)\,dv + 2\int_0^t D_s(t,s)\Big(\int_s^t |a(v)|\,dv\Big)^2\,ds\\
&=: 2L\int_0^t (x(v)-a(v))^2 g(t-v)\,dv + p_2(t)
\end{aligned}
$$

and L is defined in (24), g is defined in (23), while one can argue from (24) and $a \in L^1[0, \infty)$ that $p_2(t) \to 0$ as $t \to \infty$. Clearly, the integral on the right tends to zero as it is the convolution of an L^1-function and a function tending to zero. Also, $p(t) = 2(D(0,0) + P)(p_1(t) + p_2(t)) \to 0$ as $t \to \infty$ and so (i) is satisfied.

Looking now at Theorem 5A, since (ii) implies that $W_3(|x(t) - \Psi(t)|) \in L^1[0, \infty)$, it readily follows that $V(t) \to 0$. The stability relation follows by familiar arguments.

Clearly, Theorem 4A applies to (20) and (20_N) as well.

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