THE SEARCH FOR PERIODIC SOLUTIONS IN NEURAL NETWORKS

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Abstract. The well-known Hopfield neural network has an equilibrium set which is asymptotically stable. That network is an approximation in that the neuron has a zero threshold and, hence, charges are immediately passed on. In this expository note we present a mathematical model which suggests that for certain values of the threshold there will be periodic solutions.

1. Introduction. In this expository article we examine a physiological neural network, following a description of Hopfield [11], and present a mathematical model which takes into account both the averaging and the threshold. We discuss the equation without threshold and present reasoning which casts doubt on the existence of periodic solutions. We also present a mechanical model, attributed to van der Pol in a different context, which strongly suggests that there will be periodic solutions. The background for the problem will now be discussed.

Cohen, Grossberg, and Hopfield ([7], [10], [11]) consider systems of differential equations governing a neural network. Their work inspired a large industry, as may be seen in the survey book of Miller [15] and in ([5], [6], [8], [9]), for example. The book by Miller is interesting since both the investigators and their work is discussed in some detail.

Hopfield describes n neurons connected at a synapse by

$$C_i(du_i/dt) = \sum_{j=1}^{n} T_{ij}V_j - I_i - (u_i/R_i), \quad i = 1, 2, \ldots, n. \tag{1a}$$

Here, u_i is the charge on the ith neuron, $T_{ij} = T_{ji}$ is the efficiency of the synapse, C_i and R_i are the capacitance and resistance of the cell membrane, I_i is any other input to the ith neuron, while $V_j(u_j)$ is the input-output relation at the synapse and is sigmoidal with

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$V_j(0) = 0$, $V'_j(u_j) > 0$, and $|V_j(r)| \to 1$ as $|r| \to \infty$. For Hopfield, C_i, R_i, I_i, and T_{ij} are all constants.

Equation (1) is an approximation in two important aspects. First, charges from the jth neurons reaching the synapse sum; next, if the sum reaches a certain threshold, the neuron fires, while these charges rapidly dissipate and are gone in 3–4 miliseconds if the threshold is not reached. It is, therefore, very clear that the problem needs to be formulated as a delay equation with a distributed delay and with a fading memory.

While we do not attempt to model it mathematically, the threshold has two stages. If the neuron has not recently fired, then it has a threshold T_1; but if it has recently fired, then it has a threshold $T_2 < T_1$. Our mechanical model shows such a two-stage threshold in an interesting way.

System (1a) has been studied by computer simulation (e.g., [13]). In the process a delay is introduced owing to switching times. This delay seems to give rise to sustained oscillations, as suggested by Marcus and Westervelt [13], and further confirmed by Wu [16]. But it is well-known that (1a) does not have sustained oscillations; the equilibrium set is globally asymptotically stable, as shown by both Hopfield [11], and, more generally, by Cohen and Grossberg [7] who used an invariance principle. This introduces an interesting question. An invariance principle is generally used when the derivative of the Liapunov function is only negative semi-definite (cf. LaSalle [12]). Equation (1a) is an approximation and the natural conjecture is that with a small perturbation, sustained oscillations might appear; and this is consistent with the idea of a Liapunov function yielding asymptotic stability through a negative semi-definite derivative.

But, to the contrary, we note here that such a conjecture of sustained oscillations is unsound because (1a) is a near gradient system so that the Liapunov function yields extremely strong asymptotic stability under large (autonomous) perturbations. Moreover, if the entire right-hand-side of (1a) is averaged, this strong asymptotic stability remains.

It is to be carefully noted that the delay introduced by switching times and the distributed delay owing to the properties of the neuron are entirely independent.

We have studied (1a) in two earlier papers ([2], [3]). In the first one [2], we averaged the right-hand-side of (1a), except for the term u_i/R_i, and gave conditions under which the equilibrium set of (1a) was still asymptotically stable. Next, we studied (1a) by averaging the entire right-hand-side of (1a) and found that the equilibrium set was still asymptotically stable, using an argument parallel to an invariance principle. We point out here that that system is still nearly gradient so that it is strongly stable under perturbations.

Thus, the evidence was growing that sustained oscillations would not exist. This paper was motivated by a chance encounter with a mechanical model attributed to van der Pol which seems to strongly resemble the model described by Hopfield. And van der Pol's model not only has periodic solutions, but suggests where we should search for periodic solutions in the distributed Hopfield model.

2. Gradient systems. System (1a) is analysed by use of the Liapunov function

$$E(u) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij}V_iV_j - \sum_{i=1}^{n} I_iV_i + \sum_{i=1}^{n}\frac{1}{R_i}\int_0^{V_i} g_i^{-1}(s)\,ds \tag{2}$$

and it is readily shown that (1a) is really

$$C_i du_i/dt = -(\partial E/\partial u_i)/g_i'(u_i). \tag{1b}$$

Thus, (1b) is almost a gradient system and if $u(t)$ is a solution of (1b), then

$$\begin{aligned} dE(u(t))/dt &= -\sum_{i=1}^{n}(\partial E/\partial u_i)^2/C_ig_i'(u_i) \\ &= -\sum_{i=1}^{n} C_ig_i'(u_i)du_i/dt)^2. \end{aligned}$$

If, for brevity, we ask that $\int_0^{V_i} g_i^{-1}(s)\,ds \to \infty$ as $|V_i| \to \infty$, then all solutions are bounded. It readily follows from (3) that solutions have finite arc length. Moreover, for a positive definite Liapunov function $H(u)$ and a system

$$du/dt = f(u), \tag{4}$$

the derivative of H along a solution of (4) satisfies

$$(5) \qquad dH(u(t))/dt = (\operatorname{grad} H) \cdot f = |\operatorname{grad} H|\,|f| \cos\theta$$

where θ is the angle between $\operatorname{grad} H$ (the outward drawn normal to the set $H(u) =$ constant) and f. If $dH/dt < 0$, then f points inside the aforementioned set. But if (4) is a gradient system,

$$(6) \qquad du/dt = -\operatorname{grad} H,$$

then

$$(7) \qquad dH(u)/dt = -|\operatorname{grad} H|^2;$$

that is, $\theta = \Pi$ and it will require a very large perturbation in (6) to change from asymptotic stability to sustained oscillations.

A far more precise discussion of contours of Liapunov functions was recently given by Chamberland and Lewis [4].

In [3] we formed a weighted average of the right-hand-side of (1a), writing

$$(1c) \qquad du_i/dt = -\left(1/\sqrt{g_i'(u_i)}\right) \int_{t-T}^{t} \left\{ \left(a_i(t-s)\partial E(u(s))/\partial u_i\right) / \sqrt{g_i'(u_i)} \right\} ds$$

where

$$(8) \qquad a_i(0) > 0, \quad a_i'(t) \le 0, \quad a_i(T) = 0, \quad a_i''(t) > 0.$$

The effect of $a_i(t-s)$ is to weight the recent charges more heavily than the earlier ones, which die out entirely within T-time units. The integral sums all surviving charges. We proved the following result.

Theorem 1. *If (8) is satisfied then solutions of (1c) converge to the equilibrium set of (1a).*

The proof is based on the Liapunov functional

$$V(u) = 2E(u) - \sum_{i=1}^{n} \int_{t-T}^{t} a_i'(t-s)\left(\int_s^t \left(\partial E(u(v))/\partial u_i\right)/\sqrt{g_i'(u_i(v))}\,dv\right)^2 ds \tag{9}$$

which yields

$$dV(u(t))/dt \le -\beta \sum_{i=1}^{n} (du_i/dt)^2, \tag{10}$$

along any bounded solution, where β is a positive constant (possibly) depending on the bound on the solution. Thus, the strong asymptotic stability persists and no sustained oscillations can be expected. Clearly, we must take a closer look at Hopfield's description.

3. An average and a threshold. With E defined in (2), let

$$F(u) = E(u) - \sum_{i=1}^{n} \frac{1}{R_i} \int_0^{V_i} g_i^{-1}(s)\,ds - \sum_{i=1}^{n} I_i V_i \tag{11}$$

so that Hopfield's system is

$$C_i du_i/dt = -\frac{1}{R_i} u_i + I_i - (\partial F(u) \partial u_i)/g_i'(u_i) \tag{12}$$

which we write in the obvious way as

$$du_i/dt = -\gamma_i u_i + \overline{I_i} - f_i(u_i). \tag{13}$$

Note that if there is no charge transmitted from the neuron, then u_i exponentially approaches its resting potential from the relation

$$du_i/dt = -\gamma_i u_i + \overline{I_i}.$$

Now, consider weighting functions $a_i : [0,T] \to [0,\infty)$ satisfying (8). We average the charge to the neuron and form

$$du_i/dt = -\gamma_i u_i + \overline{I_i} - \int_{t-T}^{t} \left[a_i(t-s)(\partial F(u(s))/\partial u_i)/g_i'(u_i(s))\right] ds \tag{1d}$$

where $\int_{t-T}^{t} a_i(t-s)\,ds = 1/C_i$. Thus, (1d) has the same equilibrium set as (1a).

In [2] we considered a generalization of (1d), which included infinite delay, and proved under very restrictive conditions that all solutions approach the equilibrium set of (1a). The reader may consult [2] for a theorem for (1d) parallel to Theorem 1.

Here, we complete the model by defining

$$H_i(r) = \begin{cases} 0 & \text{if} \quad |r| < r_i \\ r & \text{if} \quad |r| \geq r_i \end{cases} \tag{14}$$

with r_i being the threshold of the ith synapse. Then

$$du_i/dt = -\gamma_i u_i + \overline{I_i} - H_i\left(\int_{t-T}^{t} \left[a_i(t-s)(\partial F(u(s))/\partial u_i)/g_i'(u_i(s))\right] ds\right) \tag{1e}$$

is a descriptive model that follows Hopfield's discussion. It is descriptive in the sense discussed by Maynard Smith [14; p. 19]. However, it falls short of our discussion in that is has only a one-stage threshold.

The system is more than complicated. It is formidable; and any serious investigation needs to be preceeded and, in some sense, directed by a mechanical model which convinces the investigator that there may be periodic solutions and where to look for them. Such a model is discussed in the next section.

4. A mechanical model. B. van der Pol was one of the early investigators of nonlinear oscillations and he constructed a number of ingenious devices to generate them. His papers are now hard to locate and some of them are in the Dutch language, but a typical example is easily accessible in [1].

An interesting model, thought to be of his design, can be found in the lobby of the Holiday Inn near the Narita airport outside Tokyo, Japan. It is placed in a beautiful setting and exhibits a striking parallel to the Hopfield description. We embellish it slightly to fit our description here. It displays periodic motion and it clearly directs us to assumptions which should lead to such motion. Still, the analysis is expected to be non-trivial. In our model we imagine that all charges come together at one synapse and when the threshold is reached the synapse delivers the charges to the neuron. This delivery causes the neuron to fire.

Several small streams of water trickle down the face of a cliff to meet and form a single stream (the charges in many neurons are delivered to a single synapse) which runs into a large wooden dipper (the synapse). The wooden dipper has a long heavy handle. A bolt runs through the handle near the cup and into a post, while the end of the handle rests on the top of another post so that the dipper handle is horizontal when the dipper is less than full of water.

The two-stage threshold is accomplished as follows. There is a small hole in the bottom of the dipper cup. If the water runs into the cup no faster than it runs out the hole, then the cup never fills (charges dissipate so that the threshold is never reached). If the water runs into the cup faster than it leaks out the hole, then the dipper fills (the threshold is reached); the full cup then over-balances the heavy handle, and very quickly the handle raises and part of the contents of the cup empties (the neuron fires) into a container below the cup (the ith neuron). Since the dipper cup only partially empties, if the water continues to run into the cup at least as quickly as before, then less water is required to fill the cup the second time and so the threshold is reached more quickly. This is the second stage threshold.

We now describe the container (the ith neuron) below the dipper cup which catches the emptied water. This container has sides which are a seive above a height I_i (the resting potential of neuron i). Water runs out the seive at a rate proportional to the volume above I_i (neuron i transmits its charge). See Figure 1.

We interpret this in two ways.

First, if the water runs too slowly, the dipper never fills and the water level in the seive container approaches I_i. If the water runs too fast, the dipper is always in the emptying position and the seive is always full. There is an interval of flow between these two extremes so that the dipper empties periodically and the level of water in the seive container rises and falls in a periodic manner. One observes this periodic motion in the mechanical model at Narita.

But for our problem we focus on the threshold. If the threshold is zero, then the dipper is always in the emptying position and the system approaches the equilibrium set. If the threshold is infinite (the end of the handle is attached to its post), the cup never empties and the level in the container approaches I_i, a new equilibrium set.

Conjecture. *There is a range for r_i in (14), say $r_i^0 \le r_i \le r_i^1$, such that (1e) has a periodic solution.*

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