PERIODIC SOLUTIONS OF VOLTERRA EQUATIONS AND ATTRACTIVITY

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Abstract. In this paper we study Volterra integral equations with a view to proving the existence of periodic solutions. Our techniques center on limiting equations, Liapunov functions, the theory of minimal solutions, and contraction mappings.

AMS (MOS) subject classification. 45DO5, 45M15

1. INTRODUCTION AND SUMMARY

In this paper we study the behavior of solutions of

$$x(t) = a(t) - \int_0^t D(t,s,x(s))\,ds, \tag{1A}$$

$$x(t) = a(t) - \int_{-\infty}^t D(t,s,x(s))\,ds, \tag{1B}$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t,s,x(s))\,ds \tag{2}$$

and their relation to each other. Equation (2) is a limiting equation of (1A), while Equation (1B) is a perturbed equation of (2). Conditions on a, p, D, and P are given later, but all of them are at least continuous.

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These equations have two very different types of solutions. Equation (1A) may have a solution $x(t)$ satisfying (1A) on $[0,\infty) =: R^+$; similarly, (1B) or (2) may have a solution $x(t)$ satisfying (2) on $(-\infty,\infty) =: R$, such as a periodic solution of (2). By contrast, given a continuous initial function φ on $[0,t_0)$, we write (1A) as

$$\begin{aligned} x(t) &= a(t) - \int_0^{t_0} D(t,s,\varphi(s))\,ds - \int_{t_0}^{t} D(t,s,x(s))\,ds \\ &=: \Phi(t) - \int_{t_0}^{t} D(t,s,x(s))\,ds. \end{aligned}$$

With Φ and D continuous, there is a solution $x(t,t_0,\varphi)$ on an interval $[t_0,\alpha)$ with $x(t,t_0,\varphi) = \varphi(t)$ for $0 \le t < t_0$, $x(t_0,t_0,\varphi) = \Phi(t_0)$, and $x(t,t_0,\varphi)$ satisfying the equation on $[t_0,\alpha)$; if the solution remains bounded then $\alpha = \infty$, as may be seen for example in [1; p. 79]. Clearly, $x(t,t_0,\varphi)$ may have a discontinuity at t_0. In the same way, for a given continuous initial function φ which is defined on $(-\infty,t_0)$, if P and $p(t) - \int_{-\infty}^{t_0} P(t,s,\varphi(s))\,ds$ are continuous, then there is a solution $x(t,t_0,\varphi)$ of (2) as just described. Under certain conditions, this last solution can be translated to the left, ultimately producing a solution of (2) which satisfies (2) on the whole line R; this is part of the theory of limiting equations. Solutions of (2) on all of R can also be obtained by contraction mappings and by other fixed point theorems, notably that of Schaefer [6], parallel to methods used in [2].

Excellent up to date collections of results for these equations are found in Corduneanu [3] and Gripenberg-Londen-Staffans [4].

We are interested in boundedness, periodicity, and convergence of solutions. Our methods include contraction mappings, Liapunov functions, minimal solutions, and limiting equations. Several lemmas are given of some independent interest, but the following overview will assist the reader in understanding the direction of the paper.

In Theorem 1 we suppose that (2) has a unique solution $x_0(t)$ on all of R which is bounded. Then it is periodic and any bounded solution of (1A) with initial function approaches $x_0(t)$ as $t \to \infty$.

Continuing the idea in Theorem 2, we suppose that (2) has a unique solution $x_0(t)$ on all of R which is bounded. We conclude that any bounded solution $x(t)$ of (1B) or (2) with bounded initial function converges to $x_0(t)$ as $t \to \infty$.

Thus, it is crucial to be able to detect a unique solution of (2) on all of R which is bounded. Theorems 3–6 show that when $P(t,s,x) = E(t,s)g(t,x)$, then there are conditions establishing such solutions of (2). Under additional conditions on periodicity and growth, we obtain a solution of (2) on all of R which is bounded, unique, periodic, and which attracts all bounded solutions of (1A) with initial functions.

Theorem 7 yields a T-periodic solution of a linear form of (2) using the theory of minimal solutions.

2. QUALITATIVE BEHAVIOR

Consider the systems of Volterra equations

$$x(t) = a(t) - \int_0^t D(t, s, x(s))\, ds, \quad t \in R^+, \tag{1A}$$

$$x(t) = a(t) - \int_{-\infty}^t D(t, s, x(s))\, ds, \quad t \in R, \tag{1B}$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t, s, x(s))\, ds, \quad t \in R, \tag{2}$$

where $a : R \to R^n$, $p : R \to R^n$, $D : R \times R \times R^n \to R^n$ and $P : R \times R \times R^n \to R^n$ are continuous, and

$$p(t+T) = p(t) \quad \text{and} \quad q(t) := a(t) - p(t) \to 0 \text{ as } t \to \infty, \tag{3}$$

where $T > 0$ is constant,

$$P(t+T, s+T, x) = P(t, s, x) \quad \text{and} \quad Q(t, s, x) := D(t, s, x) - P(t, s, x), \tag{4}$$

and for any $J > 0$ there are continuous functions $P_J : R \times R \to R^+$ and $Q_J : R \times R \to R^+$ such that

$$P_J(t+T, s+T) = P_J(t, s) \text{ if } t, s \in R,$$
$$|P(t, s, x)| \le P_J(t, s) \text{ if } t, s \in R \text{ and } |x| \le J,$$

where $|\cdot|$ denotes the Euclidean norm of R^n, and

$$|Q(t, s, x)| \le Q_J(t, s) \text{ if } t, s \in R \text{ and } |x| \le J,$$

$$\int_{-\infty}^t P_J(t+\tau, s)\, ds \to 0 \text{ uniformly for } t \in R \text{ as } \tau \to \infty, \tag{5}$$

$$\int_0^t Q_J(t, s)\, ds \to 0 \text{ as } t \to \infty \tag{6A}$$

or

(6B)

$$\int_{-\infty}^t Q_J(t, s)\, ds \to 0 \text{ as } t \to \infty, \text{ and } \int_{-\infty}^t Q_J(t+\tau, s)\, ds \to 0 \text{ uniformly for } t \in R \text{ as } \tau \to \infty.$$

First we discuss a relation between solutions of (1A) and (2).

Lemma 1. *Suppose that (3)–(5) and (6A) hold, and that (1A) has an R^+-bounded solution $x(t)$ with an initial time $t_0 \in R^+$. Then, for any non-decreasing sequence $\{s_k\}$ of nonnegative numbers with $s_k \to \infty$ as $k \to \infty$, the sequence of functions $\{x_k(t)\}$ contains a subsequence which converges to an R-bounded solution $y(t)$ of the equation*

$$(2_\sigma) \qquad x(t) = p(t+\sigma) - \int_{-\infty}^{t} P(t+\sigma, s+\sigma, x(s))\, ds, \quad t \in R$$

uniformly on any compact subset of R, where $x_k(t)$ is defined by

$$x_k(t) := \begin{cases} x(0), & t < -s_k, \\ x(t+s_k), & t \geq -s_k, \end{cases} \quad t \in R,$$

σ is a number with $0 \leq \sigma < T$, and $y(t)$ satisfies (2_σ) on R.

Proof. It is clear that the set $\{x_k(t)\}$ is uniformly bounded on R. Let $x(t)$ denote again the R-extension of the solution $x(t)$ obtained by defining $x(t) = x(0)$ for $t < 0$. From (3)–(5), (6A) and the R-boundedness of $x(t)$, it is easy to see that $x(t)$ is uniformly continuous on $[t_0, \infty)$, and since $x_k(t)$ is obtained by an s_k-translation to the left of $x(t)$, for any $j \in N$, the set $\{x_k(t)\}_{k \geq j}$ is equicontinuous on $[t_0 - s_j, \infty)$, where N denotes the set of positive integers. Thus, taking a subsequence if necessary, we may assume that the sequence $\{x_k(t)\}$ converges to a bounded continuous function $y(t)$ uniformly on any compact subset of R.

Now we show that $y(t)$ satisfies (2_σ) on R for some σ with $0 \leq \sigma < T$. For each $k \in N$, let ν_k be an integer with $\nu_k T \leq s_k < (\nu_k + 1)T$, and let $\sigma_k := s_k - \nu_k T$. Then, taking a subsequence if necessary, we may assume that $\{\sigma_k\}$ converges to some σ with $0 \leq \sigma \leq T$. From (1A) we have

(7)

$$x_k(t) = p(t+\sigma_k) + q(t+s_k) - \int_{-s_k}^{t} P(t+\sigma_k, s+\sigma_k, x_k(s))\, ds - \int_0^{t+s_k} Q(t+s_k, s, x(s))\, ds.$$

Let $J > 0$ be a number with $\|x\| := \sup\{|x(t)| : t \in R\} \leq J$. From (5), for any $\varepsilon > 0$ there is a $\tau > 0$ with

$$(8) \qquad \int_{-\infty}^{t} P_J(t+\tau, s)\, ds < \varepsilon \text{ for all } t \in R.$$

Now from (3) and (6A), for any $t \in R$ we obtain

$$\lim_{k\to\infty} q(t+s_k) = 0$$

and

$$\limsup_{k\to\infty} \left| \int_0^{t+s_k} Q(t+s_k, s, x(s))\, ds \right| \leq \limsup_{k\to\infty} \int_0^{t+s_k} Q_J(t+s_k, s)\, ds = 0.$$

Moreover, from (8), for any $t \in R$ we have

$$\begin{aligned}
&\limsup_{k\to\infty}\left|\int_{-s_k}^{t} P(t+\sigma_k, s+\sigma_k, x_k(s))\,ds - \int_{-\infty}^{t} P(t+\sigma, s+\sigma, y(s))\,ds\right| \\
&\le \limsup_{k\to\infty}\left|\int_{t-\tau}^{t} (P(t+\sigma_k, s+\sigma_k, x_k(s)) - P(t+\sigma, s+\sigma, y(s)))\,ds\right| \\
&+ \limsup_{k\to\infty}\int_{-\infty}^{t-\tau} P_J(t+\sigma_k, s+\sigma_k)\,ds + \int_{-\infty}^{t-\tau} P_J(t+\sigma, s+\sigma)\,ds < 2\varepsilon.
\end{aligned}$$

which implies $\lim_{k\to\infty}\int_{-s_k}^{t} P(t+\sigma_k, s+\sigma_k, x_k(s))\,ds = \int_{-\infty}^{t} P(t+\sigma, s+\sigma, y(s))\,ds$. Thus, letting $k \to \infty$ in (7), we obtain

$$\text{(9)} \qquad y(t) = p(t+\sigma) - \int_{-\infty}^{t} P(t+\sigma, s+\sigma, y(s))\,ds, \quad t \in R.$$

Since (2_T) is equivalent to (2_0), (9) shows that $y(t)$ is an R-bounded solution of (2_σ) with $0 \le \sigma < T$ which satisfies (2_σ) on R.

Similarly, we can prove the following lemma, which we state without proof.

Lemma 2. *Suppose that (3)–(5) and (6B) hold, and that (1B) has an R-bounded solution $x(t)$ with an initial time in R. Then, for any sequence $\{s_k\}$ with $s_k \to \infty$ as $k \to \infty$, the sequence of functions $\{x_k(t)\}$ with $x_k(t) := x(t+s_k)$, $t \in R$ contains a subsequence which converges to an R-bounded solution $y(t)$ of (2_σ) uniformly on any compact subset of R, where σ is a number with $0 \le \sigma < T$, and $y(t)$ satisfies (2_σ) on R. In particular, if (2) has an R-bounded solution $x(t)$ which satisfies (2) on R, then the same conclusion holds for any sequence $\{s_k\}$ without (6B).*

Here we note that for any ρ and σ with $0 \le \rho$, $\sigma < T$, if (2_ρ) has an R-bounded solution which satisfies (2_ρ) on R, then (2_σ) has an R-bounded solution which satisfies (2_σ) on R. From this fact and Lemmas 1 and 2, we have the following two theorems.

Theorem 1. *Suppose that (3)–(5) and (6A) hold, (2) has a unique R-bounded solution $x_0(t)$ which satisfies (2) on R. Then $x_0(t)$ is T-periodic and any R^+-bounded solution of (1A) with an initial time in R^+ is asymptotically T-periodic and approaches $x_0(t)$ as $t \to \infty$.*

Proof. Let $x_1(t)$ be a function obtained by the T-translation to the left of $x_0(t)$. Then, clearly $x_1(t)$ is also an R-bounded solution of (2) which satisfies (2) on R. Thus, from the uniqueness of R-bounded solutions which satisfy (2) on R, $x_0(t)$ and $x_1(t)$ must be identical on R, that is, $x_0(t)$ is T-periodic.

Next, let $x(t)$ be an R^+-bounded solution of (1A) with an initial time in R^+, and let $x_k(t)$ be the sequence of functions as in Lemma 1 with $s_k = kT$. Then, from Lemma 1 and the uniqueness of R-bounded solutions which satisfy (2) on R, it is easy to see that $x_k(t)$ converges to $x_0(t)$ uniformly on $[0, T]$. This implies that $x(t)$ is asymptotically T-periodic and its T-periodic part is given by $x_0(t)$. This completes the proof.

Similarly, we can prove the following theorem, which we state without proof.

Theorem 2. *In addition to the assumptions of Lemma 2, suppose that (2) has a unique R-bounded solution $x_0(t)$ which satisfies (2) on R. Then $x_0(t)$ is T-periodic and any R-bounded solution of (1B) with an initial time in R is asymptotically T-periodic and approaches $x_0(t)$ as $t \to \infty$. In particular, any R-bounded solution of (2) with an initial time in R approaches $x_0(t)$ as $t \to \infty$ without (6B).*

Among the assumptions in Theorems 1 and 2, the uniqueness of R-bounded solutions of (2) which satisfy (2) on R seems to be most crucial. Here we give two different conditions which assure the uniqueness of R-bounded solutions of (2) which satisfy (2) on R.

First we seek a condition for the equation

$$x(t) = p(t) - \int_{-\infty}^{t} E(t, s) g(s, x(s))\, ds, \quad t \in R, \tag{11}$$

where $p : R \to R^n$, $E : R \times R \to R^{n \times n}$ and $g : R \times R^n \to R^n$ are continuous, and

$$E_s(t, s) \text{ is continuous, symmetric, and } E_{st}(t, s) \text{ is continuous,} \tag{12}$$

$$\int_{-\infty}^{t} (|E(t, s)| + |E_s(t, s)|(t - s)^2 + |E_{st}(t, s)|(t - s)^2)\, ds \text{ is } R\text{-bounded,} \tag{13}$$

where $|E| = \sup\{|Ex| : x \in R^n \text{ and } |x| = 1\}$, and

$$\int_{-\infty}^{t-\tau} |E_s(t, s)|(t - s)^2\, ds \to 0 \text{ uniformly for } t \in R \text{ as } \tau \to \infty, \tag{14}$$

$$\lim_{s \to -\infty} sE(t, s) = 0 \text{ for each fixed } t, \tag{15}$$

$$E_{st}(t, s) \text{ is negative (positive) semi-definite,} \tag{16}$$

and

$$g(t, x) - g(t, y) = H(t, x, y)(x - y) \text{ if } t \in R, x \in R^n \text{ and } y \in R^n, \tag{17}$$

where $H : R \times R^n \times R^n \to R^{n \times n}$ is continuous, symmetric and positive (negative) definite, and for any $J > 0$ there are $A_J > 0$ and B_J with

$$A_J \leq |H(t, x, y)| \leq B_J \text{ if } t \in R, |x| \leq J \text{ and } |y| \leq J.$$

For (11), which is a special case of (2), we have the following theorem.

Theorem 3. *If (12)–(17) hold, then (11) has at most one R-bounded solution which satisfies (11) on R.*

Proof. Let $x_i(t)\,(i=1,2)$ be R-bounded solutions of (11) which satisfy (11) on R with $\|x_i\| \le J\,(i=1,2)$, and let $z(t) := x_1(t) - x_2(t)$, $t \in R$.

Next we consider the case where $E_{st}(t,s)$ is negative semi-definite and $H(t,x,y)$ is positive definite. Then, from (11) and (17) we have

$$z(t) = -\int_{-\infty}^{t} E(t,s)H(s)z(s)\,ds, \quad t \in R, \tag{18}$$

where $H(s) := H(s, x_1(s), x_2(s))$. Let $v(t)$ be a function defined by

$$v(t) := \int_{-\infty}^{t} \left(\int_{s}^{t} z^*(v)H(v)\,dv \right) E_s(t,s) \int_{s}^{t} H(v)z(v)\,dv\,ds, \quad t \in R,$$

where z^* denotes the transpose of z. Then, from (12)–(18) we obtain

$$\begin{aligned} v'(t) &= \int_{-\infty}^{t} z^*(t)H(t)E_s(t,s)\int_{s}^{t} H(v)z(v)\,dv\,ds \\ &\quad + \int_{-\infty}^{t} \left(\int_{s}^{t} z^*(v)H(v)\,dv \right) E_{st}(t,s) \int_{s}^{t} H(v)z(v)\,dv\,ds \\ &\quad + \int_{-\infty}^{t} \left(\int_{s}^{t} z^*(v)H(v)\,dv \right) E_s(t,s)H(t)z(t)\,ds \\ &\le 2z^*(t)H(t) \int_{-\infty}^{t} E_s(t,s) \int_{s}^{t} H(v)z(v)\,dv\,ds \\ &\le 2z^*(t)H(t) \left(\left[E(t,s)\int_{s}^{t} H(v)z(v)\,dv \right]_{-\infty}^{t} + \int_{-\infty}^{t} E(t,s)H(s)z(s)\,ds \right) \\ &= -2z^*(t)H(t)z(t) \le -2A_J|z(t)|^2, \end{aligned} \tag{19}$$

which together with the R-boundedness of $v(t)$ implies

$$Z := \int_{-\infty}^{\infty} |z(s)|^2\,ds < \infty.$$

Thus we have $\int_s^t |z(v)|^2\,dv \to 0$ uniformly for $s < t$ as $t \to -\infty$. This together with (20)

$$|v(t)| \le \int_{-\infty}^{t} |E_s(t,s)| \left(\int_{s}^{t} B_J|z(v)|\,dv \right)^2 ds \le B_J^2 \int_{-\infty}^{t} |E_s(t,s)|(t-s) \int_{s}^{t} |z(v)|^2\,dv\,ds$$

yields that $v(t) \to 0$ as $t \to -\infty$. Thus, from (19) we obtain that $v(t) \le 0$ on R. If $v(t) \equiv 0$ on R, then (19) implies $A_J|z(t)|^2 \le 0$, and hence $z(t) \equiv 0$ on R. Otherwise, for some $\beta > 0$ and $t_0 \in R$ we have

$$v(t) \le -\beta \text{ for } t \ge t_0. \tag{21}$$

For $\Pi := \sup\left\{\int_{-\infty}^{t} |E_s(t,s)|(t-s)\,ds : t \in R\right\}$ and β there is a $G > 0$ with

$$\int_G^{\infty} |z(s)|^2\,ds < \frac{\beta}{2B_J^2\Pi}.$$

Moreover, from (14), for β and G there is a $t_1 \geq G$ with

$$\int_{-\infty}^{G} |E_s(t,s)|(t-s)\,ds < \frac{\beta}{2B_J^2 Z} \text{ for } t \geq t_1.$$

Thus from (20), for $t \geq t_1$ we obtain

$$\begin{aligned}
|v(t)| &\leq B_J^2\left(\int_{-\infty}^{G} |E_s(t,s)|(t-s)\int_s^t |z(v)|^2\,dv\,ds + \int_G^t |E_s(t,s)|(t-s)\int_s^t |z(v)|^2\,dv\,ds\right)\\
&\leq B_J^2\left(\int_{-\infty}^{G} |E_s(t,s)|(t-s)\int_{-\infty}^{\infty} |z(v)|^2\,dv\,ds\right.\\
&\qquad\left.+\int_G^t |E_s(t,s)|(t-s)\int_G^t |z(v)|^2\,dv\,ds\right)\\
&< \frac{\beta}{2} + \frac{\beta}{2} = \beta,
\end{aligned}$$

which contradicts (21). Thus we have $z(t) \equiv 0$ on R.

In the case where $E_{st}(t,s)$ is positive semi-definite and $H(t,x,y)$ is negative definite, taking $-v$ instead of v, we can similarly conclude that $z(t) \equiv 0$, which completes the proof.

Next we give a condition of a contraction type. Suppose that $p : R \to R^n$ and $P : R \times R \times R^n \to R^n$ are continuous, and that for any $J > 0$ there is a continuous function $L_J : R \times R \to R^+$ with

(22) $\qquad |P(t,s,x) - P(t,s,y)| \leq L_J(t,s)|x-y|$ if $t,s \in R$, $|x| \leq J$ and $|y| \leq J$.

Then we have the following lemma.

Lemma 3. *In addition to (22), if for any $J > 0$*

(23) $$\lambda_J := \sup\left\{\int_{-\infty}^{t} L_J(t,s)\,ds : t \in R\right\} < 1$$

holds, then (2) has at most one R-bounded solution which satisfies (2) on R.

Proof. Let $x_i(t)\,(i = 1,2)$ be R-bounded solutions of (2) which satisfy (2) on R with $\|x_i\| \leq J\,(i = 1,2)$, and let $z(t) := x_1(t) - x_2(t)$, $t \in R$. Then, from (2) we have

$$z(t) = -\int_{-\infty}^{t} (P(t,s,x_1(s)) - P(t,s,x_2(s)))ds, \quad t \in R,$$

which together with (22) yields

$$|z(t)| \le \int_{-\infty}^{t} L_J(t,s)|z(s)|\,ds \le \lambda_J \|z\|, \quad t \in R. \tag{24}$$

Thus, (23) and (24) imply that $z(t) \equiv 0$ on R.

Using Theorem 2 and Lemma 3, we have the following theorem.

Theorem 4. *In addition to (3)–(5) with $q(t) \equiv 0$ and $Q(t,s,x) \equiv 0$, (22) and (23), if*

$$\lambda := \sup\{\lambda_J : J > 0\} < 1 \tag{25}$$

holds, then (2) has a unique T-periodic solution, and it is a unique R-bounded solution which satisfies (2) on R. Moreover, any R-bounded solution of (2) with an initial time $t_0 \in R$ and a bounded continuous initial function $\varphi : (-\infty, t_0) \to R^n$, approaches the T-periodic solution as $t \to \infty$.

Proof. First we prove that (2) has a unique T-periodic solution. Let $(\mathcal{P}_T, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\xi : R \to R^n$ with the supremum norm $\|\cdot\|$, and define a map H on $\mathcal{P}_T$ by

$$(H\xi)(t) := p(t) - \int_{-\infty}^{t} P(t,s,\xi(s))\,ds, \quad t \in R.$$

Then, from (3)–(5) with $q(t) \equiv 0$ and $Q(t,s,x) \equiv 0$, it is easy to see that H maps $\mathcal{P}_T$ into $\mathcal{P}_T$. Moreover, for any $\xi_i \in \mathcal{P}_T$ with $\|\xi_i\| \le J$ $(i = 1, 2)$ for some $J > 0$, we have

$$|(H\xi_1)(t) - (H\xi_2)(t)| \le \int_{-\infty}^{t} L_J(t,s)|\xi_1(s) - \xi_2(s)|\,ds \le \lambda_J \|\xi_1 - \xi_2\|, \quad t \in R,$$

which together with (25) yields

$$\|H\xi_1 - H\xi_2\| \le \lambda \|\xi_1 - \xi_2\|.$$

Thus $H : \mathcal{P}_T \to \mathcal{P}_T$ is a contraction mapping. Hence H has a unique fixed point in $\mathcal{P}_T$, which gives a unique T-periodic solution of (2), say $\pi(t)$.

Next, from Lemma 3, $\pi(t)$ is the unique R-bounded solution of (2) which satisfies (2) on R. Thus, the latter part is a direct consequence of Theorem 2.

This theorem does not necessarily give asymptotic behavior of all solutions of (2). But, for some linear equations, we can obtain asymptotic behavior of all solutions. Consider the linear equation

$$x(t) = p(t) - \int_{-\infty}^{t} P(t,s)x(s)\,ds, \quad t \in R, \tag{26}$$

where $p : R \to R^n$ and $P : R \times R \to R^{n\times n}$ are continuous. First we prove a simple lemma.

Lemma 4. *If*

$$(27) \qquad b(t) := \int_{-\infty}^{t} |P(t,s)|\, ds < 1, \quad t \in R$$

holds, then for any $t_0 \in R$ and any bounded continuous function $\varphi : (-\infty, t_0) \to R^n$, the solution $x(t) = x(t, t_0, \varphi)$ of (26) satisfies

$$|x(t)| \le X(t) := \max(\sup\{B(s) : t_0 \le s \le t\}, \sup\{|\varphi(s)| : s \le t_0\}, |x(t_0+)|), \quad t \ge t_0,$$

where

$$B(s) := \frac{1}{1-b(s)} \sup\{|p(u)| : t_0 \le u \le s\}, \quad s \ge t_0.$$

Proof. Suppose that the conclusion is false. Then there is a $\tau > t_0$ with $J := |x(\tau)| > X(\tau)$. Replacing τ if necessary, we may assume that $|x(t)| \le J$ on $(-\infty, \tau]$. Thus we have

$$\begin{aligned} |x(\tau)| &\le |p(\tau)| + \int_{-\infty}^{\tau} |P(\tau,s)||x(s)|\, ds \\ &\le \sup\{|p(s)| : t_0 \le s \le \tau\} + b(\tau)J < J = |x(\tau)|, \end{aligned}$$

which is a contradiction.

Combining Theorem 4 and Lemma 4, we have the following theorem, which we state without proof.

Theorem 5. *Suppose that $p(t+T) = p(t)$ and $P(t+T, s+T) = P(t,s)$. In addition to (27), if $b(t)$ is continuous, then (26) has a unique R-bounded solution which satisfies (26) on R, and it is T-periodic and globally attractive.*

In Theorem 1, the existence of an R^+-bounded solution of $(1A)$ with an initial time in R^+ is assumed. Here we consider a few cases where the existence of R^+-bounded solutions of (1A) with $Q(t,s,x) \equiv 0$ is assured.

First consider the equation

$$(28) \qquad x(t) = a(t) - \int_0^t P(t,s,x(s)), ds, \quad t \in R^+,$$

where $a : R^+ \to R^n$ is bounded continuous, and $P : R \times R \times R^n \to R^n$ is continuous and satisfies (22), (23) and (25). Let $(B, \|\cdot\|_+)$ be the Banach space of bounded continuous functions $\xi : R^+ \to R^n$ with the supremum norm $\|\cdot\|_+$, and define a map H on B by

$$(H\xi)(t) := a(t) - \int_0^t P(t,s,\xi(s))\, ds, \quad t \in R^+.$$

Then it is easy to see that H is a contraction mapping from B into B. Thus H has a unique fixed point, which gives a unique R^+-bounded solution of (28) which satisfies (28) on R^+. From this and Theorems 1 and 4, we have the following theorem.

Theorem 6. *Suppose that (3)–(5) with $Q(t,s,x) \equiv 0$, (22), (23) and (25) hold. Then (28) has a unique R^+-bounded solution which satisfies (28) on R^+ and (2) has a unique T-periodic solution. Moreover, any R^+-bounded solution $x(t) = x(t,t_0,\varphi)$ of (28) approaches the unique T-periodic solution of (2) as $t \to \infty$, where $t_0 \in R^+$ and $\varphi : [0,t_0) \to R^n$ is continuous.*

Proof. From the argument just before this theorem, it is easy to see that (28) has a unique R^+-bounded solution which satisfies (28) on R^+, say $\xi(t)$. Let $\xi(t)$ denote again the R-extension of the given $\xi(t)$ obtained by defining $\xi(t) := \xi(0) = a(0)$ for $t < 0$, and for any $k \in N$, let $\xi_k(t) = \xi(t + kT)$, $t \in R$. Since Theorem 4 implies that (2) has a unique T-periodic solution, say $\pi(t)$, and it is a unique R-bounded solution of (2) satisfying (2) on R, by Lemma 1, it is easily seen that $\xi_k(t)$ converges to $\pi(t)$ uniformly on $[0,T]$ as $k \to \infty$. Thus we obtain $\xi(t) - \pi(t) \to 0$ as $t \to \infty$. The latter part follows directly from Theorem 1.

Next consider the linear equation

$$x(t) = a(t) - \int_0^t P(t,s)x(s)\,ds, \quad t \in R^+, \tag{29}$$

where $a : R^+ \to R^n$ and $P : R \times R \to R^{n\times n}$ are continuous. Corresponding to Lemma 4, we have the following lemma, which we state without proof.

Lemma 5. *If (27) holds, then for any $t_0 \in R^+$ and any bounded continuous function $\varphi : [0,t_0) \to R^n$, the solution $x(t) = x(t,t_0,\varphi)$ of (29) satisfies*

$$|x(t)| \le \max(\sup\{c(s) : t_0 \le s \le t\}, \sup\{|\xi(s)| : 0 \le s \le t_0\}, |x(t_0+)|), \quad t \ge t_0,$$

where

$$c(s) := \frac{1}{1-b(s)} \sup\{|a(u)| : t_0 \le u \le s\}, \quad s \ge t_0.$$

From this lemma and Theorem 6, we obtain the following corollary.

Corollary 1. *In addition to (3), $P(t+T, s+T) = P(t,s)$ and (27), if $b(t)$ is continuous, then (29) has a unique R^+-bounded solution which satisfies (29) on R^+, and (26) has a unique T-periodic solution. Moreover, any solution $x(t) = x(t,t_0,\varphi)$ of (29) approaches the unique T-periodic solution of (26) as $t \to \infty$, where $t_0 \in R^+$ and $\varphi : [0,t_0) \to R^n$ is continuous.*

Finally consider (2) under (3)–(5) with $q(t) \equiv 0$ and $Q(t,s,x) \equiv 0$, and suppose that (2) has a unique R-bounded solution satisfying (2) on R, say $\pi(t)$. Then it is

T-periodic and it is easy to see that $\pi(t)$ is a solution of the equation

$$x(t) = p(t) + r(t) - \int_0^t P(t, s, x(s))\,ds, \quad t \in R^+, \tag{30}$$

where $r(t) := -\int_{-\infty}^0 P(t, s, \pi(s))\,ds$, $t \in R^+$. Moreover, from (5) we have that $r(t)$ is continuous and $r(t) \to 0$ as $t \to \infty$. Thus (30) is a special case of (1) with $q(t) = r(t)$ and $Q(t, s, x) \equiv 0$. From Theorem 1 and the argument in the proof of Theorem 6, we obtain the following corollary.

Corollary 2. *Suppose that (3)–(5) with $q(t) \equiv 0$ and $Q(t, s, x) \equiv 0$ hold, and that (2) has a unique R-bounded solution satisfying (2) on R, say $\pi(t)$. Then it is T-periodic and $\pi(t)$ is a unique R^+-bounded solution of (30) which satisfies (30) on R^+, and any R^+-bounded solution $x(t) = x(t, t_0, \varphi)$ of (30) approaches $\pi(t)$ as $t \to \infty$, where $t_0 \in R^+$ and $\varphi : [0, t_0) \to R^n$ is continuous.*

Now we show an example.

Example 1. Consider the scalar linear equation

$$x(t) = p(t) - \mu \int_{-\infty}^t e^{-t+s}(\sin t)\, x(s)\,ds, \quad t \in R, \tag{31}$$

where $p : R \to R$ is continuous T-periodic, and μ is a constant with $|\mu| < 1$. Equation (31) is a special case of (2) with $n = 1$ and $P(t, s, x) = \mu e^{-t+s}(\sin t)\, x$. Thus, (3) with $q(t) \equiv 0$, (4) with $Q(t, s, x) \equiv 0$, (11) with $L_J(t, s) = |\mu| e^{-t+s}$, (12) with $\lambda_J = |\mu|$, (25) with $\lambda = |\mu|$, and (27) with $b(t) = |\mu \sin t|$ hold. Thus, from Theorems 5 and 6, (31) has a unique R-bounded solution satisfying (31) on R, say $\pi(t)$, and it is T-periodic and globally attractive.

On the other hand, $\pi(t)$ is a unique R^+-bounded solution of the equation

$$x(t) = p(t) - \mu \int_{-\infty}^0 e^{-t+s}(\sin t)\, \pi(s)\,ds - \mu \int_0^t e^{-t+s}(\sin t)\, x(s)\,ds, \quad t \in R^+, \tag{32}$$

which satisfies (32) on R^+. Moreover, from Corollary 2, the T-periodic solution $\pi(t)$ of (32) is globally attractive.

In the above discussions, the existence of an R^+-bounded solution of (1A) does not necessarily imply the existence of T-periodic solutions of (2). But, if $P(t, s, x)$ is linear in x, then we can prove such an implication using the theory of minimal solutions as we see later.

Consider the equations (26) and

$$x(t) = a(t) - \int_0^t P(t, s)x(s)\,ds - \int_0^t Q(t, s, x(s))\,ds, \quad t \in R^+ \tag{33A}$$

or

$$(33B) \qquad x(t) = a(t) - \int_{-\infty}^{t} P(t,s)x(s)\,ds - \int_{-\infty}^{t} Q(t,s,x(s))\,ds, \quad t \in R,$$

where $a(t)$ and $D(t,s,x) := P(t,s)x + Q(t,s,x)$ satisfy (3)–(6) with $P_J(t,s) = J|P(t,s)|$. If (33A) (or (33B)) has an $R^+(orR)$-bounded solution with an initial time in R^+ (or R), then taking $s_k = kT$ in the proof of Lemma 1 (or 2), clearly we have that (26) has an R-bounded solution which satisfies (26) on R.

Following [5] of Hino and Murakami, let $h : R \to R^+$ be a continuous positive function with $\int_{-\infty}^{\infty} h(s)\,ds < \infty$. For any bounded continuous function $x : R \to R^n$, define a function $\lambda(x)$ by

$$\lambda(x) := \sup\left\{ \int_{-\infty}^{\infty} |x(s+t)|^2 h(s)\,ds : t \in R \right\},$$

and set

$$\Lambda = \inf \left\{ \lambda(x) : \begin{array}{c} x \text{ is an } R\text{-bounded solution of (26)} \\ \\ \text{such that } x \text{ solves (26) on } R \text{ and } \|x\| \le J \end{array} \right\},$$

where $J > 0$ is a fixed constant. Then we have the following two lemmas.

Lemma 6. *If (3)–(5) and (6A)(or (6B))hold with $D(t,s,x) = P(t,s)x + Q(t,s,x)$ and $P_J(t,s) = J|P(t,s)|$, and if (33A) (or (33B)) has an R^+ (or R)-bounded solution $x(t)$ such that its initial time is in R^+ (or R) and $\|x\|_+$ (or $\|x\|$) $\le J$ for some $J > 0$, then (26) has a minimal solution, that is, an R-bounded solution of (26) which attains the value Λ.*

Proof. Lemma 1 (or 2) assures the existence of an R-bounded solution $y(t)$ of (26) which solves (26) on R and $\|y\| \le J$. From the definition of Λ, there is a sequence $\{x_k(t)\}$ of R-bounded solutions of (26) which satisfies $\lambda(x_k) \le \Lambda + \frac{1}{k}$ and $\|x_k\| \le J$. Clearly the set of functions $\{x_k(t)\}$ is uniformly bounded on R. Moreover it is easy to see that for any $\varepsilon > 0$ there is a $\delta > 0$ with

$$|x_k(t_1) - x_k(t_2)| < \varepsilon \text{ if } k \in N \text{ and } |t_1 - t_2| < \delta,$$

which implies that the set $\{x_k(t)\}$ is equicontinuous on R. Thus the sequence $\{x_k(t)\}$ has a subsequence which converges to an R-bounded solution $c(t)$ of (26) which satisfies (26) on R and $\|c\| \le J$. Moreover, since we have $\int_{-\infty}^{\infty} |x_k(s+t)|^2 h(s)\,ds \le \lambda(x_k) \le \Lambda + \frac{1}{k}$, we obtain

$$\int_{-\infty}^{\infty} |c(s+t)|^2 h(s)\,ds \le \Lambda,$$

and hence $\lambda(c) \le \Lambda$. Thus we have $\lambda(c) = \Lambda$, because $\lambda(c) \ge \Lambda$ from the definition of Λ.

Lemma 7. *In addition to the assumptions of Lemma 6, if $c_k(t)\,(k=1,2)$ are minimal solutions of (26), then there is a sequence $\{t_k\}$ with $c_1(t+t_k)-c_2(t+t_k)\to 0$ uniformly on any compact subset of R as $k\to\infty$.*

Proof. Define functions d, $e: R\to R^n$ by

$$d(t):=\frac{c_1(t)+c_2(t)}{2}\ \text{and}\ e(t):=\frac{c_1(t)-c_2(t)}{2},\quad t\in R.$$

By the parallelogram theorem, we have

$$\int_{-\infty}^{\infty}|d(s+t)|^2h(s)\,ds+\int_{-\infty}^{\infty}|e(s+t)|^2h(s)\,ds$$
$$=\frac{1}{2}\Big(\int_{-\infty}^{\infty}|c_1(s+t)|^2h(s)\,ds+\int_{-\infty}^{\infty}|c_2(s+t)|^2h(s)\,ds\Big)\le\Lambda,$$

which implies

$$\int_{-\infty}^{\infty}|d(s+t)|^2h(s)\,ds\le\Lambda-\inf\Big\{\int_{-\infty}^{\infty}|e(s+t)|^2h(s)\,ds:t\in R\Big\}.$$

This together with the definition of Λ yields

$$\inf\Big\{\int_{-\infty}^{\infty}|e(s+t)|^2h(s)\,ds:t\in R\Big\}=0,$$

and hence, $\inf\Big\{\int_{-\infty}^{\infty}|c_1(s+t)-c_2(s+t)|^2h(s)\,ds:t\in R\Big\}=0$. From this, there is a sequence $\{t_k\}$ with

$$\lim_{k\to\infty}\int_{-\infty}^{\infty}|c_1(s+t_k)-c_2(s+t_k)|^2h(s)\,ds=0. \tag{34}$$

Since the set of functions $\{c_1(t+t_k)-c_2(t+t_k)\}$ is uniformly bounded and equicontinuous on R, taking a subsequence if necessary, we may assume that the sequence $\{c_1(t+t_k)-c_2(t+t_k)\}$ converges to a continuous function $\gamma(t)$ uniformly on any compact subset of R as $k\to\infty$. From (34), for any τ_1 and τ_2 with $\tau_1<\tau_2$ we have

$$\int_{\tau_1}^{\tau_2}|\gamma(s)|^2h(s)\,ds=0,$$

which together with the positivity of $h(t)$ on R implies $\gamma(t)\equiv 0$ on R. This completes the proof.

Now we have the following theorem.

Theorem 7. *Under the assumptions of Lemma 6, (26) has a T-periodic solution.*

Proof. Let $c(t)$ be a minimal solution of (26) assured in Lemma 6. Clearly $c(t+T)$ is also a minimal solution of (26). Thus, from Lemma 7 there is a sequence $\{t_k\}$ with $c(t+t_k) - c(t+T+t_k) \to 0$ as $k \to \infty$ uniformily on any compact subset of R. For each $k \in N$, let ν_k be an integer with $\nu_k T \le t_k < (\nu_k + 1)T$, and let $\sigma_k := t_k - \nu_k T$. Taking a subsequence if necessary, we may assume that $\sigma_k \to \sigma$ as $k \to \infty$ for some σ with $0 \le \sigma \le T$, and that for some bounded continuous function $\gamma(t)$ on R, $c(t+t_k) \to \gamma(t)$ uniformly on any compact subset of R as $k \to \infty$, since the set $\{c(t+t_k)\}$ is uniformly bounded and equicontinuous on R. Clearly $\gamma(t)$ is T-periodic. Moreover, since (5) holds with $P_J(t,s) = J|P(t,s)|$, from Lemma 2, $\gamma(t)$ is an R-bounded solution of

$$(26_\sigma) \qquad x(t) = p(t+\sigma) - \int_{-\infty}^{t} P(t+\sigma, s+\sigma)x(s)\,ds, \quad t \in R$$

for some σ with $0 \le \sigma < T$. Thus we have

$$\gamma(t) = p(t+\sigma) - \int_{-\infty}^{t} P(t+\sigma, s+\sigma)\gamma(s)\,ds, \quad t \in R.$$

For $\delta(t) := \gamma(t-\sigma)$, this equation can be rewritten as

$$\delta(t) = p(t) - \int_{-\infty}^{t} P(t,s)\delta(s)\,ds, \quad t \in R,$$

and hence $\delta(t)$ is a T-periodic solution of (26).

Among the assumptions of Theorem 7, the existence of an R^+-bounded solution of (33A) seems to be most crucial. Although a few cases are discussed just after Theorem 5 for the existence of R^+-bounded solutions of (1A) with $Q(t,s,x) \equiv 0$, for some scalar equation, a boundedness result is obtained under suitable conditions different from those assumed in the above cases. Consider the scalar equation

$$(35) \qquad x(t) = a(t) - \int_{0}^{t} E(t,s)g(s,x(s))\,ds, \quad t \in R^+,$$

where $a : R^+ \to R$, $E : R^+ \times R^+ \to R^+$ and $g : R^+ \times R \to R$ are continuous, and

$$(36) \qquad E(v,s)E(t,u) \le E(t,s)E(v,u) \text{ if } s \le u \le v \le t,$$

$$(37) \qquad \text{there is an } M \ge 0 \text{ with } E(t,s) \le ME(u,s) \text{ if } 0 \le s \le u \le t$$

and

$$(38) \qquad xg(t,x) \ge 0 \text{ if } t \in R^+ \text{ and } x \in R.$$

We can find the following result in [4, p. 620] of Gripenberg-Londen-Staffans.

Proposition. *If (36)–(38) hold, then*

$$|x(t)| \le (M+1)\sup\{|a(s)| : 0 \le s \le t\}, \quad t \in R^+,$$

where $x(t)$ is a solution of (35) which satisfies (35) on R^+.

From Theorem 7 and Proposition, we have the following corollary.

Corollary 3. *If $n = 1$, and if (3), $P(t+T, s+T) = P(t,s)$, and (36)–(38) hold, then (26) with $n = 1$ has a T-periodic solution $x(t)$ with $\|x\| \le (M+1)\|a\|_+$.*

Finally we show an example.

Example 2. Consider the scalar linear equation

$$x(t) = a(t) - \int_0^t e^{-t+s}(2+\sin t)x(s)\,ds, \quad t \in R^+, \tag{39}$$

where $a : R^+ \to R$ is continuous and satisfies (3) with $n = 1$ and $T = 2\pi$. Then (39) is a special case of (35) with $E(t,s) = e^{-t+s}(2+\sin t)$ and $g(s,x) = x$, and it is easy to see that the assumptions of Corollary 3 are satisfied with $M = 1$. Thus, Corollary 3 implies that the equation

$$x(t) = p(t) - \int_{-\infty}^t e^{-t+s}(2+\sin t)x(s)\,ds, \quad t \in R \tag{40}$$

has a 2π-periodic solution $x(t)$ with $\|x\| \le 2\|a\|_+$, while Theorems 4, 5 and 6 are not applicable to (40) since neither (25) nor (27) holds in this case.

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