PERIODIC SOLUTIONS OF A VOLTERRA EQUATION AND ROBUSTNESS

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1. Introduction. We consider the system of Volterra equation

$$x(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(s,x(s))\,ds \tag{1}$$

where $a : R \to R^n$, $D : R \times R \to R^{n\times n}$ and $g : R \times R^n \to R^n$ are all continuous,

$$a(t+T) = a(t), \quad D(t+T, s+T) = D(t,s) \quad \text{and} \quad g(t+T,x) = g(t,x), \tag{2}$$

where $T > 0$ is constant, and

$$D_s(t,s) \text{ is continuous and symmetric, } D_{st} \text{ is continuous.} \tag{3}$$

The object of this paper is to establish conditions which ensure that (1) has a T-periodic solution. These will involve convergence and sign conditions on D and g.

To this end, we first note that by (3) there is a continuous orthogonal matrix $P(t,s)$ such that

$$\Delta(t,s) := P^*(t,s)D_{st}(t,s)P(t,s), \quad (* \text{ is transpose }),$$

is a diagonal matrix. Let $\Delta(t,s) = \text{diag.}\,(\delta_j(t,s))$, and let

$$Q(t,s) = \text{diag.}\left(\sqrt{\left(\delta_j(t,s)\right)_+}\right) P^*(t,s)$$

where

$$(\delta)_+ = \max(\delta, 0).$$

We suppose that there are constants $\alpha > 1$, $0 < k < K$, $M > 0$ such that $0 < \lambda < 1$ implies that

$$2g^*(t,x)(\lambda a(t) - x) \le M - K|g(t,x)|^\alpha, \tag{4}$$

and either

$$\alpha = 2 \text{ and } \int_{-\infty}^{t} |Q(t,s)|^2 (t - s + T)(t - s)\, ds \le k, \tag{5a}$$

where $|Q| = \sup\{|Qx| : |x| = 1\}$,

$$\alpha > 2 \text{ and } \int_{-\infty}^{t} |Q(t,s)|^2 (t - s + T)(t - s)\, ds < \infty, \tag{5b}$$

or

$$1 < \alpha < 2 \text{ and diag } (\,(\delta_j(t,s))_+\,) = 0. \tag{5c}$$

Moreover, suppose that

$$\int_{-\infty}^{t} (|D(t,s)| + |D_s(t,s)|(t-s)^2 + |D_{st}(t,s)|(t-s)^2)\, ds \tag{6}$$

is continuous,

$$\lim_{s \to -\infty} (t-s)D(t,s) = 0 \text{ for fixed } t, \tag{7}$$

and there is a $B > 0$ such that for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and $t \in [0,T)$ we have

$$\sum_{j=0}^{\infty} \left(\int_{t-(j+1)T}^{t-jT} \left| D(t,s) \right|^\beta ds \right)^{1/\beta} \le B. \tag{8}$$

This condition can be weakened, as we show following our main result.

These are the conditions we will need to ensure that (1) has a T-periodic solution and we now consider some of the literature on this problem.

REMARK. *Conditions (4) and (5) can be reversed. Briefly, in the scalar case one can ask that* $2g(t,x)(\lambda a(t) - x) \ge K|g(t,x)|^\alpha - M and D_{st}(t,s) \ge 0$. *The basic inequalities* *we need are in (17) and they follow from these. Thus, we present what may be called the stable case, but the unstable case is symmetric.*

Equation (1) in the scalar case has been studied intensively when (3) is strengthened to include $D(t,s) \geq 0$, $D_s(t,s) \geq 0$, and $D_{st}(t,s) \leq 0$, and when the lower limit on the integral is zero. Substantial bibliography can be found in Gripenberg-Londen-Staffans [2]. In particular, on p. 631 of that book it is shown that

$$x(t) = a(t) - \int_0^t C(t-s)h(x(s))\,ds \tag{9}$$

has an asymptotically periodic solution when h is strictly monotone increasing, and when

$$C(t) > 0, \quad C'(t) < 0, \quad C''(t) > 0, \tag{L}$$

among other conditions. Now (L), or more generally

$$D(t,s) > 0, \quad D_s(t,s) > 0, \quad D_{st}(t,s) < 0 \tag{LL}$$

have received much attention for both (9) and

$$x'(t) = -\int_0^t C(t-s)h(x(s))\,ds, \quad {}' = d/dt, \tag{10}$$

where $xh(x) > 0$ if $x \neq 0$ and h is continuous.

An overview of (L) and (10) is found in Mac Camy and Wong [8; p.2]. In particular, it is known from transform theory that if (L) holds, then C is a positive kernel; thus, if we multiply (10) by $h(x)$ and integrate from 0 to t, then for $t > 0$ and $H(x) = \int_0^x h(s)\,ds$ we have

$$\begin{aligned} H(x(t)) - H(x(0)) &= \int_0^t x'(s)h(x(s))\,ds \\ &= -\int_0^t h(x(u)) \int_0^u C(u-s)h(x(s))\,ds\,du \leq 0 \end{aligned} \tag{11}$$

or

$$H(x(t)) \leq H(x(0)), \tag{12}$$

a stability result. Following a suggestion of Volterra, Levin [6] constructed a Liapunov function for (10) (and later for the nonconvolution case [7]) improving (12) and showing that

$$(13) \qquad (L) \text{ implies asymptotic stability for (10).}$$

See also Lakshmikantham and Leela [5; pp. 327–340]. Halanay [3] noted that the right-side of (11) was actually negative definite in a certain sense (as corrected by Mac Camy and Wong [8]) so that (11) itself would yield asymptotic stability. As there are positive kernels not satisfying (L) this improved Levin's work for the convolution case; but it had a far more important feature: no one knew how to extend Levin's Liapunov functional to (9), but Halanay's idea worked for (9) also.

The book by Gripenberg-Londen-Staffans traces much of the work with positive kernels and, in the nonconvolution case, fairly close variants of (LL) are still required for stability results.

Equation (1) was also studied in the scalar case in [1]. There, it was assumed that $D_s(t,s) \geq 0$ and $D_{st}(t,s) \leq 0$ (as in (5c)). Those conditions make the problem far easier than the one considered here. In particular, $D_s \geq 0$ makes a certain Liapunov function (defined in (16)) positive definite and bounded along a solution. When we drop those conditions here, entirely different analysis is required.

2. Periodic solutions. We will be defining a homotopy for (1) and a Liapunov function. The condition (6) is needed to ensure the existence and differentiability of that Liapunov function. Moreover, if we denote the integrand of (6) by $d(t,s)$, then a result of Hino and Murakami [4] shows that (6) is equivalent to

$$(14) \qquad \int_{-\infty}^{t} d(t+\mathcal{T}, s)\, ds \to 0 \text{ uniformly for } t \in R \text{ as } \mathcal{T} \to \infty.$$

and this will be needed to establish regularity of the homotopy.

THEOREM 1. *If (2) – (8) hold, then (1) has a T-periodic solution.*

Proof. Define an equation

$$(1_\lambda) \qquad x(t) = \lambda[a(t) - \int_{-\infty}^{t} D(t,s)g(s,x(s))\,ds]$$

and for $(P_T, \|\cdot\|)$ the Banach space of continuous T-periodic functions with the supremum norm, define a mapping H on P_T by $\varphi \in P_T$ implies that

$$(15) \qquad (H\varphi)(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(s,x(s))\,ds.$$

A simple calculation shows that $H : P_T \to P_T$. Our theorem will be established when we shown that H has a fixed point. And that will follow from a result of Schaefer [9] which we now state.

Theorem (Schaefer). *Let $(P, \|\cdot\|)$ be a normed space, H a continuous mapping of P into P which maps bounded sets into compact sets. Then either*

(i) the equation $\varphi = \lambda H\varphi$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions φ, for $0 < \lambda < 1$, is unbounded.

We establish the conditions of Schaefer's Theorem by means of three lemmas.

Lemma 1. *If H is defined by (15), then $H : P_T \to P_T$ and H maps bounded sets into compact sets.*

Proof. A change of variable shows that if $\varphi \in P_T$ then $(H\varphi)(t+T) = (H\varphi)(t)$. Let $J > 0$ be given, $\varphi \in P_T$, and $\|\varphi\| \le J$. Then there is a $Y > 0$ with $\sup\{|g(t,x)| : t \in R, |x| \le J\} = Y$. Now (14) contains

$$(14a) \qquad \int_{-\infty}^{t} |D(t+\mathcal{T}, s)|\,ds \to 0 \text{ uniformly for } t \in R \text{ as } \mathcal{T} \to \infty.$$

From this, for each $\varepsilon > 0$ there is a $\mathcal{T} > 0$ such that

$$\int_{-\infty}^{t} |D(t+\mathcal{T}, s)|\,ds < \varepsilon/5Y \text{ for all } t \in R.$$

Since a and D are uniformly continuous on R and $U = \{(t,s)|t - 2\mathcal{T} \le s \le t\}$, respectively, for the $\varepsilon > 0$ there is a δ_1 with $0 < \delta_1 < \mathcal{T}$ and

$$(14b) \qquad |a(t_1) - a(t_2)| < \varepsilon/5 \text{ if } |t_1 - t_2| < \delta_1$$

and

$$(14c) \qquad |D(t_1,s) - D(t_2,s)| < \varepsilon/5\mathcal{T}Y \text{ if } (t_1,s),(t_2,s) \in U \text{ and } |t_1 - t_2| < \delta_1.$$

Let $E = \sup\{|D(t,s)| : t - \mathcal{T} \le s \le t\}$ and let $\delta = \min(\delta_1, \varepsilon/5\mathcal{T}E)$. From (14a) – (14c), if $\varphi \in P_T$ with $\|\varphi\| \le J$, and if $0 \le t_1 < t_2 < T$ with $|t_1 - t_2| < \delta$, then we have

$$\begin{aligned}
&|(H\varphi)(t_1) - (H\varphi)(t_2)| \le |a(t_1) - a(t_2)| \\
&+ \left| \int_{-\infty}^{t_1} (D(t_1,s) - D(t_2,s)) g(s,\varphi(s))\, ds \right| \\
&+ \left| \int_{t_1}^{t_2} D(t_2,s) g(s,\varphi(s))\, ds \right| \\
&< \frac{\varepsilon}{5} + Y \int_{-\infty}^{t_1} |D(t_1,s) - D(t_2,s)|\, ds + Y \int_{t_1}^{t_2} |D(t_2,s)|\, ds \\
&< \frac{\varepsilon}{5} + Y \int_{-\infty}^{t_1 - \mathcal{T}} |D(t_1,s)|\, ds + Y \int_{-\infty}^{t_1 - \mathcal{T}} |D(t_2,s)|\, ds \\
&+ Y \int_{t_1 - \mathcal{T}}^{t_1} |D(t_1,s) - D(t_2,s)|\, ds + \frac{\varepsilon}{5} < \varepsilon
\end{aligned}$$

showing the equicontinuity of

$$\{H\varphi : \varphi \in P_T, \|\varphi\| \le J\}.$$

Lemma 2. *If H is defined by (15), then H is continuous.*

Proof. Let φ_1, $\varphi_2 \in P_T$ so that $\|\varphi_i\| \le J$ for some $J > 0$. By the uniform continuity of g, for $t \in R$ and $|x| \le J$, and by (2) and (6) we can make

$$\Big|(H\varphi_1)(t) - (H\varphi_2)(t)\Big| = \left| \int_{-\infty}^{t} D(t,s) \Big[g(s,\varphi_1(s)) - g(s,\varphi_2(s)) \Big]\, ds \right|$$

as small as we please by making $\|\varphi_1 - \varphi_2\|$ small.

Lemma 3. *There is a $\tilde{K} > 0$ such that if $0 < \lambda < 1$, if $x \in P_T$, and if x solves (1_λ), then $\|x\| \le \tilde{K}$.*

Proof. For all such x, the function

$$(16) \qquad V(t,x(\cdot)) = \lambda^2 \int_{-\infty}^{t} \left(\int_s^t g^*(v,x(v))\, dv \right) D_s(t,s) \int_s^t g(v,x(v))\, dv\, ds$$

is well-defined because of (6). Also, by (6) we can differentiate V and obtain

$$V'(t,x(\cdot)) = \lambda^2 \int_{-\infty}^{t} \Big(\int_s^t g^*(v,x(v))\,dv \Big) D_{st}(t,s) \int_s^t g(v,x(v))\,dv\,ds$$
$$+ 2\lambda^2 g^*(t,x(t)) \int_{-\infty}^{t} D_s(t,s) \int_s^t g(v,x(v))\,dv\,ds.$$

If we integrate the last term by parts and take into account (7) we have

$$2\lambda^2 g^*(t,x(t)) \Big(\Big[D(t,s) \int_s^t g(v,x(v))\,dv \Big]_{-\infty}^{t} + \int_{-\infty}^{t} D(t,s) g(s,x(s))\,ds \Big)$$
$$= 2\lambda g^*(t,x(t))\lambda \int_{-\infty}^{t} D(t,s) g(s,x(s))\,ds = 2\lambda g^*(t,x(t))[\lambda a(t) - x(t)].$$

If (5a) holds, then

$$\int_{-\infty}^{t} \Big(\int_s^t g^*(v,x(v))\,dv \Big) D_{st}(t,s) \int_s^t g(v,x(v))\,dv\,ds$$
$$\leq \int_{-\infty}^{t} \Big(\int_s^t g^*(v,x(v))\,dv \Big) Q^*(t,s)Q(t,s) \int_s^t g(v,x(v))\,dv\,ds$$
$$= \int_{-\infty}^{t} |Q(t,s) \int_s^t g(v,x(v))\,dv|^2\,ds \leq \int_{-\infty}^{t} |Q(t,s)|^2 \Big(\int_s^t |g(v,x(v))|\,dv \Big)^2 ds,$$
$$\leq \int_{-\infty}^{t} |Q(t,s)|^2 (t-s) \int_s^t |g(v,x(v))|^2\,dv\,ds$$
$$\leq \int_{-\infty}^{t} |Q(t,s)|^2 (t-s) \Big(\frac{t-s+T}{T} \Big) \int_0^T |g(v,x(v))|^2\,dv\,ds$$
$$\leq \frac{k}{T} \int_0^T |g(v,x(v))|^2\,dv$$

so that

$$V'(t,x(\cdot)) \leq (M - K|g(t,x)|^2 + \frac{k}{T} \int_0^T |g(v,x(v))|^2\,dv)\lambda.$$

Since $x \in P_T$ so is V and we have

$$0 = V(T,x(\cdot)) - V(0,x(\cdot)) \leq (MT - K \int_0^T |g(t,x(t))|^2\,dt$$
$$+ k \int_0^T |g(t,x(t))|^2\,dt)\lambda$$

or

$$(17a) \qquad \int_0^T |g(t,x(t))|^2\,ds \leq MT/(K-k).$$

If (5c) holds, then we have

$$V'(t, x(\cdot)) \le (M - K|g(t, x(t))|^\alpha)\lambda,$$

and so

$$\int_0^T |g(t, x(t))|^\alpha \, dt \le MT/K. \tag{17c}$$

If (5b) holds, then for $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and some $\hat{k} > 0$ we have

$$\begin{aligned}
&\int_{-\infty}^t |Q(t,s)|^2 \left(\int_s^t |g(v, x(v))| \, dv \right)^2 ds \\
&\le \int_{-\infty}^t |Q(t,s)|^2 (t-s)^{2/\beta} \left(\int_s^t |g(s, x(v))|^\alpha \, dv \right)^{2/\alpha} ds \\
&\le \int_{-\infty}^t |Q(t,s)|^2 (t-s)^{2/\beta} \left(\frac{t-s+T}{T} \right)^{2/\alpha} \left(\int_0^T |g(v, x(v))|^\alpha \, dv \right)^{2/\alpha} ds \\
&\le \left(\hat{k} \Big/ T^{2/\alpha} \right) \left(\int_0^T |g(v, x(v))|^\alpha \, dv \right)^{2/\alpha}
\end{aligned}$$

so that

$$V'(t, x(\cdot)) \le \left(M - K|g(t, x(t))|^\alpha + \left(\hat{k} \Big/ T^{2/\alpha} \right) \left(\int_0^T |g(v, x(v))|^\alpha \, dv \right)^{2/\alpha} \right) \lambda$$

and

$$\begin{aligned}
0 = V(T, x(\cdot)) - V(0, x(\cdot)) \le \Bigg(MT - K \int_0^T |g(t, x(t))|^\alpha \, dt \\
+ \hat{k} T^{(-\frac{2}{\alpha} - 1)} \left(\int_0^T |g(v, x(v))|^\alpha \, dv \right)^{2/\alpha} \Bigg) \lambda
\end{aligned}$$

so

$$K \int_0^T |g(t, x(t))|^\alpha \, dt - \hat{k} T^{(-\frac{2}{\alpha} - 1)} \left(\int_0^T |g(t, x(t))|^\alpha \, dt \right)^{2/\alpha} \le MT.$$

But $\alpha > 2$ implies that there is a $G > 0$ with

$$\int_0^T |g(t, x(t))|^\alpha \, dt < G. \tag{17b}$$

Hence, there is an $\tilde{M} > 0$ so that in every case (a, b, c) we have

$$\int_0^T |g(t, x(t))|^\alpha \, dt \le \tilde{M}. \tag{17}$$

Thus, from (1_λ) we see that

$$\begin{aligned}
|x(t)| &\le |a(t)| + \int_{-\infty}^t |D(t,s)|\,|g(s,x(s))|\,ds \\
&\le |a(t)| + \sum_{j=0}^{\infty} \int_{t-(j+1)T}^{t-jT} |D(t,s)|\,|g(s,x(s))|\,ds \\
&\le |a(t)| + \sum_{j=0}^{\infty} \left(\int_{t-(j+1)T}^{t-jT} |D(t,s)|^\beta \, ds \right)^{1/\beta} \left(\int_0^T |g(s,x(s))|^\alpha \, ds \right)^{1/\alpha} \\
&\le \|a\| + (\tilde{M})^{1/\alpha} B =: \tilde{K}
\end{aligned}$$

by (8). This completes the proof of Lemma 3 and proves the theorem.

Remark. Condition (8) can be reduced to

$$\sum_{j=0}^{\infty} \left(\int_{t_1-(j+1)T}^{t_1-jT} |D(t_2,s) - D(t_1,s)|^\beta \, ds \right)^{1/\beta} \le B \tag{8*}$$

for $0 \le t_1 \le t_2 \le T$ if we strengthen (4) to

(4^*)

$-2g^*(t,x)[x - \lambda a(t)] \le M - K|g(t,x)|^\alpha$ and $|g(t,x)| \to \infty$ as $|x| \to \infty$ uniformly in t.

Proof. From (17) there is a $t_1 \in [0, T]$ with

$$|g(t_1, x(t_1))|^\alpha \le \tilde{M}/T \tag{18}$$

and so by (4^*) there is an $\overline{M} > 0$ with

$$|x(t_1)| \le \overline{M}. \tag{19}$$

Let $t_2 \in [0, T]$ with

$$|x(t_2)| = \|x\| \tag{20}$$

and, to be definite, let $t_1 < t_2$. Then there is an $L > 0$ with $|a(t_2) - a(t_1)| \leq L$ and so for α and β defined with (8) we have

$$\begin{aligned}
|x(t_2) - x(t_1)| &\leq L + \int_{-\infty}^{t_1} |D(t_1, s) - D(t_2, s)|\, |g(s, x(s))|\, ds \\
&\quad + \int_{t_1}^{t_2} |D(t_2, s)|\, |g(s, x(s))|\, ds \\
&\leq L + \sum_{j=0}^{\infty} \int_{t_1-(j+1)T}^{t_1-jT} |D(t_2, s) - D(t_1, s)|\, |g(s, x(s))|\, ds \\
&\quad + \left(\int_{t_1}^{t_2} |D(t_2, s)|^{\beta}\, ds\right)^{1/\beta} \left(\int_{t_1}^{t_2} |g(s, x(s))|^{\alpha}\, ds\right)^{1/\alpha} \\
&\leq L + \sum_{j=0}^{\infty} \left(\int_{t_1-(j+1)T}^{t_1-jT} |D(t_2, s) - D(t_1, s)|^{\beta}\, ds\right)^{1/\beta} \tilde{M}^{1/\alpha} \\
&\quad + T^{1/\beta} \sup_{\substack{0 \leq s \leq T \\ 0 \leq t \leq T}} |D(t, s)| \tilde{M}^{1/\alpha} \leq \hat{B}
\end{aligned}$$

for some $\hat{B}$ by (8^*). This, (19), and (20) yield

$$\|x\| \leq \overline{M} + \hat{B} = \tilde{K}, \tag{21}$$

which establish the theorem under the revised conditions.

3. Discussion and examples. It is important to understand that in the scalar case (5c) implies

$$D_{st}(t, s) \leq 0 \tag{$5c^*$}$$

which, together with (6), implies a variant of (L).

Proposition 1. If (1) is scalar and if ($5c^*$) and (6) hold, then $D_s(t, s) \geq 0$ and $D(t, s) \geq 0$.

Proof. By way of contradiction, suppose there is a (t_0, s_0) with $s_0 < t_0$ and $d_0 := D_s(t_0, s_0) < 0$. Since D_s is continuous, there is a $\delta > 0$ with $D_s(t_0, s) \leq d_0/2$ if $|s - s_0| < \delta$. For $-d_0\delta > 0$ there is a $\mathcal{T} > 1$ such that $\int_{-\infty}^{t-\mathcal{T}} (t - s)^2 |D_s(t, s)|\, ds < d_0\delta$ for all $t \in R$. Thus, we have

$$I := \int_{-\infty}^{s_0+\delta} (s_0 + \delta + \mathcal{T} - s)^2 |D_s(s_0 + \delta + \mathcal{T}, s)|\, ds < -d_0\delta.$$

On the other hand, we obtain

$$I \geq \int_{s_0-\delta}^{s_0+\delta} (s_0+\delta+\mathcal{T}-s)^2 |D_s(s_0+\delta+\mathcal{T}, s)|\, ds$$
$$\geq -\int_{s_0-\delta}^{s_0+\delta} (d_0/2)\, ds$$
$$= -d_0\delta,$$

a condtradiction. Hence, $D_s \geq 0$.

In the same way, if there is a (t_1, s_1) with $s_1 < t_1$ and $D(t_1, s_1) < 0$, since $D_s \geq 0$, we obtain $D(t_1, s) \leq D(t_1, s_1)$ for all $s \leq s_1$, contradicting $\int_{-\infty}^{t} |D(t_1, s)|\, ds < \infty$. This proves the proposition.

Example 1. Let (1) be scalar, $\alpha = 2$, $b+2 > 0$, $D(t,s) = c(t-s-b)e^{-(t-s)}$, $c > 0$, and for the K of (4), let $0 < c(b+2)^3(b+2+T) < K$. Then (5a) holds.

Proof. We have

$$D_s(t,s) = ce^{-(t-s)}[t-s-b-1]$$

and

$$D_{st}(t,s) = ce^{-(t-s)}[b+2-(t-s)]$$

so

$$D_{st}(t,s) \geq 0 \text{ if } t-s \leq b+2 \text{ or } t-b-2 \leq s.$$

Thus, the integral in (5a) is

$$\int_{-\infty}^{t} (D_{st}(t,s))_+ (t-s)(t_s+T)\, ds$$
$$\leq \int_{t-b-2}^{t} ce^{-(t-s)}[b+2-(t-s)](t-s)(t-s+T)\, ds$$
$$\leq c(b+2)^3(b+2+T) < K.$$

Example 2. Let (1) be scalar, $\alpha = 2$, $K = 1$, $D(t,s) = c(a+\cos s)e^{-(t-s)}$, $0 < a < \sqrt{2}$, $2(1+\pi)c(\sqrt{2}-a) < 1$. Then (5a) is satisfied.

Proof. We have

$$D_s(t,s) = ce^{-(t-s)}(a + \cos s - \sin s),$$
$$D_{st}(t,s) = -c[a + \sqrt{2}\cos(s + \frac{\pi}{4})]e^{-(t-s)}.$$

Then $D_{st} \geq c(\sqrt{2} - a)e^{-(t-s)}$ and (5a) will hold if

$$\begin{aligned}
&\int_{-\infty}^{t} c(\sqrt{2} - a)e^{-(t-s)}(t-s)(t-s+2\pi)\,ds \\
&= c(\sqrt{2} - a)\int_0^\infty e^{-u}u(u+2\pi)\,du \\
&= c(\sqrt{2} - a)[2 + 2\pi] = 2c(1+\pi)(\sqrt{2} - a) < 1.
\end{aligned}$$

Note. Here, D, D_s, D_{st} can all change sign infinitely often. Moreover, as $a \to \sqrt{2}$, D can be unbounded.

Finally, if g is of polynomial growth and if $D_{st} \leq 0$, then (5c) can be satisfied.

Example 3. If (1) is scalar, $g(t,x) = x|x|$, and if $D_{st} \leq 0$, then $\alpha = \frac{3}{2}$ and $0 < K < 2$ will satisfy (5c).

4. Perturbation. As discussed in the introduction, it has long been known that solutions of

$$x'(t) = -\int_0^t C(t-s)h(x(s))\,ds \tag{10}$$

satisfy

$$\int_0^{x(t)} h(s)\,ds \leq \int_0^{x(0)} h(s)\,ds \tag{12}$$

when

$$C(t) > 0, \quad C'(t) < 0, \quad C''(t) > 0. \tag{L}$$

Levin [6] used a Liapunov functional to show that much more than (12) could be proved; in fact, he showed that $x(t)$ and some of its derivatives tend to zero. The Halanay [3] and

MacCamy and Wong [8] work showed that Levin's results held under weaker conditions than (L); but more importantly, they worked for integral equations. While a counterpart of (12) held for integral equations, no one had extended Levin's Liapunov functional to integral equations. A full development of these matters is found in [2].

In the same vein, we now investigate just how robust conditions such as (L), and in particular (5a, b, c), really are. How much can D and g be perturbed for the conclusion of Theorem 1 to still hold? In the process we show that the method itself is robust; the same Liapunov function works on more general problems.

Thus, along with (1) we consider the perturbed equation

$$x(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(s,x(s))\,ds + \int_{-\infty}^{t} E(t,s)h(s,x(s))\,ds \tag{22}$$

with a, D, g as in (1) and (2), $E : R \times R \to R^{n\times n}$ and $h : R \times R^n \to R^n$ are continuous, and

$$E(t+T, s+T) = E(t,s) \quad \text{and} \quad h(s+T, s) = h(s,x). \tag{23}$$

The matrices P, Q, $\Delta = \operatorname{diag}(\delta_j(t,s))$ are as in the introduction.

We also suppose that there are constants $\alpha > 1$, $0 < k < K$, and $M > 0$ such that $0 < \lambda < 1$ implies either

$$2g^*(t,x)(\lambda a(t) - x) \leq M - K|g(t,x)|^\alpha \tag{24a}$$

or

$$2g^*(t,x)(\lambda a(t) - x) \leq M - K|x||g(t,x)| \tag{24b}$$

and that either (5a), (5b), or (5c) holds. We strengthen (6) to

$$\int_{-\infty}^{t} (|D(t,x)| + |D_s(t,s)|(t-s)^2 + |D_{st}(t,s)|(t-s)^2 + |E(t,s)|)\,ds \tag{25}$$

is continuous and ask that when $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $0 \leq t \leq T$ then (8) holds as well as

$$\sum_{j=0}^{\infty} \left(\int_{t-(j+1)T}^{t-jT} |E(t,s)|^\beta\,ds \right)^{1/\beta} \leq \overline{E} \tag{26}$$

for some $\overline{E} > 0$.

Theorem 2. *Let (2), (3), (7), (8), (23), (24a), (25), and (26) hold, as well as (5a) or (5b). Let*

$$|h(t,x)| \le \min(|x|, |g(t,x)|) \tag{27a}$$

and for

$$\varepsilon := \sup\left\{\int_{-\infty}^{t} |E(t,s)|\, ds : t \in R\right\} < 1 \tag{27b}$$

we suppose that

$$K > k + [2\varepsilon B\sqrt{T}/(1-\varepsilon)].$$

Then (22) has a T-periodic solution.

Proof. As in the proof of Theorem 1 we define $(P_T, \|\cdot\|)$ and then extend H to

$$(H\varphi)(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(s,\varphi(s))\, ds + \int_{-\infty}^{t} E(t,s)h(s,\varphi(s))\, ds.$$

Just as before, H maps P_T into P_T continuously and maps bounded sets into compact sets. By Schaefer's theorem we need only find $\tilde{K} > 0$ such that if $x \in P_T$ solves

$$x(t) = \lambda\left[a(t) - \int_{-\infty}^{t} D(t,s)g(s,x(s))\, ds + \int_{-\infty}^{t} E(t,s)h(s,x(s))\, ds\right] \tag{22λ}$$

with $0 < \lambda < 1$, then $\|x\| \le \tilde{K}$. Thus, for V defined in (16) and $x \in P_T$ a solution of (22_λ) we have, just as before,

$$V'(t)x(\cdot)) = \lambda^2 \int_{-\infty}^{t} \left(\int_{s}^{t} g^*(v,x(v))\, dv\right) D_{st}(t,s) \int_{s}^{t} g(v,x(v))\, dv\, ds$$
$$+2\lambda^2 g^*(t,x(t)) \int_{-\infty}^{t} D(t,s)g(s,x(s))\, ds.$$

If (5b) holds, then for $\beta = \alpha/(\alpha-1)$ and some $\tilde{k}$ we have as in the proof of Theorem 1 that

$$\int_{-\infty}^{t} |Q(t,s)|^2 \left(\int_{s}^{t} |g(v,x(v))|\, dv\right)^2 ds$$
$$\le \tilde{k}\left(\int_{0}^{T} |g(v,x(v))|^\alpha\, dv\right)^{2/\alpha}$$

so

$$
\begin{aligned}
V'(t, x(\cdot)) \le\ & 2\lambda g^*(t, x(t))\Big[\lambda a(t) - x(t) + \int_{-\infty}^{t} E(t,s)h(s,x(s))\,ds\Big] \\
& + \lambda^2 \tilde{k}\Big(\int_0^T |g(v,x(v))|^\alpha\,dv\Big)^{2/\alpha} \\
\le & (M - K|g(t,x(t))|^\alpha + \tilde{k}\Big(\int_0^T |g(v,x(v))|^\alpha\,dv\Big)^{2/\alpha} \\
& + 2\varepsilon\|x\|\,|g(t,x(t))|)\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= V(T, x(\cdot)) - V(0, x(\cdot)) \\
&\le \Big(MT - K\int_0^T |g(t,x(t))|^\alpha\,dt \quad + \tilde{k}T\Big(\int_0^T |g(t,x(t))|^\alpha dt\Big)^{2/\alpha} \\
&\qquad\qquad + 2\varepsilon\|x\| \int_0^T |g(t,x(t))|\,dt\Big)\lambda
\end{aligned}
$$

so

$$
\begin{aligned}
& K\int_0^T |g(t,x(t))|^\alpha\,dt \\
& \le MT + \tilde{k}T\Big(\int_0^T |g(t,x(t))|^\alpha\,dt\Big)^{2/\alpha} + 2\varepsilon\|x\|\,T^{1/\beta}\Big(\int_0^T |g(t,x(t))|^\alpha\,dt\Big)^{1/\alpha}.
\end{aligned} \tag{28}
$$

On the other hand, (8), (22_λ), and (27b) yield

$$
\|x\| \le \|a\| + \Big(\int_0^T |g(t,x(t))|^\alpha\,dt\Big)^{1/\alpha} B + \varepsilon\|x\|
$$

or

$$
\|x\| \le \Big(\|a\| + \Big(\int_0^T |g(t,x(t))|^\alpha\,dt\Big)^{1/\alpha} B\Big)\Big/(1-\varepsilon). \tag{29}
$$

This, together with (28) implies that

$$
KI^\alpha \le MT + \tilde{k}TI^2 + 2\varepsilon T^{1/\beta}(\|a\| + BI)I/(1-\varepsilon)
$$

where

$$
I = \Big(\int_0^T |g(t,x(t))|^\alpha\,dt\Big)^{1/\alpha}.
$$

Thus, there is a $G > 0$ with

$$\left(\int_0^T |g(t,x(t))|^\alpha \, dt\right)^{1/\alpha} \le G. \tag{30}$$

Next, if (5a) holds with $K > k + (2\varepsilon B\sqrt{T})/(1-\varepsilon)$, then

$$\int_{-\infty}^t |Q(t,s)|^2 \left(\int_s^t |g(v,x(v))| \, dv\right)^2 ds \le (k/T) \int_0^T |g(t,x(t))|^2 \, dt$$

so that

$$V'(t,x(\cdot)) \le (M - k|g(t,x(t))|^2 + (k/T) \int_0^T |g(t,x(t))|^2 \, dt \\ +2\varepsilon \|x\| \, |g(t,x(t))|)\lambda$$

and

$$0 \le (MT - K \int_0^T |g(t,x(t))|^2 \, dt + k \int_0^T |g(t,x(t))|^2 \, dt \\ +2\varepsilon \|x\| \int_0^T |g(t,x(t))| \, dt)\lambda$$

so

$$\begin{aligned} K \int_0^T |g(t,x(t))|^2 \, dt &\le MT + k \int_0^T |g(t,x(t))|^2 \, dt \\ &\quad + 2\varepsilon \|x\| \sqrt{T} \left(\int_0^T |g(t,x(t))|^2 \, dt\right)^{1/2}. \end{aligned} \tag{31}$$

From (29) with $\alpha = 2$ and (31), we obtain

$$KI^2 \le MT + (2\varepsilon \|a\| \sqrt{T} I/(1-\varepsilon)) + \{k + 2\varepsilon B\sqrt{T}/(1-\varepsilon)\} I^2$$

where

$$I^2 = \int_0^T |g(t,x(t))|^2 \, dt.$$

Thus, we have (30) with $\alpha = 2$ for some $G > 0$. Consequently, from (8), (22_λ), (26), and (27a) we obtain

$$\|x\| \le \|a\| + (B + \overline{E})G =: \tilde{K}.$$

This completes the proof.

Theorem 3. *In addition to (2), (3), (5c), (7), (8), (24b), (25), (26), suppose there is a constant* $\rho > 0$ *such that for some* μ *with* $\max(\alpha - 1, \frac{1}{\alpha}) < \mu < 1$ *we have*

$$|h(t,x)| \le |x|^{1/\alpha} < |x|^{\mu} \le |g(t,x)| \le |x| \text{ if } |x| \ge \rho \tag{32a}$$

or for some $\alpha > (1+\sqrt{5})/2$ *we have*

$$|h(t,x)|^{\alpha} \le |g(t,x)| \le |x|^{1/(\alpha-1)} \text{ if } |x| \ge \rho. \tag{32b}$$

Then (22) has a T*-periodic solution.*

Proof. It suffices to find $\tilde{K}$ as in the proof of Theorem 2. Let $x \in P_T$ solve (22_λ) with $0 < \lambda < 1$, and define V as in (16). A calculation yields

$$\begin{aligned} V'(t,x(\cdot)) &\le 2\lambda^2 g^*(t,x(t)) \int_{-\infty}^{t} D(t,x) g(s,x(s))\,ds \\ &\le (M - K|x(t)|\,|g(t,x(t))| + 2\lambda g^*(t,x(t)) \int_{-\infty}^{t} E(t,s)\,h(s,x(s))\,ds)\lambda \\ &\le (M - K|x(t)|\,|g(t,x(t))| + 2\overline{E}|g(t,x(t))| \Big(\int_0^T |h(t,x(t))|^{\alpha}\,dt \Big)^{1/\alpha})\lambda \end{aligned}$$

If (32a) holds and if $|x(t)| \ge \rho$, then we have

$$V'(t,x(\cdot)) \le (M - K|x(t)|^{1+\mu} + 2\overline{E}|x(t)| \Big(\int_0^T |h(t,x(t))|^{\alpha}\,dt \Big)^{1/\alpha})\lambda$$

which implies that

$$V'(t,x(\cdot)) \le (\overline{M} - K|x(t)|^{1+\mu} + 2\overline{E}|x(t)| \Big(\int_0^T |h(s,x(s))|^{\alpha} ds \Big)^{1/\alpha})\lambda$$

for some $\overline{M} > M$ and all $t \in R$. Thus, we obtain

$$0 \le (\overline{M}T - K\int_0^T |x(t)|^{1+\mu}\,dt + 2\overline{E}\int_0^T |x(t)|\,dt \Big(\int_0^T (h(s,x(t))|^{\alpha}\,ds \Big)^{1/\alpha})\lambda$$

so

$$\begin{aligned} K\int_0^T |x(t)|^{1+\mu}\,dt &\le \overline{M}T + 2\overline{E}\int_0^T |x(t)|\,dt \Big(\int_0^T |h(t,x(t))|^{\alpha}\,dt \Big)^{1/\alpha} \\ &\le \overline{M}T + 2\overline{E}T^{\mu/(1+\mu)} \Big(\int_0^T |x(t)|^{1+\mu}\,dt \Big)^{(1+\frac{1}{\alpha})/(1+\mu)}, \end{aligned}$$

since the last exponent is less than 1 we obtain

$$\int_0^T |x(t)|^{1+\mu}\, dt \le G \tag{33}$$

for some $G > 0$. From (22_λ), (32a), and (33) we have

$$\begin{aligned}\|x\| &\le \|a\| + B\Big(\int_0^T |g(t,x(t)|^\alpha\, dt\Big)^{1/\alpha} + \overline{E}\Big(\int_0^T |h(t,x(t))|^\alpha\, dt\Big)^{1/\alpha}\\ &\le \|a\| + \Big(\int_0^T |x(t)|^\alpha\, dt\Big)^{1/\alpha} B + \overline{E}\Big(\int_0^T |x(t)|\, dt\Big)^{1/\alpha}\end{aligned}$$

and this is bounded by some $\tilde{K}$.

Next, if (32b) holds, and if $|x(t)| \ge \rho$, then we have

$$\begin{aligned}&V'(t,x(\cdot))\\ &\le (M - K|g(t,x(t))|^\alpha + 2\overline{E}|g(t,x(t))|\Big(\int_0^T |h(s,x(s))|^\alpha\, ds\Big)^{1/\alpha})\lambda\end{aligned}$$

which implies that

$$V'(t,x(\cdot)) \le (\overline{M} - K|g(t,x(t))|^\alpha + 2\overline{E}|g(t,x(t))|\Big(\int_0^T |h(s,x(s))|^\alpha\, ds\Big)^{1/\alpha})\lambda$$

for some $\overline{M} > M$ and all $t \in R$. Thus we obtain

$$\begin{aligned}K\int_0^T |g(t,x(t))|^\alpha\, dt &\le \overline{M}T + 2\overline{E}\int_0^T |g(t,x(t))|\, dt\Big(\int_0^T |h(t,x(t))|^\alpha\, dt\Big)^{1/\alpha}\\ &\le \overline{M}T + 2\overline{E}\Big(\int_0^T |g(t,x(t))|\, dt\Big)^{1+\frac{1}{\alpha}}\\ &\le \overline{M}T + 2\overline{E}T^{1-\frac{1}{\alpha^2}}\Big(\int_0^T |g(t,x(t))|^\alpha\, dt\Big)^{\frac{1}{\alpha}+\frac{1}{\alpha^2}}.\end{aligned}$$

Since the last exponent is less than 1 we obtain

$$\int_0^T |g(t,x(t))|^\alpha\, dt \le G \tag{34}$$

for some $G > 0$. From (22_λ), (32b), and (34) we have

$$\|x\| \le \|a\| + \Big(\int_0^T |g(t,x(t))|^\alpha\, dt\Big)^{1/\alpha} B + \overline{E}\Big(\int_0^T |h(t,x(t))|^\alpha\, dt\Big)^{1/\alpha}$$

which can be bounded by some $\tilde{K}$.

5. Special perturbations. In the last section we considered perturbations Eh where always $|h(t,x)| \leq |g(t,x)|$ and, implicitly, g is of polynomial order for large x. The polynomial order is dictated by the condition in V' whereby we must compare $\int_0^T |x^*(t)g(t,x(t))|\,dt$ with

$$\int_{-\infty}^{t} |Q(t,x)|^2 \left(\int_s^t |g(v,x(v))|\,dv\right)^2 ds \leq \frac{k}{T}\left(\int_0^T |g(v,x(v))|^\alpha\,dv\right)^{2/\alpha}.$$

If the former is to dominate, then we need

$$|x^* g(t,x)| \geq |g(t,x)|^\alpha \text{ with } \alpha > 1 \text{ so that } |x| \geq |g(t,x)|^{\alpha-1};$$

thus, g must be of polynomial order for large x. But if, as in Theorem 3, we ask that

$$(5c') \qquad Q(t,s) = 0,$$

then that problem vanishes and our basic inequality is

$$(35) \qquad V'(t,x(\cdot)) \leq (M - K|x(t)|\,|g(t,x(t))| + 2\lambda g^*(t,x(t)) \int_{-\infty}^{t} E(t,x)h(s,x(s))\,ds)\,\lambda.$$

It then seems clear that $|x(t)|$ must dominate $|h(t,x(t))|$ and that g can be fairly arbitrary, so long as some variant of

$$(24b) \qquad 2g^*(t,x)(\lambda a(t) - x) \leq M - K|x||g(t,x)|$$

holds. With this in mind, we begin with an example and then state a theorem. These show that as long as (5c') holds, $|h(t,x)| \leq |x|$, and (24b) is satisfied, and

$$\int_0^T |x^*(t)g(t,x(t))|\,dt \text{ can dominate } \int_0^T |g(t,x(t))|\,dt \left(\int_0^T |x(t)|^\alpha\,dt\right)^{1/\alpha}$$

for some α, then h can dominate g or g can grow exponentially.

For a given $\alpha > 1$, let $p(\alpha)$ and $q(\alpha)$ be numbers satisfying

$$\sum_{j=0}^{\infty}\Big(\int_{t-(j+1)T}^{t-jT} |E(t,s)|^{\alpha}\, ds\Big)^{1/\alpha} \le p(\alpha) \tag{36}$$

and

$$\sum_{j=0}^{\infty}\Big(\int_{t-(j+1)T}^{t-jT} |D(t,s)|^{\alpha}\, ds\Big)^{1/\alpha} \le q(\alpha), \tag{37}$$

if they exist.

Example 4. Let (2), (5c'), (7), (23), (24b), and (25) hold with $g(t,x) = bx^{1/3}+cx+dx^3$.

(i) If $d > 0$ and $K > 2p(4/3)T^{1/4}$, or

(ii) if $d = 0$ and $c > 0$ with $K > 2p(2)T^{1/2}$, or

(iii) if $d = c = 0$ and $b > 0$ with $K > 2p(4)T^{3/4}$ and if (36) holds for $\alpha = 4/3$, 2, 4, and (37) holds for $\alpha = 12/11$, 4/3, 4, 6/5, and 2 then (22) has a T-periodic solution.

Proof. When $\frac{1}{\alpha_i} + \frac{1}{\beta_i} = 1$ and $\alpha_i > 1$, then a calculation from (35) yields (here, V is defined in (16) again)

$$\begin{aligned} V' \le \lambda\{ &-K[bx^{4/3} + cx^2 + dx^4] + M \\ &+2|b|\,|x^{1/3}|p(\alpha_1)\Big(\int_0^T |x|^{\beta_1}\Big)^{1/\beta_1} \\ &+2|c|\,|x|p(\alpha_2)\Big(\int_0^T |x|^{\beta_2}\Big)^{1/\beta_2} \\ &+2|d|\,|x^3|p(\alpha_3)\Big(\int_0^T |x|^{\beta_3}\Big)^{1/\beta_3}\ \} \end{aligned}$$

with $\alpha_1 = 4$, $\alpha_2 = 2$, and $\alpha_3 = 4/3$. Thus,

$$\begin{aligned} &Kb\int_0^T x^{4/3}(t)\,dt + Kc\int_0^T x^2(t)\,dt + Kd\int_0^T x^4(t)\,dt \le MT \\ &+ 2|b|T^{3/4}p(\alpha_1)\Big(\int_0^T |x(t)|^{4/3}\,dt\Big)^{1/4}\Big(\int_0^T |x(t)|^{4/3}\,dt\Big)^{3/4} \\ &+ 2|c|T^{1/2}p(\alpha_2)\int_0^T x^2(t)\,dt \\ &+ 2|d|T^{1/4}p(\alpha_3)\int_0^T x^4(t)\,dt. \end{aligned}$$

If (i) holds, then there is an $L_i > 0$ with $\int_0^T x^4(t)\,dt \le L_i$ so by (22) we have

$$|x(t)| \le \|a\| + p(4/3)L_i^{1/4} + |b|q(12/11)L_i^{1/12} + |c|q(4/3)L_i^{1/4} + |d|q(4)L_i^{3/4}.$$

If (ii) holds, then there is an $L_{ii} > 0$ with $\int_0^T x^2(t)\,dt \le L_{ii}$ and

$$|x(t)| \le \|a\| + p(2)L_{ii}^{1/2} + |b|q(6/5)L_{ii}^{1/6} + |c|q(2)L_{ii}^{1/2}.$$

If (iii) holds, then there is an $L_{iii} > 0$ with $\int_0^T |x(t)|^{4/3}\,dt \le L_{iii}$ and

$$|x(t)| \le \|a\| + p(4)L_{iii}^{3/4} + |b|q(4/3)L_{iii}^{1/4}.$$

Let $\{a_n\}$ be a sequence of non-negative constants and suppose that some $a_n \neq 0$ and that

$$g(t,x) = \sum_{n=0}^{\infty} a_n x^{2n+1} \tag{38}$$

converges for all x; thus, the series converges uniformly and absolutely on any interval $[-L, L]$.

Theorem 4. *Let (2), (5c'), (7), (23), (24b), (25), and (38) hold. Suppose that there is an $\varepsilon > 0$ such that $0 < K \le 2$ and $\frac{1}{\alpha_n} + \frac{1}{2n+2} = 1$ imply $K + \varepsilon > 2p(\alpha_n)T^{\frac{1}{2n+2}}$, then (22) has a T-periodic solution.*

Proof. We have $xg(t,x) = \sum\limits_{n=0}^{\infty} a_n x^{2n+2}$ and

$$V' \le -K\sum_{n=0}^{\infty} a_n x^{2n+2} + M + 2\sum_{n=0}^{\infty} a_n |x|^{2n+1}\int_{-\infty}^{t} |E(t,s)|\,|x(s)|\,ds.$$

For any fixed $x \in P_T$ we can multiply term by term and get

$$V' \le -K\sum_{n=1}^{\infty} a_n x^{2n+2} + M + 2\sum_{n=0}^{\infty} a_n p(\alpha_n)|x|^{2n+1}\left(\int_0^T |x|^{2n+2}\right)^{\frac{1}{2n+2}}$$

since $p(\alpha_n)$ is bounded and the integral is bounded by the supremum norm of x. By the uniform convergence we can obtain

$$K\sum_{n=0}^{\infty} a_n \int_0^T |x(t)|^{2n+2}dt \le MT$$

$$+2\sum_{n=0}^{\infty} a_n T^{\frac{1}{2n+2}} p(\alpha_n)\int_0^T |x(t)|^{2n+2}\,dt$$

or, since $K - 2a_n p(\alpha_n) T^{\frac{1}{2n+2}} \geq \varepsilon$, we have

$$\varepsilon \int_0^T |x(t)g(t,x(t))|\,dt = \varepsilon \sum_{n=0}^{\infty} a_n \int_0^T |x(t)|^{2n+2}\,dt \leq MT. \tag{39}$$

Now some $a_n \neq 0$, say a_d, so

$$\Big| \int_{-\infty}^t E(t,s)h(s,x(s))\,ds \Big| \leq p(\alpha_d) \Big(\int_0^T |x(t)|^{2d+2} \Big)^{1/(2d+2)}. \tag{40}$$

Using (22_λ), integrating by parts, and the Schwarz inequality we have

$$(x(t) - \lambda a(t) - \lambda \int_{-\infty}^t E(t,s)h(s,x(s))\,ds)^2 \leq D(t,t)V(t,x(\cdot)).$$

But there is an $A > 0$ such that $|x| < 1$ implies $|g(t,x)| < A$ and so

$$\begin{aligned} V(t,x(\cdot)) &= \int_{-\infty}^t D_s(t,s) \Big(\int_s^t g(v,x(v))\,dv \Big)^2 ds \\ &\leq \int_{-\infty}^t D_s(t,s) \Big\{ \frac{(t-s+T)}{T} \int_0^T |g(v,x(v))|\,dv \Big\}^2 ds \\ &\leq \int_{-\infty}^t D_s(t,s)(t-s+T)^2 (2/T^2) \Big\{ T^2A^2 + [\int_0^T |x(t)g(t,x(t))|\,dt]^2 \Big\} ds. \end{aligned}$$

Hence, V is bounded, the right-hand-side of (40) is bounded, and so there is an $L > 0$ with

$$\|x\| \leq L. \tag{41}$$

In this theorem it is crucial that some $a_n \neq 0$. But there are two interesting and elementary results giving a priori bounds on $x \in P_T$ satisfying (22_λ) when $D = 0$.

Proposition 2. *Suppose that $g(t,x) = 0$, that (23), (25), (26) hold, that $|h(t,x)| \leq |x|$, and that for some n and α with $\frac{1}{\alpha} + \frac{1}{n+1} = 1$ we have $p(\alpha)$ in (36) satisfying $p(\alpha)T^{\frac{1}{n+1}} < 1$. Then there exists $\tilde{K}$ such that if $x \in P_T$ solves (22_λ), then $\|x\| < \tilde{K}$.*

Proof. From (22_λ) we have

$$\begin{aligned} |x|^{n+1} &\leq \|a\|\,|x|^n + |x|^n \int_{-\infty}^t |E(t,s)|\,|x(s)|\,ds \\ &\leq \|a\|\,|x|^n + |x|^n p(\alpha) \Big(\int_0^T |x(s)|^{n+1}\,ds \Big)^{\frac{1}{n+1}} \end{aligned}$$

so

$$\int_0^T |x(s)|^{n+1}\,ds \le \|a\|T^{\frac{1}{n+1}}\left(\int_0^T |x(s)|^{n+1}\,ds\right)^{\frac{n}{n+1}}$$
$$+T^{\frac{1}{n+1}}p(\alpha)\int_0^T |x(s)|^{n+1}\,ds.$$

Hence, there is an $L>0$ with $\int_0^T |x(s)|^{n+1}\,ds \le L$. The result now follows from (22_λ).

Proposition 3. *Suppose that $q(2)$ exists in (37). Consider (22_λ) with $g(t,x)=x$ and let $Q(t,s)=0$. If the conditions of Proposition 2 also hold for $n=1$, then there is a $\tilde{K}>0$ such that if $x\in P_T$ solves (22_λ) then $\|x\|\le\tilde{K}$.*

Proof. We define V as in (16) and find

$$V'(t,x(\cdot)) \le 2\lambda x^*(t)\lambda\int_{-\infty}^t D(t,s)x(s)\,ds$$
$$= 2\lambda x^*(t)\left[\lambda a(t) - x(t) + \int_{-\infty}^t E(t,s)h(s,x(s))\,ds\right]$$

and for each $K\in(0,1)$ there is an $M>0$ with

$$V'(t,x(\cdot)) \le \left[-Kx^*x + M + 2x^*\int_{-\infty}^t E(t,s)h(s,x(s))\,ds\right]\lambda.$$

Thus,

$$K\int_0^T |x(t)|^2\,dt \le MT + 2T^{1/2}p(2)\int_0^T |x(t)|^2\,dt$$

and since $T^{1/2}p(2)<1$, while K can be made arbitrarily near 2, it follows that there is an $L>0$ with $\int_0^T |x(t)|^2\,dt\le L$. As $p(2)$ and $q(2)$ exist, the bound on $\|x\|$ follows from (22_λ).

6. Convergence of solutions. When (2), (3), (4), (5c), (6), (7), and (8) hold then (1) has a T-periodic solution. When (5c) is weakened so that D is no longer a positive kernel, then Theorem 1 still yields periodic solutions. Sections 4 and 5 present additional perturbations of D and g which continue to yield periodic solutions. Thus, we see that positive kernels are very robust. We now point out that if g is monotone in a certain manner, then bounded solutions converge in a certain sense.

Suppose that $g(t,x)-g(t,y) = H(t,x,y)(x-y)$ defines a continuous, symmetric, positive semi-definite matrix H.

Theorem 5. *Let (3), (6), (7) hold and suppose that D_{st} is negative semi-definite, while D_s is positive semi-definite. Suppose also that $x_1(t)$ and $x_2(t)$ are solutions of (1) which are bounded. Then for each $a > 0$ there is a $B > 0$ such that $a < t$ implies that*

$$\int_a^t [x_1^*(s) - x_2^*(s)]\,[g(s,x_1(s)) - g(s,x_2(s))]\,ds < B.$$

In particular, if there is an $S > 0$ with $S \le |H(t,x,y)|$, then $\int_a^t |x_1(s) - x_2(s)|\,ds \le \overline{B}$ for some $\overline{B}$ and all $t \ge a$.

Proof. If $z = x_1 - x_2$, then z satisfies

$$z(t) = -\int_{-\infty}^t D(t,s)H(s,x_1(s),x_2(s))z(s)\,ds.$$

Also,

$$V(t) = \int_{-\infty}^t \left(\int_s^t z^*(v)H(v,x_1(v),x_2(v))dv\right) D_s(t,s) \int_s^t H(v,x_1(v),x_2(v))z(v)\,dv\,ds$$

is defined. A calculation, as with (16) in the proof of Theorem 1, yields

$$V'(t) \le -2z^*(t)H(t,x_1(t),x_2(t))z(t)$$

from which the conclusion follows.

Clearly, if the conditions of Theorem 5 are met, then (1) has at most one periodic solution.

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