PERIODIC AND ASYMPTOTICALLY PERIODIC SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS

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1. Introduction. We consider three systems of integral equations

$$x(t) = a(t) - \int_0^t D(t, s, x(s))\, ds, \tag{1}$$

$$x(t) = a(t) - \int_{-\infty}^t D(t, s, x(s))\, ds, \tag{2}$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t, s, x(s))\, ds, \tag{3}$$

where p is T-periodic, $P(t+T, s+T, x) = P(t, s, x)$ and where $a(t)$ converges to $p(t)$ and $D(t, s, x)$ converges to $P(t, s, x)$.

Under continuity and convergence conditions to be given later, if $\varphi : (-\infty, t_0) \to R^n$ is a given bounded and continuous initial function, then both (2) and (3) have solutions denoted by $x(t, t_0, \varphi)$ with $x(t, t_0, \varphi) = \varphi(t)$ for $t < t_0$, satisfying (2) or (3) on an interval $[t_0, \alpha)$, with $\alpha = \infty$ provided that the solution remains bounded. (cf. Burton [1; p. 79], Corduneanu [6], or Gripenberg-Londen-Staffans [7].) To fit that theory to (2), for example, write (2) as

$$x(t) = -\int_{t_0}^t D(t, s, x(s))\, ds + a(t) - \int_{-\infty}^{t_0} D(t, s, \varphi(s))\, ds$$

and treat the last two terms as a forcing function.

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In the same way, for a given continuous initial function $\varphi : [0, t_0) \to R^n$, (1) has a solution $x(t, t_0, \varphi)$ which agrees with φ on $[0, t_0)$ and satisfies (1) on some interval $[t_0, \alpha)$. In all cases, $x(t, t_0, \varphi)$ may have a discontinuity at t_0.

Concerning our contribution here, we first present some lemmas on limiting equations and then show that if (1) has an asymptotically T-periodic solution, then (3) has a T-periodic solution.

Next, we use Schauder's fixed point theorem to show that (1) does have an asymptotically T-periodic solution, thus yielding a T-periodic solution of (3). As a consequence, we show that if (3) has a unique solution which is bounded on $(-\infty, \infty)$, then it is periodic and any bounded solution of (1) converges to it. Parallel results between (2) and (3) are also obtained.

We also infer directly that (3) has T-periodic solutions using Schauder's theorem and a growth condition on P.

Finally, we give a detailed list of relations between solutions of (1) and (3).

Concerning the relationship between this work and the literature, it may be noted that much is known about the existence of periodic solutions of integrodifferential equation counterparts for (3); a great many of those results can be conveniently seen in the books Burton [2], Corduneanu [6], Gripenberg-Londen-Staffans [7], for example.

On p. 631 of that last reference there is an asymptotic periodic result for (1) under growth, monotonicity, and sign conditions on D and its derivatives. We have also studied (3) in ([4], [5]) under sign conditions on P and its derivatives by means of a Liapunov functional and Schaefer's fixed point theorem.

The problem of deducing the existence of a periodic solution from that of a bounded solution is an old one in the theory of ordinary and functional differential equations. Discussion and references are found in Yoshizawa [11; pp. 164–165].

Basic facts about limiting equations for differential euqtions are found in Hino and Murakami [8], Kato and Yoshizawa [9], and in Yoshizawa [11].

2. Preliminaries. Consider the systems of Volterra equations

$$x(t) = a(t) - \int_0^t D(t, s, x(s))\, ds, \quad t \in R^+, \tag{1}$$

$$x(t) = a(t) - \int_{-\infty}^t D(t, s, x(s))\, ds, \quad t \in R, \tag{2}$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t, s, x(s))\, ds, \quad t \in R, \tag{3}$$

where $a : R \to R^n$, $p : R \to R^n$, $D : R \times R \times R^n \to R^n$ and $P : R \times R \times R^n \to R^n$ are continuous, and throughout this paper suppose that

$$a(t) = p(t) + q(t), \quad p(t + T) = p(t) \quad \text{and} \quad q(t) \to 0 \text{ as } t \to \infty, \tag{4}$$

where $q : R \to R^n$ and $T > 0$ is constant,

$$D(t, s, x) = P(t, s, x) + Q(t, s, x) \text{ and } P(t + T, s + T, x) = P(t, s, x), \tag{5}$$

where $Q : R \times R \times R^n \to R^n$, and for any $J > 0$ there are continuous functions $P_J : R \times R \to R^+$ and $Q_J : R \times R \to R^+$ such that

$$P_J(t + T, s + T) = P_J(t, s) \text{ if } t, s \in R,$$

$$|P(t, s, x)| \le P_J(t, s) \text{ if } t, s \in R \text{ and } |x| \le J,$$

where $| \cdot |$ denotes the Euclidean norm,

$$|Q(t, s, x)| \le Q_J(t, s) \text{ if } t, s \in R \text{ and } |x| \le J,$$

and

$$\int_{-\infty}^t P_J(t + \tau, s)\, ds \to 0 \text{ uniformly for } t \in R \text{ as } \tau \to \infty. \tag{6}$$

In this paper, we discuss the existence of periodic and asymptotically periodic solutions of (1), (2), and (3) using the following theorem, which we state without proof.

Theorem 1 (Schauder's first theorem). *Let $(C, \|\cdot\|)$ be a normed space, and let S be a compact convex nonempty subset of C. Then every continuous mapping of S into S has a fixed point.*

Schauder's second theorem deletes the compactness of S and asks that the map be compact. (cf. Smart [10; p. 25].)

3. Asymptotically periodic solutions of (1). For any $t_0 \in R^+$, let $C(t_0)$ be a set of bounded functions $\xi : R^+ \to R^n$ such that $\xi(t)$ is continuous on R^+ except at t_0, and $\xi(t_0) = \xi(t_0+)$. For any $\xi \in C(t_0)$, define $\|\xi\|_+$ by

$$\|\xi\|_+ := \sup\{|\xi(t)| : t \in R^+\}.$$

Then clearly $\|\cdot\|_+$ is a norm on $C(t_0)$, and $(C(t_0), \|\cdot\|_+)$ is a Banach space. For any $\xi \in C(t_0)$ define a map H on $C(t_0)$ by

$$(H\xi)(t) := \begin{cases} \xi(t), & 0 \le t < t_0, \\ a(t) - \int_0^t D(t, s, \xi(s))\, ds, & t \ge t_0. \end{cases}$$

Moreover, for any $J > 0$ let $C_J(t_0) := \{\xi \in C(t_0) : \|\xi\|_+ \le J\}$. In this section, we need the following assumption.

$$\int_0^t Q_J(t, s)\, ds \to 0 \text{ as } t \to \infty. \tag{7}$$

Then we have the following lemmas.

Lemma 1. *If (4)–(7) hold, then for any $t_0 \in R^+$ and any $J > 0$ there is a continuous increasing positive function $\delta = \delta_{t_0,J}(\varepsilon) : (0, \infty) \to (0, \infty)$ with*

$$|(H\xi)(t_1) - (H\xi)(t_2)| \le \varepsilon \text{ if } \xi \in C_J(t_0) \text{ and } t_0 \le t_1 < t_2 < t_1 + \delta. \tag{8}$$

Proof. For any $\xi \in C_J(t_0)$, t_1 and t_2 with $t_0 \le t_1 < t_2$ we have

$$|(H\xi)(t_1) - (H\xi)(t_2)| \le |a(t_1) - a(t_2)| + \left| \int_0^{t_1} D(t_1, s, \xi(s))\, ds - \int_0^{t_2} D(t_2, s, \xi(s))\, ds \right|$$

$$\begin{aligned} &\le |a(t_1) - a(t_2)| + \int_0^{t_1} |P(t_1, s, \xi(s)) - P(t_2, s, \xi(s))|\, ds + \int_{t_1}^{t_2} P_J(t_2, s)\, ds \\ &\quad + \int_0^{t_1} |Q(t_1, s, \xi(s)) - Q(t_2, s, \xi(s))|\, ds + \int_{t_1}^{t_2} Q_J(t_2, s)\, ds. \end{aligned} \tag{9}$$

Since $a(t)$ is uniformly continuous on R^+ from (4), for any $\varepsilon > 0$ there is a $\delta_1 > 0$ with

$$|a(t_1) - a(t_2)| \le \tfrac{\varepsilon}{5} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_1. \tag{10}$$

From (6), for the ε there is a $\tau_1 > \max(t_0, 1)$ with

$$\int_{-\infty}^{t-\tau_1} P_J(t,s)\, ds \le \tfrac{\varepsilon}{15} \quad \text{if} \quad t \in R. \tag{11}$$

Since $P(t,s,x)$ is uniformly continuous on $U_1 := \{(t,s,x) : t - 2\tau_1 \le s \le t \text{ and } |x| \le J\}$, for the ε there is a δ_2 such that $0 < \delta_2 < 1$ and

$$|P(t_1,s,x) - P(t_2,s,x)| \le \tfrac{\varepsilon}{15\tau_1} \text{ if } (t_1,s,x), (t_2,s,x) \in U_1 \text{ and } |t_1 - t_2| < \delta_2. \tag{12}$$

From (11) and (12), if $\tau_1 \le t_1 < t_2 < t_1 + \delta_2$, then we have

$$\begin{aligned} &\int_0^{t_1} |P(t_1,s,\xi(s)) - P(t_2,s,\xi(s))|\, ds \\ &\le \int_{-\infty}^{t_1-\tau_1} P_J(t_1,s)\, ds + \int_{-\infty}^{t_1-\tau_1} P_J(t_2,s)\, ds + \int_{t_1-\tau_1}^{t_1} |P(t_1,s,\xi(s)) - P(t_2,s,\xi(s))|\, ds \le \frac{\varepsilon}{5}. \end{aligned} \tag{13}$$

On the other hand, if $t_0 \le t_1 < \tau_1$ and $t_1 < t_2 < t_1 + \delta_2$, then from (12) we obtain

$$\int_0^{t_1} |P(t_1,s,\xi(s)) - P(t_2,s,\xi(s))|\, ds \le \frac{\varepsilon}{15},$$

which together with (13) implies

$$\int_0^{t_1} |P(t_1,s,\xi(s)) - P(t_2,s,\xi(s))|\, ds \le \frac{\varepsilon}{5} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_2. \tag{14}$$

Now let $A := \sup\{P_J(t,s) : t - 1 \le s \le t\}$. Then, for the ε there is a δ_3 such that $0 < \delta_3 < \min(\frac{\varepsilon}{5A}, 1)$ and

$$\int_{t_1}^{t_2} P_J(t_2,s)\, ds \le \tfrac{\varepsilon}{5} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_3. \tag{15}$$

Next from (7), for the ε there is a $\tau_2 > \max(t_0, 1)$ with

$$\int_0^{t} Q_J(t,s)\, ds \le \tfrac{\varepsilon}{10} \text{ if } t \ge \tau_2, \tag{16}$$

which yields

(17)
$$\int_0^{t_1} |Q(t_1,s,\xi(s))-Q(t_2,s,\xi(s))|\,ds \le \int_0^{t_1} Q_J(t_1,s)\,ds+\int_0^{t_1} Q_J(t_2,s)\,ds \le \tfrac{\varepsilon}{5} \text{ if } \tau_2 \le t_1 < t_2.$$

On the other hand, since $Q(t,s,x)$ is uniformly continuous on $U_2 := \{(t,s,x) : 0 \le s \le t \le \tau_2 + 1 \text{ and } |x| \le J\}$, for the ε there is a δ_4 such that $0 < \delta_4 < 1$ and

$$|Q(t_1,s,x) - Q(t_2,s,x)| \le \tfrac{\varepsilon}{5\tau_2} \text{ if } (t_1,s,x),(t_2,s,x) \in U_2 \text{ and } |t_1 - t_2| < \delta_4,$$

which together with (17) implies

$$\int_0^{t_1} |Q(t_1,s,\xi(s)) - Q(t_2,s,\xi(s))|\,ds \le \tfrac{\varepsilon}{5} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_4. \tag{18}$$

Finally let $B := \sup\{Q_J(t,s) : 0 \le s \le t \le \tau_2 + 1\}$. Then, for the ε there is a δ_5 such that $0 < \delta_5 < \min(\frac{\varepsilon}{5B}, 1)$ and

$$\int_{t_1}^{t_2} Q_J(t_2,s)\,ds \le \tfrac{\varepsilon}{5} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_5 \le \tau_2 + 1,$$

which together with (16) yields

$$\int_{t_1}^{t_2} |Q(t_2,s,\xi(s))|\,ds \le \tfrac{\varepsilon}{5} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_5. \tag{19}$$

Thus, from (9), (10), (14), (15), (18) and (19), for the $\delta_6 := \min\{\delta_i : 1 \le i \le 5\}$ we have (8) with $\delta = \delta_6$. Since we may assume that $\delta_6(\varepsilon)$ is nondecreasing, we can easily conclude that there is a continuous increasing function $\delta = \delta_{t_0,J} : (0,\infty) \to (0,\infty)$ which satisfies (8).

Lemma 2. *If (4)–(7) hold, then for any asymptotically T-periodic function $\xi(t)$ on R^+ such that $\xi(t) = \pi(t) + \rho(t)$, $\xi, \pi \in C(t_0)$ for some $t_0 \in R^+$, $\pi(t+T) = \pi(t)$ on R^+ and $\rho(t) \to 0$ as $t \to \infty$, the function*

$$d(t) := \int_0^t D(t,s,\xi(s))\,ds, \quad t \in R^+$$

is continuous, asymptotically T-periodic, and the T-periodic part of d(t) is given by $\int_{-\infty}^{t} P(t, s, \pi(s))\, ds$.

Proof. By (5) and (6), one can easily check that the functions $d(t)$ and $\varphi(t) := \int_{-\infty}^{t} P(t, s, \pi(s))ds$ belong to the space $C(t_0)$ and that $\varphi(t+T) = \varphi(t)$ on R^+. Therefore, in order to establish the lemma, it is sufficient to show that $d(t) - \varphi(t) \to 0$ as $t \to \infty$. Let $J > 0$ be a number with $\|\xi\|_+ \le J$. Then clearly we have $\|\pi\|_+ \le J$. From (6), for any $\epsilon > 0$ there is a $\tau_1 > 0$ with

$$\int_{-\infty}^{t} P_J(t + \tau_1, s)ds < \epsilon \ \ if \ \ t \in R^+.$$

Let $t \ge \tau_1$. Then

$$\begin{aligned}
&|d(t) - \varphi(t)| \\
&= \left| \int_0^t P(t, s, \xi(s))ds - \int_{-\infty}^{t} P(t, s, \pi(s))ds + \int_0^t Q(t, s, \xi(s))ds \right| \\
&\le \int_0^{t-\tau_1} P_J(t, s)ds + \int_{-\infty}^{t-\tau_1} P_J(t, s)ds + \int_{t-\tau_1}^{t} |P(t, s, \pi(s)) - P(t, s, \xi(s))|ds + \int_0^t Q_J(t, s)ds \\
&< 2\epsilon + \int_{t-\tau_1}^{t} |P(t, s, \pi(s)) - P(t, s, \xi(s))|ds + \int_0^t Q_J(t, s)ds.
\end{aligned}$$

Since $P(t, s, x)$ is uniformly continuous on $U_3 := \{(t, s, x) : t - \tau_1 \le s \le t \ and \ |x| \le J\}$, for the ϵ there is a $\delta > 0$ with

$$|P(t, s, x) - P(t, s, y)| < \epsilon/\tau_1 \ if \ (t, s, x), (t, s, y) \in U_3 \ and \ |x - y| < \delta.$$

Moreover, since $\rho(t) \to 0 \ as \ t \to \infty$, for the δ there is a $\tau_2 > 0$ with

$$|\rho(t)| = |\xi(t) - \pi(t)| < \delta \ if \ t \ge \tau_2.$$

By (7), we can assume that

$$\int_0^t Q_J(t, s)ds < \epsilon \ if \ t \ge \tau_2.$$

Then, if $t \ge \tau_1 + \tau_2$, then $|d(t) - \varphi(t)| < 4\epsilon$. This proves that $|d(t) - \varphi(t)| \to 0$ as $t \to \infty$.

Theorem 2. *If (4)–(7) hold, and if (1) has an asymptotically T-periodic solution with an initial time t_0 in R^+, then the T-periodic extension to R of its T-periodic part is a T-periodic solution of (3). In particular, if the asymptotically T-periodic solution of (1) is asymptotically constant, then (3) has a constant solution.*

Proof. Let $x(t)$ be an asymptotically T-periodic solution of (1) with an initial time $t_0 \in R^+$ such that $x(t) = y(t) + z(t)$, $y \in C(t_0)$, $y(t+T) = y(t)$ on R^+ and $z(t) \to 0$ as $t \to \infty$. Then we have

$$y(t) + z(t) = p(t) + q(t) - \int_0^t D(t, s, x(s))\, ds, \quad t \geq t_0. \tag{20}$$

From Lemma 2, taking the T-periodic part of the both sides of (20) we obtain

$$y(t) = p(t) - \int_{-\infty}^t P(t, s, y(s))\, ds, \quad t \geq t_0.$$

From this, it is easy to see that $y(t)$ is a T-periodic solution of (3).

The latter part follows easily from the above conclusion.

In order to prove the existence of an asymptotically T-periodic solution of (1) using Schauder's first theorem, we need more assumptions. In addition to (4)–(7), suppose that for some $t_0 \in R^+$ and $J > 0$ the inequality

$$\|a\|_{t_0} + \int_0^t P_J(t, s)\, ds + \int_0^t Q_J(t, s)\, ds \leq J \text{ if } t \geq t_0 \tag{21}$$

holds, where $\|a\|_{t_0} := \sup\{|a(t)| : t \geq t_0\}$, and that there are continuous functions $L_J : R \times R \to R^+$ and $q_J : [t_0, \infty) \to R^+$ such that $L_J(t+T, s+T) = L_J(t, s)$ and

$$|P(t, s, x) - P(t, s, y)| \leq L_J(t, s)|x - y| \text{ if } t, s \in R, |x| \leq J \text{ and } |y| \leq J, \tag{22}$$

$$q_J(t) \to 0 \text{ as } t \to \infty, \tag{23}$$

and

$$|q(t)| + \int_{-\infty}^{t_0} P_J(t, s)\, ds + \int_0^{t_0} P_J(t, s)\, ds + \int_0^t Q_J(t, s)\, ds + \int_{t_0}^t L_J(t, s) q_J(s)\, ds \leq q_J(t) \text{ if } t \geq t_0. \tag{24}$$

Then we have the following theorem.

Theorem 3. *If (4)–(7) and (21)–(24) with some $t_0 \in R^+$ and $J > 0$ hold, then for any continuous initial function $\varphi_0 : [0, t_0) \to R^n$ with $\sup\{|\varphi_0(s)| : 0 \le s < t_0\} \le J$, (1) has an asymptotically T-periodic solution $x(t) = y(t) + z(t)$ such that x, $y \in C_J(t_0)$, $y(t+T) = y(t)$ on R^+, $x(t)$ satisfies (1) and $|z(t)| \le q_J(t)$ on $[t_0, \infty)$, and the T-periodic extension to R of $y(t)$ is a T-periodic solution of (3).*

Proof. Let S be a set of functions $\xi \in C_J(t_0)$ such that $\xi = \pi + \rho$, $\pi \in C_J(t_0)$, $\xi(t) = \varphi_0(t)$ on $[0, t_0)$, $\pi(t+T) = \pi(t)$ on R^+ and

$$|\rho(t)| \le q_J(t) \text{ if } t \ge t_0, \tag{25}$$

and that for the function $\delta = \delta_{t_0,J}(\varepsilon)$ in (8), $|\xi(t_1) - \xi(t_2)| \le \varepsilon$ if $t_0 \le t_1 < t_2 < t_1 + \delta$.

First we prove that S is a compact convex nonempty subset of $C(t_0)$. Since any $\xi \in C_J(t_0)$ such that $\xi(t) = \varphi_0(t)$ on $[0, t_0)$ and $\xi(t) \equiv \xi(t_0)$ on $[t_0, \infty)$ is contained in S, S is nonempty. Clearly S is a convex subset of $C(t_0)$. In order to prove the compactness of S, let $\{\xi_k\}$ be an infinite sequence in S such that $\xi_k = \pi_k + \rho_k$, $\pi_k \in C_J(t_0)$, $\pi_k(t+T) = \pi_k(t)$ on R^+ and $|\rho_k(t)| \le q_J(t)$ on $[t_0, \infty)$. From the definition of S, if k, $\ell \in N$ and $t_0 \le t_1 < t_2 < t_1 + \delta$, then we have

$$\begin{aligned} |\pi_k(t_1) - \pi_k(t_2)| &= |\pi_k(t_1 + \ell T) - \pi_k(t_2 + \ell T)| \\ &\le |\xi_k(t_1 + \ell T) - \xi_k(t_2 + \ell T)| + |\rho_k(t_1 + \ell T) - \rho_k(t_2 + \ell T)| \\ &\le \varepsilon + q_J(t_1 + \ell T) + q_J(t_2 + \ell T), \end{aligned}$$

which implies $|\pi_k(t_1) - \pi_k(t_2)| \le \varepsilon$ by letting $\ell \to \infty$, where $\delta = \delta_{t_0,J}(\varepsilon)$ is the function in (8). Hence the sets of functions $\{\pi_k\}$ and $\{\rho_k\}$ are uniformly bounded and equicontinuous on $[t_0, \infty)$. Thus, taking a subsequence if necessary, we may assume that the sequence $\{\pi_k\}$ converges to a $\pi \in C_J(t_0)$ uniformly on R^+, and the sequence $\{\rho_k\}$ converges to a $\rho \in C(t_0)$ uniformly on any compact subset of R^+. Clearly $\pi(t)$ is T-periodic on R^+, and $\rho(t)$ satisfies (25), and hence the sequence $\{\xi_k\}$ converges to the asymptotically T-periodic function $\xi := \pi + \rho$ uniformly on any compact subset of R^+ as $k \to \infty$. It is clear that $\xi \in S$. Now we show that $\|\rho_k - \rho\|_+ \to 0$ as $k \to \infty$. From (23), for any $\varepsilon > 0$ there is a $\tau \geq t_0$ with

$$q_J(t) < \tfrac{\varepsilon}{2} \text{ if } t \geq \tau,$$

which yields

$$|\rho_k(t) - \rho(t)| \leq 2q_J(t) < \varepsilon \text{ if } k \in N \text{ and } t \geq \tau. \tag{26}$$

On the other hand, since $\{\rho_k(t)\}$ converges to $\rho(t)$ uniformly on $[0, \tau]$ as $k \to \infty$, for the ε there is a $\kappa \in N$ with

$$|\rho_k(t) - \rho(t)| < \varepsilon \text{ if } k \geq \kappa \text{ and } 0 \leq t \leq \tau,$$

which together with (26) implies $\|\rho_k - \rho\|_+ < \varepsilon$ if $k \geq \kappa$. This yields $\|\rho_k - \rho\|_+ \to 0$ as $k \to \infty$, and hence $\|\xi_k - \xi\|_+ \to 0$ as $k \to \infty$. Thus S is compact.

Next we prove that H maps S into S continuously. For any $\xi \in S$ such that $\xi = \pi + \rho$, $\pi \in C_J(t_0)$, $\pi(t+T) = \pi(t)$ on R^+ and $|\rho(t)| \leq q_J(t)$ on $[t_0, \infty)$, let $\varphi := H\xi$. Then from (21), for $t \geq t_0$ we have

$$\begin{aligned} |\varphi(t)| &\leq |a(t)| + \int_0^t |P(t,s,\xi(s))|\,ds + \int_0^t |Q(t,s,\xi(s))|\,ds \\ &\leq \|a\|_{t_0} + \int_0^t P_J(t,s)\,ds + \int_0^t Q_J(t,s)\,ds \leq J, \end{aligned}$$

which together with $\xi \in C_J(t_0)$ and Lemma 1 implies that $\varphi \in C_J(t_0)$. Now from Lemma 2, φ has the unique decomposition $\varphi = \psi + \mu$, $\psi \in C_J(t_0)$, $\psi(t+T) = \psi(t)$ on R^+, and $\mu(t) \to 0$ as $t \to \infty$, where the restriction of $\mu(t)$ on $[t_0, \infty)$ is given by

$$\begin{aligned} \mu(t) := q(t) + \int_{-\infty}^{t_0} P(t,x,\pi(s))\,ds - \int_0^{t_0} P(t,s,\xi(s))\,ds - \int_0^t Q(t,s,\xi(s))\,ds \\ - \int_{t_0}^t (P(t,s,\xi(s)) - P(t,s,\pi(s)))\,ds, \quad t \geq t_0. \end{aligned}$$

Thus from (24), for $t \geq t_0$ we obtain

$$
\begin{aligned}
|\mu(t)| \leq |q(t)| &+ \int_{-\infty}^{t_0} |P(t,s,\pi(s))|\, ds + \int_0^{t_0} |P(t,s,\xi(s))|\, ds \\
&+ \int_0^t |Q(t,s,\xi(s))|\, ds + \int_{t_0}^t |P(t,s,\xi(s)) - P(t,s,\pi(s))|\, ds \\
\leq |q(t)| &+ \int_{-\infty}^{t_0} P_J(t,s)\, ds + \int_0^{t_0} P_J(t,s)\, ds + \int_0^t Q_J(t,s)\, ds + \int_{t_0}^t L_J(t,s) q_J(s)\, ds \leq q_J(t).
\end{aligned}
$$

Moreover, Lemma 1 implies that for the function $\delta = \delta_{t_0,J}(\varepsilon)$ in (8) the inequality

$$|\varphi(t_1) - \varphi(t_2)| \leq \varepsilon \text{ if } t_0 \leq t_1 < t_2 < t_1 + \delta$$

holds. Thus H maps S into S. We must prove that H is continuous. For any $\xi_i \in S\,(i = 1, 2)$ and $t \geq t_0$ we have

$$
\begin{aligned}
|(H\xi_1)(t) - (H\xi_2)(t)| &\leq \int_0^t |D(t,s,\xi_1(s)) - D(t_2,s,\xi_2(s))|\, ds \\
&\leq \int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s)|\, ds + \int_0^t |Q(t,s,\xi_1(s)) - Q(t,s,\xi_2(s))|\, ds.
\end{aligned} \tag{27}
$$

From (6), for any $\varepsilon > 0$ there is a $\tau_1 > t_0$ with

$$\int_{-\infty}^{t-\tau_1} P_J(t,s)\, ds < \tfrac{\varepsilon}{6} \text{ if } t \in R. \tag{28}$$

Since $P(t,s,x)$ is uniformly continuous on $U_3 := \{(t,s,x) : t - \tau_1 \leq s \leq t \text{ and } |x| \leq J\}$, for the ε there is a $\delta_1 > 0$ with

$$|P(t,s,x) - P(t,s,y)| < \tfrac{\varepsilon}{6\tau_1} \text{ if } (t,s,x), (t,s,y) \in U_3 \text{ and } |x - y| < \delta_1. \tag{29}$$

From (28) and (29), for the ε we obtain

$$\int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|\, ds < \tfrac{\varepsilon}{6} \text{ if } t_0 \leq t < \tau_1 \text{ and } \|\xi_1 - \xi_2\|_+ < \delta_1, \tag{30}$$

and if $t \geq \tau_1$ and $\|\xi_1 - \xi_2\|_+ < \delta_1$, then we have

$$
\begin{aligned}
&\int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|\, ds \\
\leq 2 &\int_{-\infty}^{t-\tau_1} P_J(t,s)\, ds + \int_{t-\tau_1}^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|\, ds < \tfrac{\varepsilon}{2},
\end{aligned}
$$

which together with (30) yields

$$\int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|\,ds < \tfrac{\varepsilon}{2} \text{ if } \|\xi_1 - \xi_2\|_+ < \delta_1. \tag{31}$$

Next from (7), for the ε there is a $\tau_2 > 0$ with

$$\int_0^t Q_J(t,s)\,ds < \tfrac{\varepsilon}{4} \text{ if } t > \tau_2,$$

which implies

$$\int_0^t |Q(t,s,\xi_1(s)) - Q(t,s,\xi_2(s))|\,ds \le 2\int_0^t Q_J(t,s)\,ds < \tfrac{\varepsilon}{2} \text{ if } t > \tau_2. \tag{32}$$

Since $Q(t,s,x)$ is uniformly continuous on $U_4 := \{(t,s,x) : 0 \le s \le t \le \tau_2 \text{ and } |x| \le J\}$, for the ε there is a $\delta_2 > 0$ with

$$|Q(t,s,x) - Q(t,s,y)| < \tfrac{\varepsilon}{2\tau_2} \text{ if } (t,s,x),(t,s,y) \in U_4 \text{ and } |x-y| < \delta_2,$$

which yields

$$\int_0^t |Q(t,s,\xi_1(s)) - Q(t,s,\xi_2(s))|\,ds < \tfrac{\varepsilon}{2} \text{ if } 0 \le t \le \tau_2 \text{ and } \|\xi_1 - \xi_2\|_+ < \delta_2.$$

This together with (32) implies

$$\int_0^t |Q(t,s,\xi_1(s)) - Q(t,s,\xi_2(s))|\,ds < \tfrac{\varepsilon}{2} \text{ if } \|\xi_1 - \xi_2\|_+ < \delta_2. \tag{33}$$

Thus, from (27), (31) and (33), for the $\delta := \min(\delta_1, \delta_2, \varepsilon)$ we obtain

$$\|H\xi_1 - H\xi_2\|_+ < \varepsilon \text{ if } \xi_1, \xi_2 \in S \text{ and } \|\xi_1 - \xi_2\|_+ < \delta,$$

and hence H is continuous.

Now, applying Theorem 1, H has a fixed point in S, which is a desired asymptotically T-periodic solution of (1).

The latter part is a direct consequence of Theorem 2.

From this theorem and the argument in the proof of Theorem 1 in [5], we have the following corollary.

Corollary 1. *In addition to the assumptions of Theorem 3, if the uniqueness of solutions of (1) with initial conditions* t_0 *and* φ *holds for any* φ*, where* $\sup\{|\varphi(s)| : 0 \le s < t_0\} \le J$*, then the following hold.*

(i) For any continuous initial function $\varphi : [0, t_0) \to R^n$ *with* $\sup\{|\varphi(s)| : 0 \le s < t_0\} \le J$*, the solution* $x(t) = x(t, t_0, \varphi) = y(t) + z(t)$ *of (1) is asymptotically* T*-periodic, where* x*,* $y \in C_J(t_0)$*,* $y(t+T) = y(t)$ *on* R^+*,* $|z(t)| \le q_J(t)$ *on* $[t_0, \infty)$*, and the* T*-periodic extension to* R *of* $y(t)$ *is a* T*-periodic solution of (3).*

(ii) If (3) has a unique R*-bounded solution* $\eta(t)$ *such that* $\|\eta\| \le J$ *and* $\eta(t)$ *satisfies (3) on* R*, then* $\eta(t)$ *is* T*-periodic on* R*, and any solution* $x(t) = x(t, t_0, \varphi)$ *of (1) approaches* $\eta(t)$ *as* $t \to \infty$*, provided that* $\|x\|_+ \le J$*.*

Now we show two examples of a linear equation and a nonlinear equation.

Example 1. Consider the scalar linear equation

$$x(t) = p(t) + \rho e^{-t} - \int_0^t (e^{-t+s} \cos t \sin s + \tfrac{1}{5} e^{-t-s}) x(s)\, ds, \quad t \in R^+, \tag{34}$$

where $p : R \to R$ is a continuous 2π-periodic function, and ρ is constant. Equation (34) is obtained from (1) taking $n = 1$, $T = 2\pi$, $a(t) = p(t) + \rho e^{-t}$, $q(t) = \rho e^{-t}$, $D(t, s, x) = (e^{-t+s} \cos t \sin s + e^{-t-s}/5)x$, $P(t, s, x) = e^{-t+s}(\cos t \sin s)x$ and $Q(t, s, x) = e^{-t-s}x/5$. Define a function $r : R \to R^+$ by

$$r(t) := \int_{-\infty}^t e^{-t+s} |\sin s|\, ds, \quad t \in R.$$

Then clearly $r(t)$ is a π-periodic function, and it is easy to see that

$$\alpha := \sup\{r(t) : t \in R\} \tag{35}$$

satisfies $\frac{1}{2} < \alpha < 19/20$. For $J := 20(\|p\| + |\rho|)/(19 - 20\alpha)$ with $\|p\| = \sup\{|p(t)| : t \in R\}$, we can take the following functions as P_J, Q_J and L_J.

$$P_J(t,s) := Je^{-t+s}|\sin s| \text{ if } t,s \in R,$$
$$Q_J(t,s) := \tfrac{J}{5}e^{-t-s} \text{ if } t,s \in R^+,$$

and

$$L_J(t,s) := e^{-t+s}|\sin s| \text{ if } t,s \in R.$$

It is easy to see that above functions satisfy (4)–(7) and (22). Moreover (21) with $t_0 = 0$ holds for the J, since we have $\|a\|_+ \le \|\rho\| + |\rho|$, $\int_0^t P_J(t,s)\,ds < \alpha J$ and $\int_0^t Q_J(t,s)\,ds \le J/20$ on R^+. Now define a function $q_J : R^+ \to R^+$ by

$$q_J(t) := \left(|\rho| + \alpha J + \frac{J}{5}\int_0^t \exp\left(-s - \int_0^s |\sin u|\,du\right) ds\right) \exp\left(-t + \int_0^t |\sin s|\,ds\right), \quad t \in R^+.$$

We show that (23) and (24) with $t_0 = 0$ hold. Since we have

$$\int_0^t \exp\left(-s - \int_0^s |\sin u|\,du\right) ds \le \int_0^t e^{-s}\,ds < 1 \text{ if } t \in R^+$$

and

$$\exp\left(-t + \int_0^t |\sin s|\,ds\right) = \exp\left(-\int_0^t (1 - |\sin s|)\,ds\right) \to 0 \text{ as } t \to \infty,$$

clearly (23) holds. Moreover it is easy to see that for any $t \in R^+$ we have

$$\begin{aligned} q_J(t) &= (|\rho| + \alpha J + \tfrac{J}{5})e^{-t} - \tfrac{J}{5}e^{-2t} + \textstyle\int_0^t e^{-t+s}|\sin s| q_J(s)\,ds \\ &\ge |q(t)| + \int_{-\infty}^0 P_J(t,s)\,ds + \int_0^t Q_J(t,s)\,ds + \int_0^t L_J(t,s) q_J(s)\,ds, \end{aligned}$$

that is, (24) with $t_0 = 0$ holds. Thus by Theorem 3, (34) has an asymptotically 2π-periodic solution $x(t) = y(t) + z(t)$ such that x, $y \in C_J := C_J(0)$, $y(t+2\pi) = y(t)$ and $|z(t)| \le q_J(t)$ on R^+, and the 2π-periodic extension to R of $y(t)$ is a 2π-periodic solution of the equation

$$x(t) = p(t) - \int_{-\infty}^t e^{-t+s}(\cos t \sin s)x(s)\,ds, t \in R. \tag{36}$$

Example 2. Corresponding to (34), consider the scalar nonlinear equation

$$x(t) = p(t) + \rho e^{-t} - \int_0^t (\sigma e^{-t+s} \sin s + \tau e^{-t-s}) x^2(s)\, ds,\ t \in R^+, \tag{37}$$

where $p : R \to R$ is a continuous 2π-periodic function, and ρ, σ and τ are constants such that $\|p\| + |\rho| > 0$ and $64(\|p\| + |\rho|)\sigma^2 + (4\alpha|\sigma| + |\tau|)\pi^2 < 16|\sigma|\pi$, where α is the number defined in (35). Equation (37) is obtained from (1) taking $n = 1$, $T = 2\pi$, $a(t) = p(t) + \rho e^{-t}$, $q(t) = \rho e^{-t}$, $D(t, s, x) = (\sigma e^{-t+s} \sin s + \tau e^{-t-s})x^2$, $P(t, s, x) = \sigma e^{-t+s}(\sin s)x^2$ and $Q(t, s, x) = \tau e^{-t-s} x^2$. Let J be a number defined by

$$J := \frac{2}{4\alpha|\sigma| + |\tau|}\left(1 - (1 - (\|p\| + |\rho|)(4\alpha|\sigma| + |\tau|))^{1/2}\right). \tag{38}$$

Then it is easy to see that $\|p\| + |\rho| + (\alpha|\sigma| + |\tau|/4)J^2 = J$ and $0 < 4|\sigma|J < \pi$. For this J we can take the following functions as P_J, Q_J and L_J here.

$$P_J(t, s) := J^2|\sigma|e^{-t+s}|\sin s| \text{ if } t, s \in R,$$

$$Q_J(t, s) := J^2|\tau|e^{-t-s} \text{ if } t, s \in R^+,$$

and

$$L_J(t, s) := 2J|\sigma|e^{-t+s}|\sin s| \text{ if } t, s \in R.$$

It is easy to see that these functions satisfy (4)–(7) and (22). Moreover (38) implies (21) with $t_0 = 0$, since we have $\|a\|_+ \le \|p\| + |\rho|$, $\int_0^t P_J(t, s) \le \alpha|\sigma|J^2$ and $\int_0^t Q_J(t, s)\, ds \le |x|J^2/4$ on R^+. Now define a function $q_J : R^+ \to R^+$ by

$$q_J(t) := \left(|\rho| + \alpha|\sigma|J^2 + |\tau|J^2 \int_0^t \exp\left(-s - 2|\sigma|J \int_0^s |\sin u|\, du\right) ds\right) \times \exp\left(-t + 2|\sigma|J \int_0^t |\sin s|\, ds\right),\ t \in R^+.$$

As in Example 1, it is easy to see that (23) and (24) with $t_0 = 0$ hold. Thus by Theorem 3, (37) has an asymptotically 2π-periodic solution $x(t) = y(t) + z(t)$ such that x, $y \in C_J$,

$y(t+2\pi) = y(t)$ and $|z(t)| \le q_J(t)$ on R^+, and the 2π-periodic extension to R of $y(t)$ is a 2π-periodic solution of the equation

$$x(t) = p(t) - \sigma \int_{-\infty}^{t} e^{-t+s} (\sin s) x^2(s)\, ds, \quad t \in R.$$

4. Asymptotically periodic solutions of (2). Using Theorem 1 and arguments similar to those in the previous section, we can discuss the existence of asymptotically T-periodic solutions of (2). For any $t_0 \in R$, let $C(t_0)$ denote again a set of bounded functions $\xi : R \to R^n$ such that $\xi(t)$ is continuous on R except at t_0, and $\xi(t_0) = \xi(t_0+)$. Then $\|\xi\| := \sup\{|\xi(t)| : t \in R\}$ is a norm on $C(t_0)$, and $(C(t_0), \|\cdot\|)$ is a Banach space. In this section, we need the following assumptions for $Q(t,s,x)$.

(39) $$\int_{-\infty}^{t} Q_J(t,s)\, ds \to 0 \text{ as } t \to \infty,$$

and

(40) $$\int_{-\infty}^{t} Q_J(t+\tau,s)\, ds \to 0 \text{ uniformly for } t \in R \text{ as } \tau \to \infty.$$

As in the previous section, for any $\xi \in C(t_0)$ define a map H on $C(t_0)$ by

$$(H\xi)(t) := \begin{cases} \xi(t), & t < t_0, \\ a(t) - \int_{-\infty}^{t} D(t,s,\xi(s))\, ds, & t \ge t_0. \end{cases}$$

For any $J > 0$, let $C_J(t_0)$ denote again the set $\{\xi \in C(t_0) : \|\xi\| \le J\}$. Then, corresponding to Lemmas 1 and 2, and Theorem 2, we have the following which we state without proofs.

Lemma 3. *If (4)–(6), (39) and (40) hold, then for any $t_0 \in R$ and any $J > 0$ there is a continuous increasing positive function $\delta = \delta_{t_0,J}(\varepsilon) : (0,\infty) \to (0,\infty)$ with*

(41) $$|(H\xi)(t_1) - (H\xi)(t_2)| \le \varepsilon \text{ if } \xi \in C_J(t_0) \text{ and } t_0 \le t_1 < t_2 < t_1 + \delta.$$

Lemma 4. *If (4)–(6), (39) and (40) hold, then for any asymptotically T-periodic function $\xi(t)$ on R such that $\xi(t) = \pi(t) + \rho(t)$, $\xi, \pi \in C(t_0)$ for some $t_0 \in R$, $\pi(t+T) = \pi(t)$ on*

R, and $\rho(t) \to 0$ as $t \to \infty$, the function $\int_{-\infty}^{t} D(t,s,\xi(s))\,ds$ is continuous asymptotically T-periodic on $[t_0, \infty)$, and its T-periodic part is given by $\int_{-\infty}^{t} P(t,s,\pi(s))\,ds$.

Theorem 4. *If (4)–(6), (39) and (40) hold, and if (2) has an asymptotically T-periodic solution with an initial time $t_0 \in R$, then the T-periodic extension to R of its T-periodic part on $[t_0, \infty)$ is a T-periodic solution of (3). In particular, if the asymptotically T-periodic solution of (2) is asymptotically constant, then (3) has a constant solution.*

In order to obtain a theorem similar to Theorem 3, we need more assumptions. In addition to (4)–(6), (39) and (40), suppose that for some $t_0 \in R$ and $J > 0$, (22), (23) and the following inequalities hold.

$$\|a\|_{t_0} + \int_{-\infty}^{t} P_J(t,s)\,ds + \int_{-\infty}^{t} Q_J(t,s)\,ds \le J \text{ if } t \ge t_0, \tag{42}$$

and

$$|q(t)| + \int_{-\infty}^{t} Q_J(t,s)\,ds + 2m(t) + \int_{t_0}^{t} L_J(t,s) q_J(s)\,ds \le q_J(t) \text{ if } t \ge t_0, \tag{43}$$

where $m(t) := \min(J \int_{-\infty}^{t_0} L_J(t,s)\,ds, \int_{-\infty}^{t_0} P_J(t,s)\,ds)$. Then, corresponding to Theorem 3 we have the following theorem.

Theorem 5. *If (4)–(6), (22), (23), (39), (40), (42) and (43) with some $t_0 \in R$ and $J > 0$ hold, then for any continuous initial function $\varphi : (-\infty, t_0) \to R^n$ with $\sup\{|\varphi(s)| : s < t_0\} \le J$, (2) has an asymptotically T-periodic solution $x(t) = y(t) + z(t)$ such that $x,\ y \in C_J(t_0)$, $x(t)$ satisfies (2) and $|z(t)| \le q_J(t)$ on $[t_0, \infty)$, and $y(t)$ is a T-periodic solution of (3).*

This theorem can be proved easily from Lemmas 3 and 4 by similar arguments to those in the proof of Theorem 3 letting S be a set of functions $\xi \in C_J(t_0)$ such that $\xi = \pi + \rho$, $\pi \in C_J(t_0)$, $\xi(t) = \varphi(t)$ if $t < t_0$, $\pi(t+T) = \pi(t)$ on R, $\rho(t)$ satisfies (25), and that for the function $\delta = \delta_{t_0,J}(\varepsilon)$ in (31), $|\xi(t_1) - \xi(t_2)| \le \varepsilon$ if $t_0 \le t_1 < t_2 < t_1 + \delta$.

Next, corresponding to Corollary 1, we have the following corollary.

Corollary 2. *In addition to the assumptions of Theorem 5, if the uniqueness of solutions of (2) with initial conditions* t_0 *and* φ *holds for any* φ*, where* $\sup\{|\varphi(s)| : s < t_0\} \le J$*, then the following hold.*

(i) For any continuous initial function $\varphi : (-\infty, t_0) \to R^n$ *with* $\sup\{|\varphi(s)| : s < t_0\} \le J$*, the solution* $x(t) = x(t, t_0, \varphi) = y(t) + z(t)$ *of (2) is asymptotically* T*-periodic, where* x*,* $y \in C_J(t_0)$*,* $|z(t)| \le q_J(t)$ *on* $[t_0, \infty)$*, and* $y(t)$ *is a* T*-periodic solution of (3).*

(ii) If (3) has a unique R*-bounded solution* $\eta(t)$ *such that* $\|\eta\| \le J$ *and* $\eta(t)$ *satisfies (3) on* R*, then* $\eta(t)$ *is* T*-periodic, and any solution* $x(t) = x(t, t_0, \varphi)$ *of (2) approaches* $\eta(t)$ *as* $t \to \infty$ *provided that* $\|x\| \le J$*.*

Finally, corresponding to Example 1 we show an example of a linear equation.

Example 3. Consider the scalar linear equation

$$x(t) = p(t) - \rho e^{-t} - \int_{-\infty}^{t} e^{-t+s}\left(\sin s + \frac{1}{20}e^{-s^2}\right)x(s)\,ds, \quad t \in R, \tag{44}$$

where $p : R \to R$ is a continuous 2π-periodic function, and ρ is constant. It is easy to see that (4)–(6), (22), (39), (40) and (42) hold. Now define a function q_J on R^+ by

$$q_J(t) := \left(|\rho| + 2\alpha J + \frac{J}{20} + \frac{J}{20}\int_0^t (1-2s)\exp(s - s^2 - \int_0^s |\sin u|\,du)\,ds\right) \times \exp\left(-t + \int_0^t |\sin s|\,ds\right),\ t \in R^+.$$

Using an integration by parts, we can easily see that

$$-1 < \int_0^t (1-2s)\exp\left(s - s^2 - \int_0^s |\sin u|\,du\right) ds < e + e^{1/4} - 1,$$

which implies that $q_J(t) > 0$ on R^+ and satisfies (23). As in Example 1, it is easy to see that (43) with $t_0 = 0$ holds. Moreover, since (44) is a linear equation, the uniqueness of solutions of (44) with initial conditions t_0 and φ holds for any φ with $\sup\{|\varphi(s)| : s < 0\} \le J$. Thus by Corollary 2, for any continuous initial function $\varphi : (-\infty, 0) \to R$ with $\sup\{|\varphi(s)| : s < 0\} \le J$, the solution $x(t) = x(t, t_0, \varphi) = y(t) + z(t)$ of (44) is asymptotically

2π-periodic, where x, $y \in C_J$, $|z(t)| \le q_J(t)$ on R^+, and $y(t)$ is a 2π-periodic solution of (36).

5. Periodic solutions. Although Theorems 3 and 5 assure the existence of T-periodic solutions of (3), we can prove directly the existence of T-periodic solutions of (3) under weaker assumptions than those in Theorems 3 and 5 using Schauder's first theorem.

Let $(\mathcal{P}_T, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\xi : R \to R^n$ with the supremum norm. For any $\xi \in \mathcal{P}_T$, define a map H on $\mathcal{P}_T$ by

$$(H\xi)(t) := p(t) - \int_{-\infty}^{t} P(t, s, \xi(s))\, ds, \quad t \in R.$$

Then, by a method similar to the method used in the proof of Lemma 1, we can prove the following lemma which we state without proof.

Lemma 5. *If (4)–(6) with $Q(t, s, x) \equiv 0$ hold, then for any $J > 0$ there is a continuous increasing positive function $\delta = \delta_J(\varepsilon) : (0, \infty) \to (0, \infty)$ with*

$$|(H\xi)(t_1) - (H\xi)(t_2)| \le \varepsilon \text{ if } \xi \in \mathcal{P}_T,\ \|\xi\| \le J \text{ and } |t_1 - t_2| < \delta. \tag{45}$$

Now we have the following theorem.

Theorem 6. *In addition to (4)–(6) with $Q(t, s, x) \equiv 0$, suppose that for some $J > 0$ the inequality*

$$\|p\| + \int_{-\infty}^{t} P_J(t, s)\, ds \le J \text{ if } t \in R \tag{46}$$

holds. Then (3) has a T-periodic solution $x(t)$ with $\|x\| \le J$.

Proof. Let S be a set of functions $\xi \in \mathcal{P}_T$ such that $\|\xi\| \le J$ and for the function $\delta = \delta_J(\varepsilon)$ in (45), $|\xi(t_1) - \xi(t_2)| \le \varepsilon$ if $|t_1 - t_2| < \delta$.

First we can prove that S is a compact convex nonempty subset of $\mathcal{P}_T$ by a method similar to the method used in the proof of Theorem 3.

Next we prove that H maps S into S. For any $\xi \in S$, let $\varphi := H\xi$. Then, clearly $\varphi(t)$ is T-periodic. In addition, from (46) we have

$$|\varphi(t)| \le \|p\| + \int_{-\infty}^{t} P_J(t,s)\,ds \le J \text{ if } t \in R,$$

and hence $\|\varphi\| \le J$. Moreover, Lemma 5 implies that for the δ in (45) we obtain

$$|\varphi(t_1) - \varphi(t_2)| \le \varepsilon \text{ if } \xi \in \mathcal{P}_T,\ \|\xi\| \le J \text{ and } |t_1 - t_2| < \delta.$$

Thus H maps S into S.

The continuity of H can be proved similarly as in the proof of Theorem 3.

Finally, applying Theorem 1 we can conclude that H has a fixed point x in S, which is a T-periodic solution of (3) with $\|x\| \le J$.

Remark 1. (i) In addition to the continuity of the map H, we can easily prove that H maps each bounded set of $\mathcal{P}_T$ into a compact set of $\mathcal{P}_T$. Thus Theorem 6 can be proved using Schauder's second theorem.

(ii) Theorem 1 in [4] also assures the existence of T-periodic solutions of (3) under suitable assumptions including differentiability conditions on P, while Theorem 6 does not require any differentiability condition on P.

6. Relation of (1) and (3). In Theorems 2 and 4, we showed relations between an asymptotically T-periodic solution of (1) or (2) and a T-periodic solution of (3). Moreover, concerning a relation between (1) and (3) we have the following theorem.

Theorem 7. *Under the assumptions (4)–(7), the following five conditions are equivalent:*

(i) Equation (3) has a T-periodic solution.

(ii) For some $q(t)$ and $Q(t,s,x) \equiv 0$, (1) has a T-periodic solution which satisfies (1) on R^+.

(iii) For some $q(t)$ and $Q(t,s,x) \equiv 0$, (1) has an asymptotically T-periodic solution with an initial time in R^+.

(iv) For some $q(t)$ and $Q(t,s,x)$, (1) has a T-periodic solution which satisfies (1) on R^+.

(v) For some $q(t)$ and $Q(t,s,x)$, (1) has an asymptotically T-periodic solution with an initial time in R^+.

Proof. First we prove that (i) implies (ii). Let $\pi(t)$ be a T-periodic solution of (3), and let

$$q(t) := -\int_{-\infty}^{0} P(t,s,\pi(s))\,ds, \quad t \in R^+.$$

Then, clearly $q(t)$ is continuous and $q(t) \to 0$ as $t \to \infty$. Thus it is easy to see that for the $q(t)$ and $Q(t,s,x) \equiv 0$, (1) has a T-periodic solution $\pi(t)$, which satisfies (1) on R^+.

Next, it is clear that (ii) and (iii) imply (iii) and (v) respectively. Moreover, from Theorem 2, (v) yields (i).

Finally, since it is trivial that (ii) implies (iv), we prove that (iv) yields (ii). Let $\psi(t)$ be a T-periodic solution of (1) with some $q(t)$ and $Q(t,s,x)$ which satisfies (1) on R^+, and let

$$r(t) := -\int_{0}^{t} Q(t,s,\psi(s))\,ds, \quad t \in R^+.$$

Then, clearly $r(t)$ is continuous and $r(t) \to 0$ as $t \to \infty$. Thus it is easy to see that for $a(t) = p(t)+q(t)+r(t)$ and $Q(t,s,x) \equiv 0$, (1) has a T-periodic solution $\psi(t)$ which satisfies (1) on R^+.

Remark 2. From the proof of Theorem 7, it is easy to see that the equivalence among (i)–(iii) can be obtained without (7).

In [5], we discussed a relation between the equation

$$x(t) = a(t) - \int_0^t E(t,s)x(s)\,ds - \int_0^t Q(t,s,x(s))\,ds, \quad t \in R^+ \tag{47}$$

and the linear equation

$$x(t) = p(t) - \int_{-\infty}^{t} E(t,s)x(s)\,ds, \quad t \in R, \tag{48}$$

where a, p and Q satisfy (4) and (7), and $E : R \times R \to R^{n\times n}$ is a continuous function such that $E(t+T, s+T) = E(t,s)$ and $\int_{-\infty}^{t} |E(t+\tau, s)|\, ds \to 0$ uniformly for $t \in R$ as $\tau \to \infty$, where $|E| := \sup\{|Ex| : |x| = 1\}$. Concerning (47) and (48), we state a theorem. For the proof, see Lemma 1 and Theorem 7 in [5].

Theorem 8([5]). *Under the above assumptions for (47) and (48), the following hold.*

(i) If (47) has an R^+-bounded solution with an initial time in R^+, then (48) has an R-bounded solution which satisfies (48) on R.

(ii) If (48) has an R-bounded solution which satisfies (48) on R, then (48) has a T-periodic solution.

Now we have the following theorem concerning a relation between (47) and (48).

Theorem 9. *Under the above assumptions for (47) and (48), the following eight conditions are equivalent:*

(i) Equation (48) has a T-periodic solution.

(ii) For some $q(t)$ and $Q(t,s,x) \equiv 0$, (47) has a T-periodic solution which satisfies (47) on R^+.

(iii) For some $q(t)$ and $Q(t,s,x) \equiv 0$, (47) has an asymptotically T-periodic solution with an initial times in R^+.

(iv) For some $q(t)$ and $Q(t,s,x) \equiv 0$, (47) has an R^+-bounded solution with an initial time in R^+.

(v) For some $q(t)$ and $Q(t,s,x)$, (47) has a T-periodic solution which satisfies (47) on R^+.

(vi) For some $q(t)$ and $Q(t,s,x)$, (47) has an asymptotically T-periodic solution with an initial time in R^+.

(vii) For some $q(t)$ and $Q(t,s,x)$, (47) has an R^+-bounded solution with an initial time in R^+.

(viii) Equation (48) has an R-bounded solution which satisfies (48) on R.

Proof. The equivalence among (i)–(iii), (v) and (vi) is a direct consequence of Theorem 7. From this and the trivial implication from (iii) to (iv), it is clear that (i) and (iv) imply (iv) and (vii) respectively. Next, from Theorem 8(i), (vii) yields (viii). Moreover, from Theorem 8(ii), (viii) implies (i), which completes the proof.

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