LINEAR INTEGRAL EQUATIONS AND PERIODICITY

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1. Introduction, results, and problems.

Consider the scalar equations

$$x'(t) = p(t) - \int_{t-T}^{t} C(t-s)x(s)\,ds,\ C''(t) > 0, \tag{1}$$

and

$$x(t) = \alpha x(t-h) + p(t) - \int_{t-T}^{t} C(t-s)x(s)\,ds,\ C''(t) \geq 0,\ |\alpha| < 1, \tag{2}$$

where

$$p(t+T) = p(t),\ \int_0^T p(s)\,ds = 0,\ h \text{ is constant.} \tag{3}$$

Theorem 1. *Equations (1) and (2) have unique T-periodic solutions.*

The motivation for this theorem is a set of five works [9-14] concerning

$$x'(t) = -\int_{t-T}^{t} C(t-s)g(x(s))\,ds,\ xg(x) > 0 \text{ if } x \neq 0, \tag{V}$$

and

$$C(T) = 0,\ C(t) \geq 0,\ C'(t) \leq 0,\ C''(t) > 0,\ 0 \leq t \leq T, \tag{V^*}$$

together with a companion problem

$$x'(t) = -\int_0^t C(t-s)g(x(s))\,ds,\ xg(x) \geq 0 \text{ if } x \neq 0, \tag{L}$$

and

$$(L^*) \qquad\qquad C(t) > 0,\ C'(t) \le 0,\ C''(t) \ge 0.$$

Equation (V) seems to have been first investigated by Volterra in [20] and in the fourth chapter of [21] to model a population problem. Levin and Nohel [14] model reactor dynamics using (V) with conditions (V^*) and show that solutions tend to zero. Hale and Lunel [8; pp. 145–6] apply sophisticated techniques to (V) using (V^*) to find limit sets. In addition to its application to populations and reactors, it is also pointed out in [8; pp. 145–6] that the derivative of (V), namely,

$$x'' + C(0)g(x) = -\int_{t-T}^{t} C'(t-s)g(x(s))\,ds$$

is a one-dimensional model of viscoelasticity with x the strain and C the relaxation function.

Obviously the condition $C(T) = 0$ in (V^*) is severe and greatly limits the problem. But there is some motivation for it. Equation (V) is a problem with a weighted and distributed memory. That integral can be written as

$$\int_{-T}^{0} C(-s)g(x(s+t))\,ds$$

so that under (V^*) we see that C weights g most heavily at the present time, while the weight decreases in an exceedingly regular way until it vanishes at T time units ago.

While much has been written since about (V), no one seems to have been able to obtain qualitative theory about (V) without asking that $C(T) = 0$.

By contrast, much can be said about the companion equation (L) without asking that C vanish. A typical work is that of Levin [9] where he shows that solutions of (L) tend to zero when (L^*) holds. Levin's paper inspired the positive kernel work of Halanay [6] which was corrected and extended by MacCamy and Wong [16]. Out of that has proceeded much work which can be found summarized in Corduneanu [4] and Gripenberg, Londen, and Staffans [5]. In some of those references we encounter the integral as

$$\int_{-\infty}^{t} C(t-s)g(x(s))\,ds$$

which can represent (V) when C has compact support.

This is the merest sketch of relevant work and the interested reader is urged to consult at least ([3], [9–17]) for a fuller view.

It needs to pointed out that (V) is a much deeper problem than (L). Indeed, when $g(x) = x$, then the solution space of (L) is one-dimensional when $t_0 = 0$, while that of (V) is generally infinite dimensional.

One of our other interests in the problem was to try to delete at least the condition $C(T) = 0$ in (V^*) and to prove that the remainder was sufficient for asymptotic stability as it is for (L).

We have failed to do that, but we have obtained clear periodic results in the linear case. It is, of course, well-known, by way of Perron's theorem, that there is a connection between boundedness of all solutions under all bounded perturbations and the concept of asymptotic stability of the unperturbed system (see Hale [7; p. 152] and Burton [1; p. 114]).

We pose three problems along these lines and comment on them in the form of remarks as we go through the proofs of our theorem.

PROBLEM 1. Can the conclusion of our theorem be changed to asymptotic stability when $p(t) \equiv 0$?

PROBLEM 2. Can the integral in (1) and (2) have the limits $t-h$ to t, where $h \neq T$?

PROBLEM 3. Can the theorem be proved when x in the integral is replaced by $g(x)$ as in (V)?

Finally, we motivated (V) and (L) by physical applications and these are, of course, important. But historically these equations have been of much more interest from a purely mathematical point of view. In so many examples used to illustrate stability theory for delay–differential equations two properties are present:

(A) The equation contains an ODE part which dominates the equation and/or

(B) A Liapunov function is found with a negative definite derivative which enables investigators to drive solutions to zero.

Problem (V) and (L) have served as examples in which (A) certainly fails, while the standard Liapunov functional has a derivative which is only semi-definite so that sophisticated techniques are still required to prove asymptotic stability.

2. Proofs. Our theorem rests on two fixed point theorems. The first is an old result of Schaefer [19] (cf. Smart [18; p. 29]) which will yield the periodic solution for (1).

Theorem. *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, $P : \mathcal{B} \to \mathcal{B}$ be continuous and map bounded sets into compact sets. Either*

(i) the equation $x = \lambda P x$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.

The proof of the periodic solution for (2) rests on the following result of Burton-Kirk [2].

Theorem. *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $A, B : \mathcal{B} \to \mathcal{B}$, B a contraction with contraction constant $\alpha < 1$, and A continuous with A mapping bounded sets into compact sets. Either*

(i) $x = \lambda B(x/\lambda) + \lambda A x$ has a solution in $\mathcal{B}$ for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

We begin with (2) and consider the homotopy equation

$$x(t) = \alpha x(t-h) + \lambda \left[p(t) - \int_{t-T}^{t} x(s) C(t-s) ds \right] \tag{2_λ}$$

$0 < \lambda < 1$. Let $\mathcal{P}_T^0, \|\cdot\|)$ be the Banach space of continuous T-periodic functions having mean value zero with the supremum norm. Define $P : \mathcal{P}_T^0 \to \mathcal{P}_T^0$ by $\varphi \in \mathcal{P}_T^0$ implies that

$$(P\varphi)(t) = \alpha\varphi(t-h) + \lambda\left[p(t) - \int_{t-T}^{t} \varphi(s)C(t-s)\,ds\right] \tag{4}$$

and define $A : \mathcal{P}_T^0 \to \mathcal{P}_T^0$ by $\varphi \in \mathcal{P}_T^0$ implies that

$$(A\varphi)(t) = p(t) - \int_{t-T}^{t} \varphi(s)C(t-s)\,ds. \tag{5}$$

Thus, in the Burton-Kirk Theorem we have $(B\varphi)(t) = \alpha\varphi(t-h)$. To see that $P : \mathcal{P}_T^0 \to \mathcal{P}_T^0$, note that $\varphi \in \mathcal{P}_T^0$ implies that $P\varphi$ is T-periodic and

$$\begin{aligned} \int_0^T \int_{u-T}^{u} \varphi(s)C(u-s)\,ds\,du &= \int_0^T \int_{-T}^{0} \varphi(s+u)C(-s)\,ds\,du \\ &= \int_{-T}^{0} \int_0^T \varphi(s+u)\,du\; C(-s)\,ds = 0 \end{aligned}$$

for each fixed s; hence $P\varphi \in \mathcal{P}_T^0$.

REMARK 1. This argument fails in the nonlinear case.

We may note that A is continuous and maps bounded sets into equicontinuous sets. Clearly, $(B\varphi)(t) = \alpha\varphi(t-h)$ is a contraction. Thus, all that remains to prove that (2) has a solution in $\mathcal{P}_T^0$ is to find an a priori bound on fixed points of P in $\mathcal{P}_T^0$. Thus, let $x \in \mathcal{P}_T^0$ solve (2_λ) and define

$$V(t) = \lambda \int_{t-T}^{t} C_s(t-s)\left(\int_s^t x(u)\,du\right)^2 ds \tag{6}$$

so that

$$\begin{aligned} V'(t) = -\lambda C_s(T)&\left(\int_{t-T}^{t} x(u)\,du\right)^2 \\ +\lambda \int_{t-T}^{t} C_{st}(t-s)&\left(\int_s^t x(u)\,du\right)^2 ds \\ +2\lambda x \int_{t-T}^{t} C_s(t-s)&\int_s^t x(u)\,du\,ds \end{aligned}$$

REMARK 2. The first term in V' is zero because $x \in \mathcal{P}_T^0$. This would fail in the nonlinear case and it would fail if T were replaced by an h (as in Problem 2). Note that

(V^*) would ask $C_s(T) > 0$, while $C_{st} \le 0$; thus, whether our conditions hold or (V^*) holds, we now get

$$\begin{aligned} V'(t) &\le 2\lambda x \int_{t-T}^{t} C_s(t-s) \int_s^t x(u)\,du\,ds \\ &= 2\lambda x \left[C(t-s) \int_s^t x(u)\,du \Big|_{t-T}^{t} + \int_{t-T}^{t} C(t-s)x(s)\,ds \right] \end{aligned}$$

REMARK 3. The first term on the right is zero since $x \in \mathcal{P}_T^0$; this would fail in the nonlinear case and when T is replaced by h. When (V^*) holds, that first term is zero because $C(T) = 0$. Thus, whether our conditions hold or (V^*) holds, we now have

$$\begin{aligned} V'(t) &\le 2x\lambda \int_{t-T}^{t} C(t-s)x(s)\,ds \\ &= 2x\Big[-x + \alpha x(t-h) + \lambda p(t)\Big] \end{aligned}$$

from (2_λ). Now, if $\varepsilon > 0$ and $-2 + 2|\alpha| + \varepsilon < 0$, then

$$\begin{aligned} V'(t) &\le -2x^2 + |\alpha|(x^2 + x^2(t-h)) + \varepsilon x^2 + p^2/\varepsilon \\ &= [-2 + |\alpha| + \varepsilon]x^2 + |\alpha| x^2(t-h) + p^2(t)/\varepsilon. \end{aligned}$$

REMARK 4. Had we used (V^*), at this point if $p \equiv 0$, then V' is negative definite and an easy asymptotic stability result would follow. But our condition gave this V' only in case $x \in \mathcal{P}_T^0$, so we have nothing in the form of asymptotic stability.

It is clear that $x \in \mathcal{P}_T^0$ implies that $V(T) = V(0)$ and so

$$\begin{aligned} 0 = V(T) - V(0) &\le \Big[-2 + |\alpha| + \varepsilon\Big] \int_0^T x^2(s)\,ds \\ &+ |\alpha| \int_0^T x^2(s-h)\,ds + T\|p\|^2/\varepsilon \\ &= [-2 + 2|\alpha| + \varepsilon] \int_0^T x^2(s)\,ds + T\|p\|^2/\varepsilon \end{aligned}$$

and there is an $M > 0$ with

$$\int_0^T x^2(s)\,ds \le M^2. \tag{7}$$

From (2_λ) and (7) we have

$$|x(t)| \leq |\alpha||x(t-h)| + \|p\| + \left[\int_{t-T}^{t} C^2(t-s)\,ds \int_{t-T}^{t} x^2(s)\,ds\right]^{1/2}$$

so that

$$\|x\|(1-|\alpha|) \leq \|p\| + M\left[\int_{-T}^{0} C^2(-s)\,ds\right]^{1/2},$$

a suitable a priori bound. This proves that (2) has a T-periodic solution. To see that it is unique, if there are two, then the difference satisfies

$$x(t) = \alpha x(t-h) - \int_{t-h}^{t} \lambda C(t-s)x(s)\,ds,\ \lambda = 1.$$

We may proceed to get an a priori bound on all possible periodic solutions of this equation just as before and see that in (7) we can let $M \to 0$.

We now turn to (1) which is parallel to the preceeding proof, but considerably more difficult.

Lemma 1. *Consider*

$$\int_{t-T}^{t} x(s)C(t-s)\,ds = \int_{-T}^{0} x(t+s)C(-s)\,ds$$

so that C is defined only on $[0,T]$ and $C''(t) > 0$. Then there is a $\beta > 0$ with $C''(t) > \beta|C'(t)|$ on $[0,T]$.

We now proceed to apply Schaefer's theorem. Let $0 \leq \lambda \leq 1$ and suppose that $x \in \mathcal{P}_T^0$ satisfies

$$x'(t) = \lambda\left[p(t) - \int_{t-T}^{t} x(s)C(t-s)\,ds\right]. \tag{1_λ}$$

Now define $P : \mathcal{P}_T^0 \to \mathcal{P}_T^0$ by $\varphi \in \mathcal{P}_T^0$ implies that

$$\begin{aligned}(P\varphi)(t) &= \lambda \int_0^t [p(s) - \int_{s-T}^{s} \varphi(u)C(s-u)\,du]\,ds \\ &-(\lambda/T)\int_0^T \int_0^r \left[p(s) - \int_{s-T}^{s} \varphi(u)C(s-u)\,du\right] ds\,dr.\end{aligned} \tag{8}$$

The integrand in the first term of $P\varphi$ is periodic with mean value zero; hence, that first term is periodic. The second term subtracts the mean value of the first term. Hence, $\varphi \in \mathcal{P}_T^0$ implies $P\varphi \in \mathcal{P}_T^0$. Moreover, a fixed point of P in $\mathcal{P}_T^0$ satisfies (1_λ).

Since $P\varphi \in \mathcal{P}_T^0$, it is clear that P maps bounded sets into equicontinuous sets and P is continuous. Schaefer's theorem will yield a periodic solution of (1) if we can establish an a priori bound.

If $x \in \mathcal{P}_T^0$ satisfies (8) (hence (1_λ)), define

$$V(t) = x^2(t) + \lambda \int_{t-T}^{t} C_s(t-s)\Big(\int_s^t x(u)du\Big)^2 ds$$

so that (arguing as before with $x \in \mathcal{P}_T^0$)

$$\begin{aligned} V' = {} & 2x\Big[\lambda p - \lambda \int_{t-T}^{t} C(t-s)x(s)\,ds\Big] \\ & + \lambda \int_{t-T}^{t} C_{st}(t-s)\Big(\int_s^t x(u)du\Big)^2 ds \\ & + 2\lambda x \int_{t-T}^{t} C_s(t-s)\Big(\int_s^t x(u)du\Big)\,ds. \end{aligned}$$

The last term is

$$2\lambda x\Big[C(t-s)\int_s^t x(u)\,du\Big|_{t-T}^{t} + \int_{t-T}^{t} C(t-s)x(s)\,ds\Big]$$

so that (using Lemma 1) we now have

$$V' \le 2\lambda x p - \lambda\beta \int_{t-T}^{t} |C_s(t-s)|\Big(\int_s^t x(u)\,du\Big)^2 ds. \tag{9}$$

Lemma 2. *There is a* $\gamma > 0$ *with*

$$\lambda\beta \int_{t-T}^{t} |C_s(t-s)|\Big(\int_s^t x(u)\,du\Big)^2 ds \ge \gamma(x' - \lambda p(t))^2.$$

To prove this, from (1_λ) we have

$$\begin{aligned} (x' - \lambda p(t))^2 &= \lambda^2\Big(\int_{t-T}^{t} C(t-s)x(s)\,ds\Big)^2 \\ &= \lambda^2\Big[-C(t-s)\int_s^t x(u)\,du\Big|_{t-T}^{t} + \int_{t-T}^{t} C_s(t-s)\int_s^t x(u)\,du\,ds\Big]^2 \\ &\le \lambda^2 \int_{t-T}^{t} |C_s(t-s)|\,ds \int_{t-T}^{t} |C_s(t-s)|\Big(\int_s^t x(u)\,du\Big)^2 ds \end{aligned}$$

from which Lemma 2 follows.

Thus, from this and (9) we have

$$\begin{aligned} V'(t) &\leq 2\lambda x p(t) - \gamma(x' - \lambda p(t))^2 \\ &= 2\lambda x p(t) - \gamma(x')^2 + 2\gamma x' p(t) - \lambda\gamma p^2(t) \end{aligned}$$

or

$$V'(t) \leq -\mu(x')^2 + 2|x|\,|p(t)| + M \tag{10}$$

for some $\mu > 0$ and $M > 0$.

Now $x \in \mathcal{P}_T^0$ implies that $x(t_0) = 0$ for some t_0 so that the inequality

$$\|x\| \leq \min |x(t)| + \int_0^T |x'(s)|\, ds$$

yields

$$\|x\|^2 \leq T \int_0^T |x'(s)|^2\, ds \tag{11}$$

by the Schwarz inequality. Also, $x \in \mathcal{P}_T^0$ yields $V(T) = V(0)$ so that (10) and (11) imply

$$\begin{aligned} 0 = V(T) - V(0) &\leq -\mu \int_0^T |x'(s)|^2\, ds + 2T\|x\|\,\|p\| + MT \\ &\leq -\mu(\|x\|^2/T) + 2T\|x\|\,\|p\| + MT \end{aligned}$$

or

$$\|x\|^2 \leq J \tag{12}$$

for some $J > 0$.

By Schaefer's theorem we see that (1) has a T-periodic solution. The uniqueness proceeds just as it did for (2). That will complete the proof of our theorem.

3. A nonlinear problem. We now show that the condition $C(T) = 0$ can be replaced in the periodic case by conditions on C' and C'' at $t = 0$ and $t = T$. Consider the equation

$$x(t) = p(t) - \int_{t-T}^{t} C(t-s)g(x(s))\,ds \tag{13}$$

in which

$$C(0) > C(T) > 0,\ C'(t) \le 0,\ C''(t) \ge 0, \tag{14}$$

$$xg(x) > 0 \text{ if } x \ne 0, \ \text{ and } p(t+T) = p(t) \tag{15}$$

A similar discussion can be given for

$$x'(t) = p(t) - \int_{t-T}^{t} C(t-s)g(x(s))\,ds$$

but we would then need a minimal growth condition on $\int_0^x g(s)\,ds$.

Now (13) requires that C be defined for $0 \le t \le T$ and so we can define $L(t)$ for $t \ge 0$ by

$$L(t+nT) = k^n C(t) \text{ for } 0 \le t \le T,\ n = 0, 1, 2, \cdots \tag{16}$$

The condition we impose on L is that

$$L(t) > 0,\ L'(t) \le 0,\ \text{and } L''(t) \ge 0 \text{ for } t \ge 0. \tag{17}$$

This will be satisfied in case

$$k := C(T)/C(0),\ kC'(0^+) = C'(T^-),\ kC''(0^+) = C''(T^-). \tag{18}$$

Theorem 2. *If (14) – (17) hold, then (13) has a T-periodic solution.*

Proof. We first show that for any $M > 0$ the equation

$$x(t) = p(t) - M\int_{-\infty}^{t} L(t-s)g(x(s))\,ds \tag{19}$$

has a T-periodic solution. Then we will show that there is an $M > 0$ such that the T-periodic solution of (19) also satisfies (13).

By (16) the function L is exponentially decaying and so are its derivatives. If $(\mathcal{P}_T, \|\cdot\|)$ is the Banach space of continuous T-periodic functions with the supremum norm and if $P : \mathcal{P}_T \to \mathcal{P}_T$ is defined by

$$(P\varphi)(t) = \lambda\Big[p(t) - M\int_{-\infty}^{t} L(t-s)g(\varphi(s))\,ds\Big],\ 0 < \lambda < 1, \tag{20}$$

then P is continuous and maps bounded sets into equicontinuous sets.

Next, if $x \in \mathcal{P}_T$ is a fixed point of P, then define

$$V(t) = \lambda M \int_{-\infty}^{t} L_s(t-s)\Big(\int_s^t g(x(v))\,dv\Big)^2\,ds$$

so that by (17) we have

$$\begin{aligned}
V'(t) &\le 2\lambda M g(x)\int_{-\infty}^{t} L_s(t-s)\int_s^t g(x(v))\,dv\,ds\\
&= 2\lambda M g(x)\Big[L(t-s)\int_s^t g(x(v))\,dv\Big|_{-\infty}^{t} + \int_{-\infty}^{t} L(t-s)g(x(s))\,ds\Big]\\
&= 2g(x)\Big[\lambda M\int_{-\infty}^{t} L(t-s)g(x(s))\,ds\Big]\\
&= 2g(x)\Big[-x + \lambda p(t)\Big]\\
&= -2xg(x) + 2\lambda p(t)g(x).
\end{aligned}$$

There are then positive constants J and C_1 with

$$V' \le -C_1|g(x)| + J.$$

As $x \in \mathcal{P}_T$ implies $V \in \mathcal{P}_T$ we have

$$0 = V(T) - V(0) \le -C_1\int_0^T |g(x(s))|ds + JT. \tag{21}$$

Next, for this $x \in P_T$ satisfying $Px = x$ we have

$$
\begin{aligned}
x &= \lambda\Big[p(t) - M\int_{-\infty}^{t} L(t-s)g(x(s))\,ds\Big] \\
&= \lambda\Big[p(t) - M\sum_{i=0}^{\infty}\int_{t-(i+1)T}^{t-iT} L(t-s)g(x(s))\,ds\Big] \\
&= \lambda\Big[p(t) - M\sum_{i=0}^{\infty}\int_{t-T}^{t} L(t-s+iT)g(x(s+iT))\,ds\Big] \\
&= \lambda\Big[p(t) - M\sum_{i=0}^{\infty}\int_{t-T}^{t} C(t-s)k^i g(x(s))\,ds\Big]
\end{aligned}
$$

or

$$
x(t) = \lambda\Big[p(t) - \frac{M}{1-k}\int_{t-T}^{t} C(t-s)g(x(s))\,ds\Big] \tag{22}
$$

Using (21) and (22) we then have

$$
\begin{aligned}
|x(t)| &\le \|p\| + [MC(0)/(1-k)]\int_{t-T}^{t} |g(x(s))|\,ds \\
&\le \|p\| + [MC(0)/(1-k)]JT/C_1,
\end{aligned}
$$

an a priori bound for $\|x\|$.

By Schaefer's Theorem (19) has a T-periodic solution which also solves (22) for $\lambda = 1$ and for every $M > 0$. Take $M = 1-k$ to see that (13) has the same T-period solution. This completes the proof.

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