AN EXTENSION OF KRASOVSKII'S STABILITY THEORY

Dedicated to Professor N. N. Krasovskii

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1. Introduction. Let $F : [0,\infty) \times C_H \to R^n$ be continuous and take bounded sets into bounded sets and consider the system of functional differential equations

$$x'(t) = F(t, x_t) \qquad ' = d/dt \tag{1}$$

where $x_t(s) = x(t+s)$ for $-h \le s \le 0$ and h is a positive constant. Here, $(C, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-h, 0] \to R^n$ with the supremum norm and for $H > 0$ then C_H is the H-ball in C. Thus, from standard theory (e.g., [3; pp. 186–191]), if $\phi \in C_H$ and $t_0 \ge 0$, then there is a solution $x(t, t_0, \phi)$ of (1) on an interval $[t_0, t_0 + \alpha)$ and, if there is an $H_0 < H$ with $|x(t, t_0, \phi)| \le H_0$, then $\alpha = \infty$.

It is to be understood that F takes bounded (t, ϕ) sets into bounded sets in R^n. Thus, no assumption is being made that $|F(t, \phi)|$ is bounded for ϕ bounded. Also, for an ordinary differential equation, when a function is written without its argument, that argument is t; but in the PDE case, it is (t, x).

In the study of stability for ordinary, functional, and partial differential equations by means of Liapunov functions or functionals there is the challenging problem of showing that a solution does not move too rapidly in a region in which the Liapunov function has a negative derivative. Marachkov [14] faced this problem for an ordinary differential equation $x' = f(t,x)$ by asking that $|f(t,x)|$ be bounded for x bounded. That condition was adopted by Krasovskii for functional differential equations (see Theorem K1 below). It has two faults. First, there are too many interesting systems which do not satisfy the condition. But, what is perhaps more important, while there is a growing theory of stability which applies in a unified way to ordinary, functional, and partial differential equations, the Marachkov condition does not seem to extend well to partial differential equations. On the other hand, Krasovskii proved a second result (see Theorem K2 below) which we now generalize and show that it is natural for partial differential equations as well.

For a continuous functional $V : [0,\infty) \times C_H \to [0,\infty)$ which is locally Lipschitz in ϕ, we follow Yoshizawa [16; p. 186] and define the derivative of V along a solution of (1) by

$$V'_{(1)}(t, x_t(t_0,\phi)) = \limsup_{\delta \to 0^+} \frac{1}{\delta}\{V(t+\delta, x_{t+\delta}(t_0,\phi)) - V(t, x_t(t_0,\phi))\}.$$

We shall also use continuous strictly increasing functions $W_i : [0,\infty) \to [0,\infty)$ with $W_i(0) = 0$, called wedges. The next two results are by Krasovskii [10; pp. 151–155].

THEOREM K1. Suppose there is a continuous functional $V : [0,\infty) \times C_H \to [0,\infty)$ which is locally Lipschitz in ϕ, and a constant M such that

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(\|\phi\|)$,

(ii) $V'_{(1)}(t,x_t) \le -W_3(|x(t)|)$,

and

(iii) $|F(t,\phi)| \le M$ if $t \ge 0$ and $\|\phi\| < H$.

Then $x = 0$ is uniformly asymptotically stable (UAS).

THEOREM K2. Let $|\cdot|_2$ denote the L^2-norm on C. Suppose there is a continuous functional $V : [0,\infty) \times C_H \to [0,\infty)$ which is locally Lipschitz in ϕ and wedges W_i such that

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_3(|\phi|_2)$

and

(ii) $V'_{(1)}(t, x_t) \leq -W_4(|x(t)|)$.

Then $x = 0$ is asymptotically stable (AS).

In [2] we showed that the conclusion in Theorem K2 is actually UAS.

In [5] Makay and the author showed that in Theorem K1 we can reduce (i), have $|F(t,\phi)| \leq M(t+1)\ln(t+2)$, and retain a conclusion of AS. In [3; p. 270] we showed that the condition $|F(t,\phi)| \leq M$ could be transferred to a condition of bounded difference quotient on part of V and still retain the conclusion of UAS. That work suggests that the problem of showing that a solution does not move too fast through regions in which V' is negative can be approached either by restricting the derivative of $x(t)$ or $\partial V/\partial t$ when we try to prove UAS. We now understand that it is more general to focus on restrictions on V since that will apply to a wider class of problems.

In Theorem K2, Krasovskii only obtained AS, but he used properties in (i) which yield uniform stability as well. Thus, it is reasonable to ask if we could reduce the upper bound on V in Theorem K2 and still retain AS. We show here that this can be done.

2. Asymptotic stability. The following definitions are given for reference.

DEF. The zero solution of (1) is said to be *stable* if for each $\epsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\phi \in C_\delta, t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \epsilon$. The zero solution of (1) is *asymptotically stable* (AS) if it is stable and if for each $t_0 \geq 0$ there is an $\eta > 0$ such that $\phi \in C_\eta$ implies that $|x(t, t_0, \phi)| \to 0$ as $t \to \infty$.

THEOREM 1. Let $r, V : [0,\infty) \to [0,\infty)$, let $h > 0$, let $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$ with $\alpha + \beta > 0$, and let $\{t_n\}$ and $\{\lambda_n\}$ be positive sequences with $t_{n+1} \geq t_n + 2h$ and $\sum 1/\lambda_n = \infty$. Suppose there are wedges W_i such that

(i) $0 \leq V(t)$, $V'(t) \leq 0$,

and for $t \in [t_n - 2h, t_n]$ we have both

(ii) $V(t) \leq \lambda_n \left[\alpha W_2(r(t)) + \beta W_3\left(\frac{1}{h}\int_{t-h}^{t} W_4(r(s))ds\right)\right] + \gamma W_5(r(t))$

and

(iii) $V'(t) \leq -\alpha W_2(r(t)) - \beta W_3\big(W_4(r(t))\big)$, W_3 convex downward.

Then $V(t) \to 0$ as $t \to \infty$.

COR. 1. Suppose there is a continuous and locally Lipschitz functional $V : [0, \infty) \times C_H \to [0, \infty)$ and positive sequences $\{t_n\}$ and $\{\lambda_n\}$ with $t_{n+1} \geq t_n + 2h$ and $\sum 1/\lambda_n = \infty$. Suppose also that there are constants $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$ with $\alpha + \beta > 0$ and wedges W_i with

(i) $W_1(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$, $V'_{(1)}(t, x_t) \leq 0$

and that for $t \in [t_n - 2h, t_n]$ we have both

(ii) $V(t, \phi) \leq \lambda_n \left[\alpha W_2(|\phi(0)|) + \beta W_3 \left(\frac{1}{h} \int_{-h}^{0} W_4(|\phi(s)|)ds\right)\right] + \gamma W_5(|\phi(s)|)$

and

(iii) $V'_{(1)}(t, x_t) \leq -\alpha W_2(|x(t)|) - \beta W_3(W_4(|x(t)|))$, W_3 convex downward.

Then $x = 0$ is AS.

COR. 2. Suppose there is a continuous and locally Lipschitz functional $V : [0, \infty) \times C_H \to [0, \infty)$, constants $M > 0$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ with $\alpha + \beta > 0$, and wedges W_i such that

(i) $$W_1(|\phi(0)|) \leq V(t, \phi) \leq M(t + 1) \ln(t + 2)\big[\alpha W_2(|\phi(0)|) + \beta W_3 \left(\frac{1}{h} \int_{-h}^{0} W_4(|\phi(s)|)ds\right)\big] + \gamma W_5(|\phi(0)|)$$

and

(ii) $V'_{(1)}(t, x_t) \leq -\alpha W_2(|x(t)|) - \beta W_3(W_4(|x(t)|))$, W_3 convex downward.

Then $x = 0$ is AS.

3. Proofs. To prove Theorem 1, consider the sequence $S_n = [t_n - h, t_n]$. Let $\{q_n\}$ be chosen so that $r(q_n) \leq r(t)$ on S_n. We claim that $r(q_n) \to 0$ as $n \to \infty$. For if this is false, then there is a subsequence and an $\epsilon > 0$ with $|r(t)| \geq \epsilon$ on $\{S_{n_k}\}$ so that $V'(t) \leq -\alpha W_2(\epsilon) - \beta W_3(W_4(\epsilon))$ on $\{S_{n_k}\}$ and so $V(t) \to -\infty$ as $t \to \infty$, a contradiction. Let $J_n = [q_n - h, q_n]$ and note that $t_n - 2h \leq q_n - h, q_n \leq t_n$, and the J_n are disjoint.

If $V(t) \nrightarrow 0$ as $t \to \infty$, then there is a $c \geq 0$ with $V(t) \geq c$. It then follows that for large n, say $n \geq N$, we have

$$c \leq V(q_n) \leq \lambda_n \left[\alpha W_2(r(q_n)) + \beta W_3 \left(\frac{1}{h} \int_{q_n - h}^{q_n} W_4(r(s)) ds \right) \right] + W_5(r(q_n))$$
$$\leq (c/2) + \lambda_n \left[\alpha W_2(r(q_n)) + \beta W_3 \left(\frac{1}{h} \int_{q_n - h}^{q_n} W_4(r(s)) ds \right) \right].$$

Case 1. Suppose that for a given $n \geq N$ we have

$$(*) \qquad \alpha W_2(r(q_n)) \geq \beta W_3 \left(\frac{1}{h} \int_{q_n - h}^{q_n} W_4(r(s)) ds \right).$$

Then $c/(4\alpha\lambda_n) \leq W_2(r(q_n)) \leq W_2(r(t))$ on S_n. Hence, on S_n we have $V'(t) \leq -\alpha W_2(r(t)) \leq -c/(4\lambda_n)$ so $V(t_n) - V(t_n - h) \leq -ch/(4\lambda_n)$.

Case 2. Suppose that (*) fails for some $n \geq N$. Then we have $W_3\left(\frac{1}{h} \int_{q_n - h}^{q_n} W_4(r(s)) ds \right) \geq c/(4\beta\lambda_n)$. But an integration of (iii) and use of Jensen's inequality yields $V(q_n) - V(q_n - h) \leq -\beta h \, W_3\left(\frac{1}{h} \int_{q_n - h}^{q_n} W_4(r(s)) ds \right) \leq -ch/4\lambda_n$. Since $\sum 1/\lambda_n = \infty$, $V(t) \to -\infty$, a contradiction. This proves Theorem 1.

To prove Cor. 1, we note that (i) implies that $x = 0$ is stable. Thus, let $\epsilon < H/2$ and $t_0 \geq 0$ be given. There is then a $\delta > 0$ such that $\phi \in C_\delta$ implies that $|x(t, t_0, \phi)| < \epsilon$ for $t \geq t_0$ and, hence, $x(t, t_0, \phi)$ exists for $t \geq t_0$. Let $r(t) = |x(t, t_0, \phi)|$ and $V(t) = V(t, x(t, t_0, \phi))$. By Theorem 1, $V(t) \to 0$ as $t \to \infty$. Thus, $|x(t, t_0, \phi)| \to 0$ as $t \to \infty$.

Cor. 2 is proved by taking $t_n = t_0 + 2nh$, $\lambda_n = M(t_n + 1) \ln(t_n + 2)$, and proceeding as in the proof of Cor. 1.

4. Examples.

EXAMPLE 1. Let $0 < h < \frac{1}{4}$ and write the equation

$$x'' + tx' + tx(t - h) = 0$$

as

$$
\begin{aligned}
x' &= y - x \\
y' &= -(t-1)(y-x) - tx((t-h) \\
&= -(t-1)y + (t-1)x + t\int_{t-h}^{t} x'(s)ds - tx \\
&= -(t-1)y - x + t\int_{t-h}^{t} x'(s)ds
\end{aligned}
$$

or

$$
\begin{aligned}
x' &= y - x \\
y' &= -(t-1)y - x + t\int_{t-h}^{t} (y(s) - x(s))ds.
\end{aligned}
$$

Define

$$
V = tx^2 + 2y^2 + 2\int_{-h}^{0}\int_{t+s}^{t} (v-s)(y(v)-x(v))^2 dv\, ds
$$

so that

$$
\begin{aligned}
V' &= x^2 + 2tx(y-x) + 4y\left[-(t-1)y - x + t\int_{t-h}^{t} (y(s)-x(s))ds\right] \\
&\quad + 2\int_{-h}^{0} (t-s)(y(t)-x(t))^2 ds - 2\int_{-h}^{0} t(y(t+s)-x(t+s))^2 ds \\
&\le x^2 + 2txy - 2tx^2 - 4(t-1)y^2 - 4xy + 4ty\int_{t-h}^{t} (y(s)-x(s))ds \\
&\quad + 2h(t+h)(y-x)^2 - 2t\int_{t-h}^{t} (y(s)-x(s))^2 ds \\
&\le -(2t-1)x^2 + (2t-4)xy - 4(t-1)y^2 + 2thy^2 + 2t\int_{t-h}^{t} (y(s)-x(s))^2 ds \\
&\quad + 4h(t+h)(x^2+y^2) - 2t\int_{t-h}^{t} (y(s)-x(s))^2 ds \\
&\le [-2t + 1 + t - 2 + 4h(t+h)]x^2 + [-4t + 4 + t - 2 + 2th + 4h(t+h)]y^2 \\
&\le -\mu(x^2+y^2)
\end{aligned}
$$

for some $\mu > 0$ if t is large enough.

Thus, we take $\gamma = 0$, $W_1 = W_2 = W_4$, $W_1(r) = r^2$, $W_3(r) = r$, $\alpha = \beta = \frac{1}{2}$, and M an appropriate constant. The conditions of Cor. 2 can then be readily satisfied.

EXAMPLE 2. Let $\alpha > h$ and $\alpha + h < 1$. Then there is an $a > 0$ such that any solution of

$$u_{tt} = tu_{xx}(t-h,x) + \alpha t u_{xxt}, \quad u(t,0) = u(t,1) = 0,$$

which exists on $[a,\infty)$ satisfies

$$\int_0^1 (u_x^2 + u_t^2)dx \to 0 \quad \text{as } t \to \infty.$$

PROOF. We will define two Liapunov functionals and add them together. Write the equation as

$$u_{tt} = tu_{xx} - t\int_{t-h}^{t} u_{xxt}(s,x)ds + \alpha t u_{xxt}$$

and define

$$V_1(t) = \int_0^1 \left[tu_x^2 + u_t^2 + \int_{-h}^{0}\int_{t+s}^{t}(v-s)u_{tx}^2(v,x)dv\,ds\right]dx$$

along any solution which exists on $[a,\infty)$, where a is to be determined. Denote the last term in V_1 by P and formally differentiate V_1 to obtain

$$V_1'(t) = \int_0^1 \left\{u_x^2 + 2tu_xu_{xt} + 2u_t\left[tu_{xx} - t\int_{t-h}^{t} u_{xxt}(s,x)ds + \alpha t u_{xxt}\right] + P'(t)\right\}dx.$$

Upon integration by parts and use of the boundary condition we obtain

$$\begin{aligned} V_1'(t) = &\int_0^1 \{u_x^2 - 2tu_{xx}u_t + 2tu_{xx}u_t \\ &+ 2t\int_{t-h}^{t} u_{tx}(t,x)u_{xt}(s,x)ds - 2\alpha t u_{tx}^2 + P'(t)\}dx \\ \leq &\int_0^1 \{u_x^2 + htu_{tx}^2 + t\int_{t-h}^{t} u_{xt}^2(s,x)ds - 2\alpha t u_{tx}^2 \\ &+ h(t+h)u_{tx}^2 - \int_{t-h}^{t} tu_{tx}^2(s,x)ds\}dx \end{aligned}$$

or

$$V_1'(t) \leq \int_0^1 [u_x^2 - (2\alpha t - 2ht - h^2)u_{tx}^2]dx.$$

As $\alpha > h$, there is an $a_1 > 0$ and an $\xi > 0$ such that if $t \geq a_1$ then

$$V_1'(t) \leq \int_0^1 [u_x^2 - t\xi u_{tx}^2]dx.$$

Next, define

$$V_2(t) = \int_0^1 \left[(u+u_t)^2 + 2\int_{-h}^0 \int_{t+s}^t (v-s)u_{tx}^2(v,x)dv\,ds\right]dx$$

so that

$$\begin{aligned} V_2'(t) &= \int_0^1 \Big\{ 2(u+u_t)\Big[u_t + tu_{xx} - t\int_{t-h}^t u_{xxt}(s,x)ds + \alpha t u_{xtx}\Big] \\ &\quad + 2h(t+h)u_{tx}^2 - 2t\int_{t-h}^t u_{tx}^2(s,x)ds\Big\}dx \\ &\leq \int_0^1 \Big\{ 2uu_t - 2tu_x^2 + 2t\int_{t-h}^t u_x(t,x)u_{xt}(s,x)ds - 2\alpha t u_x u_{tx} \\ &\quad + 2u_t^2 - 2tu_{tx}u_x + 2t\int_{t-h}^t u_{tx}(t,x)u_{xt}(s,x)ds - 2\alpha t u_{tx}^2 \\ &\quad + 2h(t+h)u_{tx}^2 - 2t\int_{t-h}^t u_{tx}^2(s,x)ds\Big\}dx \\ &\leq \int_0^1 \{u^2 + 3u_t^2 - 2tu_x^2 + \alpha t u_x^2 + \alpha t u_{tx}^2 + tu_x^2 + tu_{tx}^2 \\ &\quad - 2\alpha t u_{tx}^2 + htu_x^2 + 2t\int_{t-h}^t u_{tx}^2(s,x)ds + htu_{tx}^2 \\ &\quad + 2h(t+h)u_{tx}^2 - 2t\int_{t-h}^t u_{tx}^2(s,x)ds\}dx \\ &\leq \int_0^1 \{u^2 - (t - \alpha t - ht)u_x^2 - (\alpha t - 3 - t - 3ht - 2h^2)u_{tx}^2\}dx \end{aligned}$$

since $\int_0^1 u_t^2 dx \leq \int_0^1 u_{tx}^2 dx$. As $\alpha + h < 1$, there are positive constants a_2, A, B so that $t \geq a_2$ implies $V_2'(t) \leq \int_0^1 [-Atu_x^2 + Btu_{tx}^2]dx$. Thus, there is a $C > 0$, $a > \max[a_1, a_2]$, and $\mu > 0$ so that if $t \geq a$ then

$$V(t) = V_1(t) + CV_2(t)$$

satisfies

$$V'(t) \leq -\mu \int_0^1 [u_x^2 + u_{tx}^2]dx =: -2\mu r(t).$$

We may then take $W_2 = W_3 = W_4$ with $W_2(s) = ks$ for some $k > 0$. Then let $t_n = 2nh$ and $\lambda_n = 2Knh$ for some $K > 0$. Also, take $\gamma = 0$. By Theorem 1 we see that $V(t) \to 0$ as $t \to \infty$; in particular,

$$\int_0^1 (u_x^2 + u_t^2)dx \to 0 \text{ as } t \to \infty.$$

REMARK. A second Liouville transformation (cf. [3; p. 65]) will map the equation of Example 2 into a form

$$w_{tt} = F(t) + \alpha w_{xxt}$$

where F contains a delay, and, hence, an initial function. An integration will then yield

$$w_t = H(t) + \alpha w_{xx}$$

which is a forced heat equation and can be readily solved by the variation of parameters formula and the method of steps. The details require a degree of smoothness in the initial functions. Thus, existence theory can be established; but, again, our bounds are *a priori*. This problem is considered in [7] with $r = 0$. In some sense u_{xxt} introduces a memory; when $r > 0$ the memory is explicit.

5. Relation to the literature. Theorem K1 was crippled by the condition

(iii) $|F(t, \phi) \leq M$ if $\|\phi\| < H$;

yet, it was the standard on UAS until 1977, as may be seen in Hale [9; p. 105]. When condition (i) in Theorem K1 is changed to

(i)′ $W_1(|\phi(0)|) \leq V(t, \phi)$, $\quad V(t, 0) = 0$,

then the conclusion is AS and that, along with Theorem K2, remained the standard for many years. Investigators have been quite interested in reducing the boundedness condition on F and the upper bound on V.

Busenberg and Cooke [6] consider systems of the form

$$x' = F(t, x_t) - G(t, x(t))$$

where it is assumed that for given positive numbers η and γ, there exists $T > 0$ such that

$$\int_t^{t+T} |F(s,\phi)|ds < \eta \text{ for all } t \geq 0 \text{ and all } \|\phi\| < \gamma,$$

so long as $x^T DG(t,x) \geq 0$ for some positive definite matrix D. These conditions can replace (iii) of Theorem K1; in effect, F need not be bounded, but must be bounded on average.

Hatvani and the author [4] avoid (iii) of Theorem K1 by asking for an L^1-norm of x_t in the derivative of V. But still a bound of the type in Theorem K2, condition (i), is required.

Makay [13] shows by example that it is unlikely that condition (iii) of Theorem K1 can be entirely eliminated.

Becker, Zhang, and the author [1] continue the work in [4] (see p. 156) and give an example in which the upper bound on V can grow of order $\sqrt{t}$.

Lakshmikantham, Matrosov, and Sivasundaram (for example) focus on relations of the form $V'(t) \leq f(t, V(t))$ with a view to showing that $V(t) \to 0$ whenever solutions of $r' = f(t,r)$ tend to zero. Such work does not necessarily concern itself with the relations displayed here. One frequently finds that by carefully modifying the Liapunov functionals in the examples, then suitable differential inequalities can be obtained. But it is very unclear to us how to deal with Theorem 1 itself as a differential inequality.

At base, our work stemmed from a conjecture derived from a paper of Smith [15] concerning

$$x'' + a(t)x' + x = 0, \quad a(t) \geq a_0 > 0,$$

in which he showed that solutions tend to zero provided that $a(t)$ behaves well and does not grow much faster than $t \ln t$. Moreover, his work was sharp in a certain sense. Makay and the author [5] showed that a similar bound could replace (iii) in Theorem K1. Since [3; p. 270] suggested that the bound on F could be replaced by a similar bound on $\partial V/\partial t$, Theorem 1 became a natural conjecture. A significant bonus was obtained when it was noted that Theorem 1 had natural applications to partial differential equations, while Theorem K1 does not extend in the way we would like. For example, in Example 2 we have $V' \leq 0$ and so V is bounded; but the bound on V translates into very little in terms of bounds on the variables in the equation. Hale [8] formally advanced Theorem K1 (and extensions) to PDEs. Boundedness conditions in R^n translate into complex compactness conditions in Sobolev spaces, leading one to essentially autonomous systems. Nothing of the kind is needed when we focus on bounds on V instead of on F.

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