INTEGRAL EQUATIONS WITH A DELAY

T.A. Burton
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901

1. Introduction. The past ([1–3], [5], [7; pp. 133–6, 154–8], [11], [15]), the present ([4], [6], [12–13]), and the future ([8–9], [16–17]) literature on delay equations is replete with attempts to show that all solutions of variants of

$$x'(t) = -a(t)x(t) + g(t, x(t - r(t)))$$

tend to zero as $t \to \infty$. Most of this work uses Liapunov functions or functionals V and two problems are always encountered which we attempt to overcome.

1) We need $a(t) \geq 0$ and $a(t) \geq a_0 > 0$ on intervals I_j with $\cup I_j$ having infinite measure.
2) We need $a(t)$ and $g(t, x)$ bounded for x bounded so that if $|x(t_n)| \geq b > 0$, then $|x(t)| \geq b/2$ for $t_n \leq t \leq t_n + \alpha$ for some $\alpha > 0$. This allows us to integrate the derivative of the Liapunov function, V, over $\cup I_j$ and drive V to 0.

Indeed, this problem has been a main driving force behind the development of Liapunov's direct method. The cited literature focuses on reducing 2), with varying success. But $a(t) \geq 0$ is almost always required.

In this note we consider

$$x'(t) = -a(t)h(x(t))x(t) + g(t, x(t - r(t))) + f(t) \tag{1}$$

with a view to proving boundedness, stability, and the existence of periodic solutions. Instead of using Liapunov's direct method, we transform (1) to an integral equation having

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the rather curious property that it is of delay type: $x(t)$ depends on $x(s)$ for $s \leq t - r(t)$. Thus, when $r(t) \geq r^* > 0$, then the integral equation is actually a type of difference equation involving an integral of the past.

In this way, we employ a contraction type argument and avoid both problems 1) and 2) when $h(x) = 1$, and avoid some of 1) and all of 2) when $h(x) \geq h_1 > 0$. In the former case we are able to let $\limsup_{t\to\infty} a(t) = +\infty$ and $\liminf_{t\to\infty} a(t) = -\infty$. We apply the theory to show that all solutions of

$$x'(t) = (4t \sin t^2 - 2t)x(t) + g(t, x(t - r(t)))$$

tend to zero when g has a linear bound.

The results are motivated by (1) and stated for that equation but they clearly will apply to general integral equations of the forms of (1a), (12), and (17).

2. Periodic solutions: the half-linear case. Suppose there is a $T > 0$ with

$$a(t+T) = a(t), \quad r(t+T) = r(t), \quad g(t+T, x) = g(t, x), \text{ and } f(t+T) = f(t) \tag{2}$$

for all $t \in R$. Let $h(x) = 1$ and write (1) as

$$(x(t)\mathrm{e}^{\int_0^t a(s)\,ds})' = \mathrm{e}^{\int_0^t a(s)\,ds}[g(t, x(t - r(t))) + f(t)]$$

so that upon integration from $t - T$ to t we have

$$\begin{aligned} x(t) &= x(t-T)\mathrm{e}^{-\int_{t-T}^t a(s)\,ds} \\ &\quad + \int_{t-T}^t \mathrm{e}^{-\int_u^t a(s)\,ds}[g(u, x(u - r(u))) + f(u)]\,du \end{aligned} \tag{1a}$$

where we now ask that $\int_0^T a(s)ds > 0$. Then

$$exp[-\int_{t-T}^t a(s)\,ds] =: C < 1. \tag{3}$$

Theorem 1. *Let (2) and (3) hold, let a, g, r, and f be continuous, and let $h(x) = 1$. Suppose there are positive constants L, M, and R with the property that if x is a T-periodic function with $|x(t)| \leq M$ then*

$$\left| \int_{t-T}^{t} e^{-\int_u^t a(s)\,ds} g(u, x(u - r(u)))\,du \right| \leq RM, \tag{4}$$

$$\left| \int_{t-T}^{t} e^{-\int_u^t a(s)\,ds} f(u)\,du \right| \leq L, \tag{5}$$

and

$$M(C + R) + L \leq M. \tag{6}$$

Then (1) has a T-periodic solution.

Proof. Let $(P_T, \|\cdot\|)$ be the Banach space of continuous T-periodic scalar functions with the supremum norm.

Define $H : P_T \to P_T$ by

$$\begin{aligned} (H\varphi)(t) &= \varphi(t-T)C + \int_{t-T}^{t} \mathrm{e}^{-\int_u^t a(s)\,ds}[g(u, \varphi(u - r(u))) + f(u)]\,du \\ &=: (B\varphi)(t) + (A\varphi)(t). \end{aligned} \tag{7}$$

As $C < 1$, B is a contraction and A maps bounded sets into compact sets by Ascoli's theorem. If $S = \{\varphi \in P_T | \|\varphi\| \leq M\}$ and $\varphi \in S$, then (4) - (6) yield $|(H\varphi)(t)| \leq MC + RM + L \leq M$. Clearly, H is continuous. As the bounds on $B\varphi$ and $A\varphi$ depend only on the norm of φ, these are precisely the conditions required in Krasnoselskii's theorem (cf. Smart [10; p. 31] to ensure that there is a fixed point in P_T. That fixed point solves (1) and (1a). This completes the proof.

We could write (7) as

$$(H\varphi)(t) = \{1/(1-C)\} \int_{t-T}^{t} [g(u, \varphi(u - r(u))) + f(u)] \exp[-\int_u^t a(s)\,ds]\,du$$

and use Schauder's fixed point theorem.

REMARK 1. If $a(t) \equiv a > 0$, if $h(x) = 1$, if $f(t) \equiv 0$, and if $g(t,x) = b(t)x$, then for any $T > 0$ we have $\exp[-\int_{t-T}^{t} a(s)\,ds] = \mathrm{e}^{-aT} =: C < 1$ and for $|x(t)| \le M$, then

$$\Big\{1/(1-C)\Big\} \int_{t-T}^{t} g(u, x(u - r(u))) \exp[-\int_{u}^{t} a(s)\,ds]\,du\Big| \le$$
$$\Big\{M/(1-C)\Big\} \int_{t-T}^{t} |b(u)| \exp[-a(t-u)]\,du$$

(and if $\|\cdot\|$ denotes a supremum)

$$\begin{aligned} &\le \{M\|b\|/a(1-C)\} \exp[-a(t-u)]\Big|_{t-T}^{t} \\ &= \{M\|b\|/a(1-C)\}[1 - \mathrm{e}^{-aT}] = M\|b\|/a < M \end{aligned}$$

if $\|b\| < a$. This is exactly the classical condition for asymptotic stability using a Liapunov functional (cf. Hale and Lunel [7; p. 135]).

This leads us to the next section. We want to show that when $f(t) = 0$, then (4), (5), and (6) imply that $x(t)$ tends to zero.

3. Attractors: the half-linear case. We now drop all periodic assumptions. Let $0 \le r(t) \le r_0$ and suppose there is an $M_0 \ge 0$ and a continuous function $\lambda : [M_0, \infty) \to [0, \infty)$ with

$$\lambda(u) < u \text{ for } u > M_0, \tag{8}$$

together with positive constants α, β, J, $M > M_0$, and R such that if $|x(s)| \le M$ for $t - r_0 - J \le s \le t - r(t)$ then

$$\Big| \int_{t-J}^{t} g(u, x(u - r(u))) \exp[-\int_{u}^{t} a(s)\,ds]\,du\Big| \le RM, \tag{9}$$

$$\exp[-\int_{t-J}^{t} a(s)\,ds] \le \alpha, \qquad \Big|\int_{t-J}^{t} f(u) \exp[-\int_{u}^{t} a(s)\,ds]\,du\Big| \le \beta, \tag{10}$$

and

$$(\alpha + R)M + \beta < \lambda(M). \tag{11}$$

REMARK 2. At first, (11) seems very restrictive; but the reader is urged to reserve judgment until after studying Examples 1, 2, and their accompanying remarks. Our work is directed at large $a(t)$. Thus, we first make J very small to find a suitable R; then we utilize a later Remark 5 to make t large and find a suitably small α. In Example 2 we reverse the idea.

Denote the composition of λ k times by λ^k. It should be noted here that $a(t)$ is not assumed to be bounded in any way. On the other hand, (9) essentially asks that $|g(t,x)| \leq \mu(t)|x|$ for some function μ.

Theorem 2. *Let $h(x) = 1$, (8) – (11) hold for each $M > M_0$, and suppose that $\lambda^k M \to M_0$ as $k \to \infty$ for all $M > M_0$. Then each solution of (1) satisfies $|x(t)| < M_0 + 1$ for all large t.*

Proof. For a given $t_0 \in R$ and continuous initial function $\varphi : [t_0 - r_0, t_0] \to R$, there is a solution $x(t) = x(t, t_0, \varphi)$ with $x(t) = \varphi(t)$ on $[t_0 - r_0, t_0]$. We can use (9) to show that $x(t)$ can be continued for all future time. Indeed, the only way in which solutions of (1) can fail to be defined past some L is for $\limsup_{t \to L^-} |x(t)| = +\infty$. But in (12) if $\|x_t\| = \sup_{s \leq t} |x(s)|$, then

$$|x(t)| \leq C\|x_t\| + R\|x_t\| + \beta \leq \lambda(\|x_t\|) < \|x_t\|$$

for $\|x_t\| > M_0$. Thus, whenever $|x(t)| = \|x_t\| > M_0$ we have a contradiction.

Fix $x(t)$ on $[t_0, \infty)$ and from considerations in Section 2 write

$$\begin{aligned} x(t) &= x(t-J)\exp[-\int_{t-J}^{t} a(s)\,ds] \\ &\quad + \int_{t-J}^{t} [g(u, x(u-r(u))) + f(u)]\exp[-\int_{u}^{t} a(s)\,ds]\,du. \end{aligned} \tag{12}$$

Using (9) – (11) estimate $x(t)$ by

$$|x(t)| \le \alpha|x(t-J)| + \beta + \left| \int_{t-J}^{t} \mathrm{e}^{-\int_u^t a(s)\,ds} g(u, x(u-r(u)))\,du \right|. \tag{13}$$

For $t_1 > J + r_0 + t_0$, consider the intervals

$$I_i = [t_i, t_{i+1}] = [t_1 + (i-1)(J + r_0), t_1 + i(J + r_0)]$$

for $i = 1, 2, \ldots$. Now there is an $M > M_0$ with $|x(t)| < M$ for $t < t_1$. Using (9), (11), and (13) notice that:

i) As long as $t \ge t_1$ and $|x(s)| < M$ for $s \le t$ then

$$|x(t)| \le \alpha M + RM + \beta < \lambda(M) < M.$$

ii) Hence, $|x(t)| < \lambda(M)$ for all $t \ge t_1$.

Inductively, $|x(t)| < \lambda^k M$ on I_{k+1} since $x(t)$ depends only on $x(s)$ for $t - J - r_0 \le s < t$. This completes the proof. □

REMARK 3. If $f(t) = 0$, then $\beta = 0$; thus in (11) we have $\lambda(M) = (\alpha + R)M$ so $\lambda^k M = (\alpha + R)^k M$ and $|x(t)| \le (\alpha + R)^k M$ on I_k. If we ask that $f(t) = 0$, that (9) holds for each $M > 0$, and that $-\int_u^t a(s)\,ds$ is bounded above for $u < t$, then we can prove that the zero solution is uniformly asymptotically stable.

REMARK 4. If (9) holds for select $M > 0$, then we can say that any solution $x(t)$ with $|x(t)| < M$ for $t < t_1$ (see the proof of Theorem 2), satisfies $|x(t)| < \lambda(M)$ for $t \ge t_1$. If $g(t, 0) = 0$ and g has a linear bound, then for a sufficiently small initial function we can show that

$$|x(t)| < M \text{ for } t < t_1; \text{ this yields } |x(t)| < M \text{ for } t \ge t_1.$$

REMARK 5. Conditions (8) – (10) need only hold for $t \ge t^*$, some $t^* \in R$; since solutions can be defined for all future time, we begin our arguments to the right of t^*.

EXAMPLE 1. Suppose that $0 \le r(t) \le r_0$ and $|g(t, x)| \le \mu|x|$ for $\mu > 0$ and $r_0 > 0$. Then every solution of

$$x'(t) = (4t \sin t^2 - 2t)x(t) + g(t, x(t - r(t))) \tag{14}$$

tends to zero as $t \to \infty$.

Proof. First, find $J > 0$ with $\mu e^4 J < 1/2$. Then, take t so large, say $t \geq t^* + r_0$, that $e^{4-2tJ+J^2} \leq C < 1/2$. For such t we now have

$$\exp \int_u^t (4s \sin s^2 - 2s)\, ds =$$
$$\exp[-2\cos t^2 - t^2 + 2\cos u^2 + u^2] \leq$$
$$e^4 \text{ for } u \leq t.$$

Next, for those same t we can satisfy (10) since

$$\begin{aligned}
&\exp \int_{t-J}^t (4s \sin s^2 - 2s)\, ds = \\
&\exp[-2\cos t^2 - t^2 + 2\cos(t-J)^2 + (t-J)^2] \leq \\
&\exp[4 - 2tJ + J^2] \leq C < 1/2,
\end{aligned}$$

so (10) holds. Taking $|x(t)| \leq M$ we see that (9) is satisfied since

$$\left| \int_{t-J}^t g(u, x(u - r(u))) \exp[-\int_u^t a(s)\, ds]\, du \right| \leq \mu M e^4 J < M/2,$$

and the conditions of Theorem 2 with Remark 5 are satisfied.

4. The nonlinear equation. We now consider

$$x'(t) = -a(t)h(x(t))x(t) + g(t, x(t - r(t))) + f(t) \tag{15}$$

in which a, f, g, h, and r are continuous and there are positive constants α, h_1, r_0, and J with

$$a(t) \geq 0, \quad h(x) \geq h_1, \quad 0 \leq r(t) \leq r_0, \quad \exp[-h_1 \int_{t-J}^t a(s)\, ds] \leq \alpha. \tag{16}$$

For any $J > 0$ we have

$$\begin{aligned}
x(t) = x(t-J) &\exp[-\int_{t-J}^t a(s)h(x(s))\, ds] \\
&+ \int_{t-J}^t [g(u, x(u - r(u))) + f(u)] \exp[-\int_u^t a(s)h(x(s))\, ds]\, du.
\end{aligned} \tag{17}$$

Let $M_0 \geq 0$ and $\lambda : [M_0, \infty) \to [0, \infty)$ be a continuous function with $\lambda(M) < M$ for $M > M_0$. Assume that there are positive constants L, $M \geq M_0$, R such that if $|x(s)| \leq M$ for $t - r_0 - J \leq s \leq t - r(t)$ then

$$\int_{t-J}^{t} |g(u, x(u - r(u)))| \exp[-h_1 \int_{u}^{t} a(s)\, ds] du \leq RM, \tag{18}$$

$$\int_{t-J}^{t} |f(u)| \exp[-h_1 \int_{u}^{t} a(s)\, ds] du \leq \beta, \tag{19}$$

and

$$(\alpha + R)M + \beta < \lambda(M). \tag{20}$$

Since the stable part, $-a(t)h(x)x$, is nonlinear it is an unpleasant surprise to need

$$|g(t, x)| \leq \mu(t)|x| \tag{21}$$

for some continuous function μ. But something like this is needed unless we are prepared to ask that $a(t) \geq a_0 > 0$. For if $a(t) = 0$ on an arbitrarily short interval and if $r(t)$ is zero at some point in that same interval, then unrestricted growth of g will yield solutions with finite escape time. The proof of the next result is identical to that of Theorem 2. That proof would fail at the point where we showed that no solution has finite escape time unless we have a strong growth condition on g or a sign condition.

Theorem 3. *Let (16) – (21) hold for each $M \geq M_0$ and suppose that $\lambda^k M \to M_0$ as $k \to \infty$ for each $M > M_0$. Then every solution of (15) satisfies $|x(t)| < M_0 + \varepsilon$ for all large t and each $\varepsilon > 0$.*

REMARK 6. The next example is completely different from Example 1 and indicates the versatility of the theorem. Our first estimate yields R independent of J; we take μ to be constant for brevity, but a review of the proof shows that it works if $|\mu(t)| \leq ka(t), 0 < k < 1$. We use a large function for $a(t)$ for simplicity; but a review of the proof shows that any nonnegative $a(t)$ with divergent integral will work. This is the type of classical condition discussed in Hale and Lunel [7;p. 134-5]; but those results need $a(t)$ bounded, while we do not.

EXAMPLE 2. If $|g(t,u)| \leq \mu|u|$ then all solutions of

$$x'(t) = -3t^2(1+\sin t^3)(x(t)+x^3(t)) + g(t, x(t-r(t)))$$

tend to zero as $t \to \infty$.

Proof. In view of Remark 5, in (18) we take $t^2 \geq P$ where $P > 2\mu e^2$. Recall that $h_1 = 1$ and let $|x(s)| \leq M$ to obtain from (18) that

$$\begin{aligned}
&\int_{t-J}^{t} |g(u, x(u-r(u)))| \exp[-\int_u^t a(s)\,ds] \leq \\
&\mu M \int_{t-J}^{t} \exp[-3\int_u^t s^2(1+\sin s^3)\,ds]\,du \\
&\leq \mu M \int_{t-J}^{t} \exp -[(t^3-u^3) + \int_u^t 3s^2 \sin s^3\,ds]\,du \\
&\leq \mu M \int_{t-J}^{t} \exp[-(t^3-u^3)+2]\,du \\
&\leq \mu M e^2 \int_{t-J}^{t} \exp[-(t-u)(t^2+tu+u^2)]\,du \\
&\leq \mu M e^2 \int_{t-J}^{t} \exp[-P(t-u)]\,du \\
&\leq \frac{\mu M e^2}{P}\left[1-e^{-PJ}\right] < \frac{M}{2}.
\end{aligned}$$

Here, $J > 0$ is arbitrary.

Next, we examine (16) and write

$$\begin{aligned}
\exp[-\int_{t-J}^{t} h_1 a(s)\,ds] &\leq e^2 \exp[-J(t^2 + t(t-J) + (t-J)^2)] \\
&\leq e^2 \exp[-Jt^2] < 1/2
\end{aligned}$$

if $t^2 \geq 2\mu e^2$ and J is large enough. This completes the proof. □

References

1 Becker, Leigh C., Burton, T.A., and Zhang, S. Functional differential equations and Jensen's inequality, J. Math. Anal. Appl. **138** (1989), 137–156.

2 Busenberg, S.N. and Cooke, K.L., Stability conditions for linear non-autonomous delay differential equations, Quart. Appl. Math. **42** (1984), 295–306.

3 Burton, T.A., Casal, A., and Somolinos, A., Upper and lower bounds for Liapunov functionals, Funkcialaj Ekvacioj, **32** (1989), 23–55.

4 Burton, T.A. and Hering, R.H., Liapunov theory for functional differential equations, Rocky Mountain J. Math. **24** (1994), 3–17.

5 Burton, T.A. and Hatvani, Laszlo, Stability theorems for nonautonomous functional differential equations by Liapunov functionals, Tohoku Math. J., **41** (1989), 65–104.

6 Burton, T.A. and Makay, G., Asymptotic stability for functional differential equations, Acta Math. Hungar. **65(3)** (1994), 243–251.

7 Hale, J. and Lunel, S.M.V., Introduction to Functional Differential Equations, Springer, New York, 1993.

8 Hatvani, L., Asymptotic stability conditions for a linear nonautonomous delay differential equation, preprint.

9 Hatvani, L. and Krisztin, T., Asymptotic stability for a differential-difference equation containing both delayed and undelayed terms, preprint.

10 Smart, D.R., Fixed Point Theorems, Cambridge Univ. Press, Cambridge, 1980.

11 Wang, Tingxiu, Equivalent conditions on stability of functional differential equations, J. Math. Anal. Appl. **170** (1992), 138–157.

12 Wang, Tingxiu, Upper bounds for Liapunov functionals, integral Lipschitz condition and asymptotic stability, Differential and Integral Equations **7** (1994), 441–452.

13 Wang, Tingxiu, Asymptotic stability and the derivatives of solutions of functional differential equations, Rocky Mountain J. Math. **24** (1994), 403–427.

14 Wang, Tingxiu, Weakening the condition $W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(\|\phi\|)$ for uniform asymptotic stability, Nonlinear Analysis **23** (1994), 251–264.

15 Yoshizawa, T., Asymptotic behaviors of solutions of differential equations, in Differential equations:qualitative theory, Colloq. Math. Soc. Janos Bolyai **47**, North-Holland, Amsterdam, 1987, 1141–1164.

16 Zhang, Bo, Asymptotic stability in functional differential equations by Liapunov functionals, Trans. Amer. Math. Soc., to appear.

17 Zhang, Bo, A stability theorem in functional differential equations, Differential and Integral Equations, to appear.