AN INTEGRAL EQUATION OF ADVANCED AND RETARDED TYPE

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ABSTRACT: In this note we consider an integral equation of both advanced and retarded type. We use a Liapunov functional and a fixed point theorem to prove that there is a periodic solution.

AMS(MOS) subject classifications. 45DO5

1. INTRODUCTION

Functional equations with a delay have been considered for decades and both theory and applications are well established. Equations which are purely advanced are also considered simply by a time reversal. By contrast, virtually nothing has been said about equations which are both advanced and retarded in nature. The purpose of this paper is to present an example of an integral equation which is both advanced and retarded and to establish a strong qualitative result for it in the form of the existence of a periodic solution using fixed point theory. This example should suggest that ample theory exists for treating advanced equations as readily as retarded ones.

It is natural to ask: what is the application of equations of advanced and retarded type? It is possible to give a descriptive rationale parallel to those so often given in population biology in which the population is predicated on future occurrences; at this point such applications are tenuous, at best. It is possible to formulate an investment scheme in terms of selling stocks short and buying futures contracts. Again, this would be over stating the case. In fact, this is an example from pure mathematics of an interesting property generalizing one aspect of a very long line of important work which we soon briefly describe. In particular, this paper addresses a gap in the theory of functional equations: we have long studied equations without a delay, with a delay, and neutral equations. Only the most incurious mathematician could fail to ask about the advanced and retarded case.

In a series of papers Levin and Nohel [6–9] consider equations of the form

$$x'(t) = -\int_0^t D(t,s)f(x(s))\,ds \tag{A}$$

and

$$x'(t) = -\int_{t-h}^t D(t,s)f(x(s))\,ds \tag{B}$$

where

$$D(t,s) \geq 0, \quad D_s(t,s) \geq 0, \quad D_{st}(t,s) \leq 0, \quad xf(x) > 0 \text{ if } x \neq 0, \tag{C}$$

and for (2)

$$D(t, t-h) = 0. \tag{D}$$

They obtain strong boundedness and stability results which have been generalized in a number of ways using the theory of positive kernels and transform theory. Summaries are found in Corduneanu [3] and in Gripenberg-Londen-Staffans [4]. In the convolution case for (B), Hale [5; pp. 72–3] obtained interesting results on limit sets and periodic solutions using dynamical system theory which depends on h being constant and positive.

Since $h > 0$, both (A) and (B) are delay differential equations. In the linear case (A) has only one linearly independent solution, but (B) is infinite dimensional. Moreover, if h is taken as a function of t and if $h(t)$ can become negative, then the theory for (B) becomes far more complicated and there seem to be no nice qualitative results.

Equation (B) has been offered as a model for a circulating fuel nuclear reactor and both have been used as population models ([9], [12]), but they are basically equations with memory and occupy an important place in pure mathematics, as seen in [3] and [4]. Conditions (C) and (D) give the problems their character and counterparts for integral equations have been studied in ([1], [2], [8]), for example.

In this paper we consider a scalar equation

$$x(t) = \lambda \left[a(t) - \sum_{i=1}^{n} \int_{t-f_i(t)}^{t} D_i(t,s) g(s, x(s))\, ds \right], \quad 0 < \lambda \leq 1, \tag{1}$$

with $a(t)$, $f_i'(t)$, $D_i(t,s)$, and $g(t,x)$ all continuous. The goal is to prove that there is a periodic solution for $\lambda = 1$. Here $f_i(t)$ is allowed to change sign. The analysis is based on the construction of a Liapunov function to obtain a priori bounds on periodic solutions, followed by application of a fixed point theorem of Schaefer.

2. A PERIODIC SOLUTION

We suppose there is a $T > 0$ with

$$a(t+T) = a(t), \quad f_i(t+T) = f_i(t), \quad D_i(t+T, s+T) = D_i(t,s), \quad g(t+T, x) = g(t,x). \tag{2}$$

In the introduction we denoted partial derivatives by $D_{st}(t,s)$. Here, because of the subscript on D we write $D_i(t,s)_{ts}$ instead. As $f_i(t)$ will be allowed to become negative, (C) and (D) must be changed. We suppose that

$$\begin{cases} D_i(t,s)_{st} \leq 0 \text{ if } f_i(t) \geq 0 \quad \text{and} \quad t - f_i(t) \leq s \leq t \\ \\ D_i(t,s)_{st} \geq 0 \text{ if } f_i(t) \leq 0 \quad \text{and} \quad t \leq s \leq t - f_i(t), \end{cases} \tag{3}$$

$$D_i(t, t - f_i(t)) = 0, \quad D_i(t, t - f_i(t))_s(1 - f_i'(t)) \geq 0, \tag{4}$$

$$\exists M > 0 \text{ with } 2g(t, x)[\lambda a(t) - x] \leq M - |g(t, x)|. \tag{5}$$

Remark 1. The character of (B) is determined by (C) and (D). The integral is a weighted memory which vanishes at the lower limit by (D); ours does the same. In (B) the memory is weighted most heavily by the present value of $x(t)$ with less weight on the past. In the same way, when $f_i(t) < 0$, our integral weights the present value of $x(t)$ more heavily than the future value. No real difficulties occur if some $f_i(t) = \pm\infty$. Here, (3) is the third condition in (C), taking into account the sign of $f_i(t)$; in particular, D_i must be a function of its lower limit, as may be seen in the following example. Also, the first part of (4) is the second part of (C) when we again take the sign of $f_i(t)$ into account; this can be seen in comparing the derivative of our subsequent Liapunov function (6) with the derivative of Levin's Liapunov function for (B). Finally, (5) is a relaxation of the last part of (C). It seems curious that we did not ask that D_i be positive, as in (C); in fact, if f_i becomes unbounded, that condition may be induced by the others. In conclusion, we offer (3)-(5) as the natural extension of (C) and (D) for (B). Our work, then, is a suggestion that the results established over the last forty years for (B) may have a natural counterpart for equations of advanced and retarded type.

Example. Consider

$$\int_{t-f(t)}^{t} D(t, s)g(s, x(s))\, ds = \int_{t-\sin t}^{t} -(t - \sin t - s)^n g(s, x(s))\, ds$$

for $n > 2$ and n odd. Clearly, (4) is satisfied. Also

$$D_{st}(t, s) = n(n - 1)(t - \sin t - s)^{n-2}(1 - \cos t)$$

which satisfies (3).

Theorem. *If (2) – (5) hold, then (1) has a T-periodic solution for $\lambda = 1$.*

Proof. Let $(P, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\varphi : R \to R$ with the supremum norm. For $\varphi \in P$ define

$$(H\varphi)(t) = a(t) - \sum_{i=1}^{n} \int_{t-f_i(t)}^{t} D_i(t, s)g(s, \varphi(s))\, ds.$$

Clearly, $H\varphi$ is a continuous function of t. Notice also by a change of variable that $(H\varphi)(t+T) = (H\varphi)(t)$ so that $H : P \to P$. Moreover, H is continuous in φ. Finally, it follows from the uniform continuity of D_i and g on bounded sets, together with Ascoli's theorem, that H maps bounded sets into compact sets. Under precisely these conditions, a result of Schaefer ([10], [11; p. 29]) states that if there is an a priori bound on all periodic solution of (1) for $0<\lambda<1$, then there is a periodic solution for $\lambda=1$.

We now prove that there is a $B>0$ such that $x\in P$ and $x=\lambda Hx$ implies $\|x\|\le B$ for $0<\lambda<1$. For such an $x\in P$ define

$$V(t)=\lambda^2\sum_{i=1}^{n}\int_{t-f_i(t)}^{t}D_i(t,s)_s\Big(\int_s^t g(v,x(v))\,dv\Big)^2\,ds \tag{6}$$

so that

$$\begin{aligned}V'&=\lambda^2\sum_{i=1}^{n}-D_i(t,t-f_i(t))_s\Big(\int_{t-f_i(t)}^{t}g(v,x(v))\,dv\Big)^2\,ds(1-f_i'(t))\\&+\lambda^2\sum_{i=1}^{n}\int_{t-f_i(t)}^{t}D_i(t,s)_{st}\Big(\int_s^t g(v,x(v))\,dv\Big)^2\,ds\\&+2\lambda^2 g(t,x)\sum_{i=1}^{n}\int_{t-f_i(t)}^{t}D_i(t,s)_s\int_s^t g(v,x(v))\,dv\,ds\end{aligned}$$

and that last term can be written as

$$\begin{aligned}&2\lambda^2 g(t,x)\sum_{i=1}^{n}\Big[D_i(t,s)\int_s^t g(v,x(v))\,dv\Big|_{t-f_i(t)}^{t}\\&\qquad+\int_{t-f_i(t)}^{t}D_i(t,s)g(s,x(s))\,ds\Big]\\&=2\lambda g(t,x)\sum_{i=1}^{n}\int_{t-f_i(t)}^{t}\lambda D_i(t,s)g(s,x(s))\,ds\\&=2\lambda g(t,x)[\lambda a(t)-x(t)]\end{aligned}$$

from (1), (4). By (3), (4) the first two terms of V' are not positive and we have by (5)

$$V'(t)\le\lambda[M-|g(t,x)|]. \tag{7}$$

Now $x\in P$ implies $V\in P$ so

$$0=V(T)-V(0)\le\lambda\Big[MT-\int_0^T|g(t,x(t))|\,dt\Big];$$

but $\lambda>0$ and so

$$\int_0^T|g(t,x(t))|\,dt\le MT. \tag{8}$$

Next, $f_i\in P$ implies that there is an $h>0$ with $|f_i(t)|\le h$. Also, there is a $Q>0$ such that $-h\le s\le t\le T+h$ implies that $|D_i(t,s)|\le Q$. From (1) and (8) we then have

$$|x(t)|\le\|a\|+\sum_{i=1}^{n}Q\Big|\int_{t-f_i(t)}^{t}|g(s,x(s))|\,ds\Big|$$

for $0 \leq t \leq T$. There is then an $N > 0$ with

$$\left| \int_{t-f_i(t)}^{t} |g(s, x(x))| \, ds \right| \leq N \int_0^T |g(s, x(s))| \, ds$$

so that

$$|x(t)| \leq \|a\| + QNnMT =: B.$$

This completes the proof. □

Remark 2. Schaefer's Theorem and the Liapunov functional used in the proof of the theorem can be used to show the existence of a solution whenever (3) – (5) hold without (2).

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