A General Stability Result of Marachkov Type

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1. Introduction

In this paper we prove a result concerning the behavior of functions satisfying certain relations on their derivatives. While it is a result on Liapunov's direct method, it is not stated in terms of any differential equation and, for this reason, it applies equally well to ordinary, functional, and partial differential equations, as we show by examples. The result addresses the classical problem of determining limit sets of solutions when the differential equation is unbounded in t, when the Liapunov function is not radially unbounded, and when the derivative of the Liapunov function is not negative definite in a convenient space. It has its roots in the classical theorem of Marachkov [13], which is its corollary, and it deals with limit set problems of Krasovskii [10; p. 66-68], Hale ([6], [7]), Henry [9], LaSalle [11], and Yoshizawa [15] in the cases in which the differential equation does not define a dynamical system. It may be stated as follows.

THEOREM 1. Let $V, g, \eta : [t_0, \infty) \to [0, \infty)$, $H : [0, \infty) \to [0, \infty)$ with H increasing, $[H, g', V']$ continuous,

(I). $\int_t^{t+1} \eta(t)dt \to 0$ as $t \to \infty$,

(II). $V'(t) \leq -g(t)$,

(III). when $g(t) \leq 1$ either

((a)) $g'(t) \leq H(V(t))(1 + \eta(t))$

or

((b)) $g'(t) \geq -H(V(t))(1 + \eta(t))$.

Then $g(t) \to 0$ as $t \to \infty$.

The proof will be given later, as will illustrative examples. But it is necessary to first place the result in perspective in order that its significance be understood.

2. The setting

The classical theory of Liapunov's direct method proceeds as follows. Let D be an open subset of R^n with $0 \in D$ and let $f : [0, \infty) \times D \to R^n$ be continuous with $f(t, 0) \equiv 0$. Then

$$x' = f(t, x) \tag{1}$$

is a system of ordinary differential equations and for each $(t_0, x_0) \in [0, \infty) \times D$, there is a solution $x(t, t_0, x_0)$ on some interval $[t_0, \alpha)$; if there is a compact subset of D in which $x(t, t_0, x_0)$ remains, then $\alpha = \infty$. Since $f(t, 0) = 0$, $x(t) = 0$ is a solution and stability theory begins by studying solutions starting near $x = 0$.

DEF. 1. The solution $x = 0$ is <u>stable</u> if for each $\varepsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[|x_0| < \delta, t \geq t_0]$ imply that $|x(t, t_0, x_0)| < \varepsilon$.

DEF. 2. The solution $x = 0$ is <u>asymptotically stable</u> if it is stable and if for each $t_0 \geq 0$ there is a $\mu > 0$ such that $|x_0| < \mu$ implies that $x(t, t_0, x_0) \to 0$ as $t \to \infty$.

Liapunov [12] introduced a technique for studying stability by means of a function $V : [0, \infty) \times D \to [0, \infty)$, having continuous first partial derivatives (a Lipschitz condition was later shown to suffice, cf., Yoshizawa [16, p. 4]) which would serve as a generalized norm on solutions of (1). While V need bear no relation to (1), solutions of (1) exist so $V(t, x(t, t_0, x_0))$ is a well-defined function of t and we may compute the derivative by the chain rule:

$$V'_{(1)}(t) := dV(t, x(t, t_0, x_0))/dt = \text{grad}V \cdot f + \partial V/\partial t. \tag{2}$$

DEF. 3. A wedge is a function $W : [0, \infty) \to [0, \infty)$ which is continuous, strictly increasing, and satisfies $W(0) = 0$.

THEOREM 2 (Liapunov). Suppose there is a differentiable function $V : [0, \infty) \times D \to$

$[0,\infty)$ and a wedge W_1 with

(i) $$W_1(|x|) \le V(t,x), V(t,0) = 0,$$

and

(ii) $$V'_{(1)}(t,x) \le 0.$$

Then $x = 0$ is stable.

It is not enough to add that

(iii) $$V'_{(1)}(t,x) \le -W_2(|x|)$$

to ensure asymptotic stability, but it is too much to add (iii) and

$$W_1(|x|) \le V(t,x) \le W_3(|x|)$$

for that will imply uniform asymptotic stability.

Ultimately, Marachkov [13] offered a compromise that has had far-reaching consequences. His contribution may be stated as follows.

THEOREM 3 (Marachkov). Suppose that $x = 0$ is stable and that there is a constant J, a differentiable function $V : [0,\infty) \times D \to [0,\infty)$, and a wedge W_3 with

(iv) $$V'_{(1)}(t,x) \le -W_3(|x|)$$

and

(v) $$|f(t,x)| \le J \text{ if } t \ge 0 \text{ and } x \in D.$$

Then $x = 0$ is asymptotically stable.

This result is a corollary of Theorem 1 in the sense that if the conditions of Theorem 3 hold then (I), (II), (III) hold and $g(t) \to 0$ implies that $x(t, t_0, x_0) \to 0$.

Marachkov's result started a long line of fruitful investigation. In the vast majority of examples, condition (iii) fails and the investigator is then faced with trying to salvage something.

Krasovskii [10; p. 67] asked that (1) and V be periodic in t (so that (iv) holds when D is bounded), $V'_{(1)}(t,x) \le 0$, and that there is a set M with $V'_{(1)}(t,x) < 0$ if $(t,x) \in [0,\infty) \times [D - M]$, while $x_0 \in M$ implies that $x(t,t_0,x_0)$ leaves M. He concluded asymptotic stability.

Yoshizawa [15] asked that $D = R^n$, $V(t,x) \ge 0$,

(vi) $$V'_{(1)}(t,x) \le -U(x) \le 0.$$

where $U(x)$ is positive definite with respect to a closed set Ω, and

(vii) $$f(t,x) \text{ is bounded for } x \text{ bounded.}$$

He concluded that all bounded solutions approach Ω as $t \to \infty$.

Our Theorem 1 is very close to Yoshizawa's result, but we relax (vi) and our result holds for far more general systems.

A close refinement of Yoshizawa's result is found in the work of LaSalle [11] (who had significant earlier and later work on this problem). His work introduces the idea of an "invariance principle." LaSalle defines $E = \{x : U(x) = 0, x \in \overline{D}\}$ and $E_\infty = E \cup \{\infty\}$ where U is defined in (vi). He proves that:

(a). if $x(t)$ remains in D, if (v) and (vi) hold, then $x(t) \to \infty$ or $x(t) \to E_\infty$ on its maximal right-interval of definition,

and

(b). if $x(t)$ remains in D on $[t_0,\infty)$ and if $U'(x(t))$ is bounded above or below, then $x(t) \to E_\infty$.

Our theorem was motivated by (a) and (b) in two ways. First, for general systems of partial differential equations, there seems to be no way to formulate and effectively use the idea that $f(t,x)$ is bounded for x bounded. Indeed, in his closing remarks LaSalle mentions that his work can not be extented to partial differential equations. Next, in similar examples we could not satisfy the condition that $U'(x(t))$ be bounded. Theorem 1 generalizes both of these ideas in a very useful way. It turns out that $U'(x(t))$ need be bounded only where $U(x)$ is small and that $V' \leq 0$ means that V is bounded so this can be used to show U' bounded. Moreover, $|V'|$ is an L^1 function on $[t_0, \infty)$. We will say more on this in the examples.

Haddock [4], Hatvani [8], the author [1], and many others extended the notions in LaSalle's presentation. But LaSalle also extended Krasovskii's idea concerning the set M in a very fruitful way for the autonomous case; and that had a significant impact on the theory for partial differential equations since some of them can be written as abstract ordinary differential equations and, therefore, his result could be extended almost without change (cf. Hale [7] and Henry [9; p. 91]).

LaSalle [11] studied

$$x' = f(x) \tag{1*}$$

with a function $V(x)$, both defined on D, with $V'_{(1^*)}(x) \leq 0$ and considered

$$E = \{x : V'(x) = 0, x \in D\}$$

and

$$M \text{ the largest invariant set in } E.$$

He proved that bounded solutions approach M as $t \to \infty$.

His result was extended by Hale ([6], [7]) to functional differential equations and certain partial differential equations. Almost always the form of the equation was crucial in the extensions. In particular, functions in the differential equation which were unbounded in t seemed difficult to handle in any way. The results of Hale depend crucially on the equation defining a dynamical system and consequent compactness arguments. Our use of V and η in Theorem 1 (iii) with V' being a logical candidate for η avoids all the compactness problems encountered by Hale. On the other hand, our conclusion is parallel to that of Yoshizawa or LaSalle's first result, while Hale's results deal with the refinements of invariant sets. The invariance principle was also studied by Haddock and Terjeki [5] using Liapunov functions instead of functionals.

It is to be emphasized that this is only a brief survey with a view to presenting only enough detail to place our result in a perspective that allows its relation to the literature be seen and to indicate where it can be applied. The literature on this subject is vast.

3. Proof of Theorem 1.

By way of contradiction, assume that $g(t) \not\to 0$ as $t \to \infty$ and, to be definite, suppose that (a) holds; the proof when (b) holds is very similar. Then there is an $\varepsilon > 0$ and $\{t_n\} \to \infty$ with $g(t_n) \geq \varepsilon$ and $t_n + 1 < t_{n+1}$.

Consider the intervals $s_n = [t_n - 1, t_n]$. Either $g(t) \geq \varepsilon/2$ on s_n or there is a subinterval $[a_n, b_n] \subset s_n$ with $g(a_n) = \varepsilon/2$, $g(b_n) = \varepsilon$, $\varepsilon/2 \leq g(t) \leq \varepsilon$ on $[a_n, b_n]$. Since $V'(t) \leq 0$ we have $V(t) \leq V(t_0)$ and, since H is increasing, it follows that for $t \in [a_n, b_n]$ we have

$$g'(t) \leq H(V(t))(1 + \eta(t)) \leq H(V(t_0))(1 + \eta(t)).$$

Thus,

$$\begin{aligned} \varepsilon/2 &= g(b_n) - g(a_n) = \int_{a_n}^{b_n} g'(s)ds \leq \int_{a_n}^{b_n} [H(V(t_0))(1 + \eta(t))]dt \\ &= H(V(t_0))[b_n - a_n + \int_{a_n}^{b_n} \eta(t)dt] \end{aligned}$$

Since the integral tends to zero as $n \to \infty$, there is an integer N, say $N = 1$ by renumbering, and a $Q > 0$ with $b_n - a_n \geq Q$ if $n \geq N = 1$. Thus, if $t > b_n$ then

$$0 \leq V(t) \leq V(t_0) - \sum_{i=1}^{n} \int_{a_i}^{b_i} g(s)ds \leq V(t_0) - nQ\varepsilon/2,$$

a contradiction for large n. This completes the proof.

We now show that Marachkov's result (Theorem 3) follows from Theorem 1. Let $x(t)$ be a solution of (1) on $[t_0, \infty)$ with $x(t) \in D$ and $|x(t)| \leq 1$. Note that

$$W_4(|x|) = \int_0^{|x|} W_3(s)ds = W_3(\xi)|x| \leq W_3(\xi) \leq W_3(|x|)$$

since W_3 is increasing and $|x| \le 1$, where $0 < \xi < |x|$. Hence,

$$V'(t,x) \le -\int_0^{|x(t)|} W_3(s)ds =: -g(t)$$

so

$$\begin{aligned} g'(t) &= W_3(|x(t)|)|x(t)|' \le W_3(|x(t)|)|x'(t)| \\ &\le W_3(1)|x'(t)| \le W_3(1)J =: H(V(t)), \end{aligned}$$

where H is a constant function. Hence, by Theorem 1, $g(t) \to 0$ so $|x(t)| \to 0$ and the proof is complete.

4. Examples.

The example used most often to test the stability results in the work we have cited is

$$x'' + f(t,x,x')x' + g(x) = 0,$$

its delay counterpart

$$x'' + f(t,x,x')x' + g(x(t-r)) = 0,$$

or its PDE counterpart

$$u_{tt} = g(u_x)_x - f(t,u,u_x)u_t.$$

And these are of much interest to us here.

REMARK. We have only briefly mentioned the role of $\eta(t)$ in Theorem 1. In applications it is to be noted that $V'(t) \le -g(t) \le 0$ and $V(t) \ge 0$ implies that $|V'(t)| \in L^1[0,\infty)$, so $|V'(t)|$ is a logical candidate for $\eta(t)$, as we will see.

EXAMPLE 1. Consider the equation

(E1) $$x'' + f(t,x,x')x' + r(x) = 0$$

and suppose there is a continuous increasing function $q : [0,\infty) \to [0,\infty)$ with

(i) $$r^2(x) \le q(G(x)), G(x) = \int_0^x r(s)ds,$$

a differentiable function $p : [0, \infty) \to [0, \infty)$ with

(ii) $$f(t, x, x') \geq p(t), p'(t) \leq P(1 + f(t, x, x')), \text{ and } p(t) \leq P$$

for a constant $P > 0$, and

(iii) $$xr(x) > 0 \text{ if } x \neq 0.$$

Then $g(t) := p(t)(x'(t))^2 \to 0$ as $t \to \infty$.

PROOF. Write (E1) as

$$\begin{aligned} x' &= y \\ y' &= -f(t, x, y)y - r(x) \end{aligned}$$

and define

$$V(t) = y^2 + 2G(x)$$

along any solution $(x(t), y(t))$. A calculation yields

$$V'(t) = -2f(t, x, y)y^2 \leq -p(t)y^2$$

so y is bounded; as $x' = y$, it follows that $x(t)$ is bounded by a linear function and so the solution can be defined for all future time. Let

$$W(t) = p(t)y^2$$

and obtain

$$\begin{aligned} W'(t) &= p'(t)y^2 + 2p(t)y[-f(t, x, y)y - r(x)] \\ &\leq p'(t)y^2 - 2p(t)r(x)y \\ &\leq p'(t)y^2 + p(t)(r^2(x) + y^2) \\ &\leq P(1 + f(t, x, y))y^2 + p(t)[q(G(x)) + y^2] \\ &\leq PV + P|V'| + Pq(V) + |V'| \\ &\leq P(V + q(V)) + (P + 1)|V'| \\ &\leq H(V)(1 + |V'|) \end{aligned}$$

where $|V'| \in L^1[0,\infty)$ and H is appropriately chosen. The conclusion now follows from Theorem 1.

REMARK. If we had attempted to use LaSalle's result (see (b) following (v) and (vi)), we would have needed $f(t,x,y) \geq C > 0$. Then we would have needed f bounded above or $r(x)$ bounded. Our f need not be bounded and neither does $r(x)$.

This equation has been widely studied and a good discussion with references is found in Yoshizawa [17]. From that discussion we see that to obtain a similar conclusion authors have needed $p(t)$ to be integrally positive. Moreover, most discussions require conditions to ensure that $x(t)$ is bounded; our conditions do not. On the other hand, other investigators conclude that $y(t) \to 0$; we do not as may be seen from the example

$$f(t,x,y) = t^3(|\sin t| - \sin t), p(t) = (|\sin t| - \sin t)^2 \text{ and } r(x) = x.$$

In that case, solutions satisfy $x'' + x = 0$ on intervals of length π, while $p(t)y^2(t) \to 0$ as $t \to \infty$.

EXAMPLE 2. The equation

$$\text{(E2)} \qquad u_{tt} = r(u_x)_x + q(t)u_{xtx}, u(t,0) = u(t,1) = 0,$$

was studied by Greenberg, MacCamy, and Mizel [3] when $q(t)$ is a positive constant. We suppose here that there is a constant ϕ with

$$\text{(i)} \qquad r^2(s) \leq \phi(G(s)+1), G(s) = \int_0^s r(v)dv,$$

there is a function $p(t) \leq q(t)$ with

$$\text{(ii)} \qquad 0 \leq p(t) \leq P, p'(t) \leq P(1+q(t)) \text{ for some } P > 0,$$

and

$$\text{(iii)} \qquad sr(s) > 0 \text{ if } s \neq 0,$$

and $p'(t)$ and $r(x)$ are continuous.

Under these conditions,

$$\text{(iv)} \qquad p(t)\int_0^1 u_t^2(t,x)dx \to 0 \ \text{ as } \ t\to\infty$$

for any solution $u(t,x)$ of (E2) on $[t_0,\infty)$.

PROOF. If $u(t,x)$ is a solution on $[t_0,\infty)$, define

$$V(t)=\int_0^1 [u_t^2+2G(u_x)]dx$$

so that

$$\begin{aligned} V'(t) &= 2\int_0^1 [u_t u_{tt}+r(u_x)u_{xt}]dx \\ &= 2\int_0^1 \{u_t[r(u_x)_x+q(t)u_{xtx}]-r(u_x)_x u_t\}dx \end{aligned}$$

(by integration by parts and the boundary condition)

$$\begin{aligned} &= 2q(t)\int_0^1 u_t u_{xtx}dx = -2q(t)\int_0^1 u_{tx}^2 dx \\ &\le -p(t)\int_0^1 u_t^2 dx =: -g(t). \end{aligned}$$

Note that $|V'(t)|\in L^1[t_0,\infty)$. Then

$$\begin{aligned} g'(t) &= p'(t)\int_0^1 u_t^2dx+2p(t)\int_0^1 u_t[r(u_x)_x+p(t)u_{xtx}]dx \\ &= p'(t)\int_0^1 u_t^2dx-2p(t)\int_0^1 r(u_x)u_{tx}dx-2p^2(t)\int_0^1 u_{tx}^2dx \\ &\le P(1+q(t))\int_0^1 u_t^2dx+p(t)\int_0^1 [r^2(u_x)+u_{tx}^2]dx \\ &\le P(1+q(t))\int_0^1 u_t^2dx+p(t)\int_0^1 u_{tx}^2dx+P\int_0^1 \phi[G(u_x)+1]dx \\ &\le P(1+\phi)V(t)+[Pq(t)+p(t)]\int_0^1 u_{tx}^2dx \\ &\le H(V(t))(1+(P+1)|V'(t)|) \end{aligned}$$

for an appropriate function H. The conclusion now follows from Theorem 1.

Background on the next example is found in Nakagiri [14].

EXAMPLE 3. Consider the control system

$$\text{(E3)} \qquad \begin{cases} u_t &= \Delta u+\int_{t-h}^t p(s,X)u(s,X)ds+c(X)f(\sigma) \\ \sigma' &= \int_\Omega b(X)\cdot\nabla u dX+\int_\Omega d(X)\int_{t-h}^t p(s,X)u(s,X)dsdX-rf(\sigma), \end{cases}$$

(i) $$u(t,X) = 0 \text{ on } \partial\Omega$$

where Ω is a domain with smooth boundary, h is a positive constant, $X = (x, y, z)$, Δ is the Laplacian, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$,

(ii) $$c \text{ and } b \text{ are bounded } L^2 \text{ functions,}$$

(iii) $$\sigma f(\sigma) > 0 \text{ if } \sigma \neq 0,$$

(iv) $df(\sigma)/d\sigma$ and p are continuous, $|df(\sigma)/d\sigma| \leq \Gamma(f(\sigma))$ for some continuous function Γ,

(v) $$\lambda_1 \text{ is the first eigenvalue of } -\Delta u \text{ on } H_0^1(\Omega),$$

k, γ, α, and β are positive constants with $\alpha + \beta = 1$, $k > 1$,

(vi) $$\beta\lambda_1 - (h/2) - khp^2(t,X) \geq \gamma > 0,$$

and there is a constant $\mu > 0$ with

(vii) $$4r > \int_\Omega [(b^2(X)/\alpha) + (c^2(X)/\gamma)]dX + \mu.$$

Under these conditions, for any solution $(u(t,X), \sigma(t))$ on $[t_0, \infty)$,

$$|\sigma(t)| + \int_\Omega [u^2(t,X) + \int_{-h}^0 \int_{t+s}^t p^2(v,X)u^2(v,X)dvds]dX \to 0 \text{ as } t \to \infty.$$

PROOF. If (u, σ) is a solution on $[t_0, \infty)$, define

(viii) $$V(t) = \int_\Omega [\frac{1}{2}u^2 + k\int_{-h}^0 \int_{t+s}^t p^2(v,X)u^2(v,X)dvds]dX + \int_0^\sigma f(s)ds.$$

After a lengthy calculation, use of the divergence theorem and the boundary conditions, and the fact that $\lambda_1 \int_\Omega u^2 dX \leq \int_\Omega |\nabla u|^2 dx$, we arrive at positive constants λ and c_1 with

(ix) $$\begin{aligned} V'(t) &\leq -\lambda \int_\Omega [u^2 + |\nabla u|^2 + f^2(\sigma) + \int_{t-h}^t p^2(s,X)u^2(s,X)ds]dX \\ &\leq -c_1 f^2(\sigma) =: -g(t). \end{aligned}$$

For $df(\sigma)/d\sigma = f*(\sigma)$ we then have positive constants K_1 and K_2 with

$$\begin{aligned} g'(t) &= 2c_1 f(\sigma) f*(\sigma)\sigma' \\ &\leq K_1|f(\sigma)f*(\sigma)|\{K_2 + \int_\Omega |\nabla u|^2 dX + r|f(\sigma)| \\ &+ \int_\Omega \int_{t-h}^t p^2(s,X)u^2(s,X)dsdX\}. \end{aligned}$$

Because of $|V'|$ being integrable on $[t_0,\infty)$, we see that

$$\int_\Omega \{|\nabla u|^2 + \int_{t-h}^t p^2(s,X)u^2(s,X)ds\}dX =: \gamma(t)$$

is an L^1-function. Thus, $g'(t) \leq H(g(t))(1+\gamma(t))$, for an appropriate function H when we take (iv) and the definition of $g(t)$ into account. Since the inequality need only hold for $g(t) \leq 1, H$ can be constant. By Theorem 1 it follows that $g(t) \to 0$. This means that $f^2(\sigma) \to 0$ and, as $\sigma f(\sigma) > 0$ if $\sigma \neq 0$, either $\sigma(t) \to 0$ or $|\sigma(t)| \to \infty$.

By way of contradiction, suppose that $\sigma(t) \to \infty$. Then $V'(t) \leq 0$ implies that $\int_0^\infty f(s)ds = \xi > 0$ and $V'(t) \leq -\lambda(V(t)-\xi)$ so that if $W(t) = V(t) - \xi$, then

$$W' \leq -\mu|\sigma'|W^{1/2}$$

for $\mu > 0$. Thus, $\sigma(t)$ is bounded and, therefore, tends to zero. Since $\gamma(t)$ is integrable, $V(t)$ tends to zero along a sequence. Since $V(t)$ decreases, V tends to zero. This completes the proof.

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