A FIXED POINT THEOREM OF KRASNOSELSKII

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Abstract. Krasnoselskii's fixed point theorem asks for a convex set M and a mapping $Pz = Bz + Az$ such that: (i) $Bx + Ay \in M$ for each $x, y \in M$; (ii) A is continuous and compact; (iii) B is a contraction. Then P has a fixed point. A careful reading of the proof reveals that (i) need only ask that $Bx + Ay \in M$ when $x = Bx + Ay$. The proof also yields a technique for showing that such x is in M.

1. Introduction and result. Two main results of fixed point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result (cf. [1] or [6; p. 31]).

Theorem 1. *Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map M into S such that*

(i) $Ax + By \in M (\forall x, y \in M)$,

(ii) *A is continuous and AM is contained in a compact set,*

(iii) *B is a contraction with constant $\alpha < 1$.*

Then there is a $y \in M$ with $Ay + By = y$.

This is a captivating result and it has a number of interesting applications. It was motivated by an observation that inversion of a perturbed differential operator may yield the sum of a compact and contraction operator.

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But the result has a major weakness. Given operators A and B, it may be possible to find sets M and M^* with $A : M \to M$ and $B : M^* \to M^*$, but if the sets are bounded (which is frequently needed if AM is to be in a compact set), then it is often impossible to arrange matters so that $M = M^*$ and $Bx + Ay \in M$.

The point of this note is that a careful reading of the proof reveals two items:

a. The quantifiers in (i) are too stringent. What is actually needed is that for fixed $y \in M$, if x is the unique fixed point of the contraction mapping $x \to Bx + Ay$, then $x \in M$. This observation in very useful in applications; moreover, subsequent investigators seeking to extend the result have not noticed it, as may be seen for example in a recent such work by O'Regan [3;p. 2].

b. The proof of Theorem 1 hinges on the fact that $(I - B)$ has a continuous inverse. In showing that, one writes

$$\|(I-B)x-(I-B)y\| = \|(x-y)-(Bx-By)\|$$
$$\geq \|x-y\| - \|Bx-By\| \geq (1-\alpha)\|x-y\|.$$

Clearly,

$$\|(I-B)x-(I-B)y\| \leq (1+\alpha)\|x-y\|.$$

Together we have

$$(1-\alpha)\|x-y\| \leq \|(I-B)x-(I-B)y\| \leq (1+\alpha)\|x-y\| \tag{1}$$

and, in particular,

$$(1-\alpha)\|x\| \leq \|(I-B)x\| \leq (1+\alpha)\|x\|. \tag{2}$$

These relations rest on the contraction property alone. We note that a tightening of (2) allows us to confirm the requirement that $x = Bx + Ay$ yields $x \in M$. This is illustrated in an example.

Theorem 2. *Let M be a closed, convex, and nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that $A : M \to S$ and $B : S \to S$ such that:*

(i) B is a contraction with constant $\alpha < 1$,

(ii) A is continuous, AM resides in a compact subset of S,

(iii) $[x = Bx + Ay, y \in M] \Rightarrow x \in M$.

Then there is a $y \in M$ with $Ay + By = y$.

Remark 1. It will be clear that B need only be defined on a set $H \subset S$ such that $M \subset H$ and if $[y \in M$ and $x \in H]$ then $Bx + Ay \in H$.

Proposition. *Let (i) and (ii) of Theorem 2 hold. Suppose there is an $r > 0$ so that $M = \{y \in S | \, \|y\| \leq r\}$ and AM is in M.*

If (2) is strengthened to

$$\|x\| \leq \|(I - B)x\| \tag{2*}$$

then item (iii) of Theorem 2 holds.

Example. Let $0 < \alpha < 1$ and consider the scalar integral equation

$$\begin{aligned} x(t) = &-\alpha \sin^2 t[x^3(t)/(1 + 2x^2(t))] + p(t) \\ &+ \int_{-\infty}^{t} D(t - s)g(x(s))\, ds \end{aligned} \tag{3}$$

where p, D, and g are continuous, $p(t + 2\pi) = p(t)$.

Suppose that there is an $r > 0$ such that

$$|x| \leq r \Rightarrow |g(x)| \leq r - \|p\| \tag{4}$$

and that

$$\int_{-\infty}^{t} |D(t - s)|\, ds \leq 1 \text{ and } \int_{-\infty}^{t} |D'(t - s)| ds < \infty. \tag{5}$$

Then (3) has a 2π-periodic solution.

Here,

$$(Bx)(t) = -\alpha \sin^2 t[x^3(t)/(1+2x^2(t))]$$

and

$$(Ay)(t) = p(t) + \int_{-\infty}^{t} D(t-s)g(y(s))\,ds,$$

while $(S, \|\cdot\|)$ is the Banach space of continuous 2π-periodic functions with the supremum norm, and $M = \{y \in S | \, \|y\| \le r\}$.

Clearly, $(2)^*$ holds. We see no way to establish a set M so that (i) of Theorem 1 holds. For each set M which we construct, we find some x, $y \in M$ with Ay, $Bx \in M$, but $Ay + Bx \notin M$.

Krasnoselskii's result has been of continuing interest. In 1971, Reinermann [4] obtained two theorems related to Theorem 1. He asked that $A+B : M \to M$, while A, $B : M \to S$.

Just this year, O'Regan [3] states that he has extended Reinermann's result by assuming that:

(i) $A + B : M \to S$,

(ii) $A + B$ is condensing, and

(iii) if $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ is a sequence in $\partial M x[0,1]$ converging to (x, λ) with $x = \lambda(A+B)x$ and $0 < \lambda < 1$, then $\lambda_j(A+B)\, x_j \in M$ for large j.

The idea of using condensing maps in conjunction with Theorem 1 goes back to 1967 with Sadovskii [5], who still maintains an interest in the subject. The reader can find the definition and properties in [3].

We do not see how any of these come close to (iii) of Theorem 2.

2. Proofs. To prove Theorem 2, we follow Krasnoselskii's proof as given by Smart [6; p. 32]. Smart first proves:

Lemma. *If B is a contraction mapping of a subset X of a normed space S into S, then*

$I - B$ is a homeomorphism on X to $(I - B)X$. If $(I - B)X$ is precompact, so is X.

Next, for each fixed $y \in M$, the map of $S \to S$ defined by

$$z \to Bz + Ay$$

is a contraction with unique fixed point z so that $z = Bz + Ay$; by (iii), $z \in M$. Hence, $(I - B)z = Ay$ or $z = (I - B)^{-1}Ay \in M$ for each $y \in M$. Now AM resides in a compact subset of S, while $(I - B)^{-1}$ is continuous, and so $(I - B)^{-1}AM$ resides in a compact subset of the closed set M. (For a proof of this in general metric spaces, see Kreyszig [2; p. 412, 656].) By Schauder's second theorem [6; p. 25], $(I - B)^{-1}A$ has a fixed point $y \in M : y = (I - B)^{-1}Ay$. This proves Theorem 2.

To prove the proposition, if $x = Bx + Ay$, then $(I - B)x = Ay$; thus, by the first part of $(2)^*$,

$$\|x\| \leq \|(I - B)x\| = \|Ay\| \leq r$$

since $y \in M \Rightarrow Ay \in M \Rightarrow \|Ay\| \leq r$. Hence, $x \in M$.

We now show that the conditions of Theorem 2 hold for the example. Recall that A, B, S, and M are defined in the example.

First, if $y \in M$, then $\|y\| \leq r$ and so

$$\begin{aligned} \|Ay\| &\leq \|p\| + \int_{-\infty}^{t} |D(t - s)|[r - \|p\|]\, ds \\ &\leq \|p\| + 1[r - \|p\|] = r. \end{aligned}$$

and a change of variable shows that $(Ay)(t + 2\pi) = (Ay)(t)$. Hence, $A : M \to M$.

It is an elementary exercise to show that A maps M into an equicontinuous set. Also, continuity of A on M is easily verified.

B is a contraction with constant α. Clearly, $\|(I - B)x\| \geq \|x\|$.

This completes the proof. $\square$

3. Concluding remarks. The proposition does not represent the only way in which (iii) of Theorem 2 can be verified.

If $Px = Bx + Ay$ is a contraction with fixed point z and if φ is any point, then $\|z - \varphi\| \leq \frac{1}{1-\alpha}\|\varphi - P\varphi\|$ (cf. Smart [6; p. 3]). In a given problem, clever choice of φ can establish that $z \in M$. Application of fixed point theory is an art. Most nice results are based on some clever selection. But if the imagination fails, $P^k\varphi \to z$ so there is always the alternative of trying to iterate P.

The equation $x = Bx + Ay$, $y \in M$, may have properties so that it can be shown that there is an a priori bound on solutions in the set S; that bound may yield $x \in M$.

But there is a far more definite idea which the reader may find attractive.

Conjecture. The proposition is still true if $(2)^*$ is replaced by $x \neq 0 \Rightarrow$

$$\|(I - B)x\| < \|x\|. \tag{2^{**}}$$

We have been unable to prove the conjecture. But a certain symmetry in the problem suggests it is true. It may be a simple retraction argument, when viewed properly.

REFERENCES

1. Krasnoselskii, M.A., *Amer. Math. Soc. Transl.* (2) **10**(1958), 345–409.
2. Kreyszig, Erwin, *Introductory Functional Analysis with Applications*, Wiley, New York, 1978.
3. O'Regan, D., Fixed point theory for the sum of two operators, *Appl. Math. Lett.* **9**(1996), 1–8.
4. Reinermann, J., Fixpunktsätze vom Krasnoselski-typ, *Math. Z.* **119**(1971), 339–344.
5. Sadovskii, B.N., A fixed-point principle, *Func. Anal. and Applications* **1**(1967), 151–3.
6. Smart, D.R., *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1980.