DIFFERENTIAL INEQUALITIES AND EXISTENCE THEORY FOR DIFFERENTIAL, INTEGRAL, AND DELAY EQUATIONS

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1. Introduction. This paper is concerned with simple, concise, and unifying proofs of global existence of solutions for

$$x' = f(t, x) \tag{1}$$

where $f : [0, \infty) \times R^n \to R^n$ is continuous, for

$$x(t) = a(t) + \int_0^t D(t, s, x(s))ds \tag{2}$$

where $D : [0, \infty) \times R \times R^n \to R^n$ and $a : [0, \infty) \to R^n$ are both continuous, and for functional differential equations

$$x' = f(t, x_t) \tag{3}$$

with both finite and infinite delay. Local existence results are given as corollaries.

Classical existence theory is first local, then piecemeal, and then awkward, as we explain in the next section. In this paper we use Schaefer's fixed point theorem to prove the existence of a global solution of each equation in one step. Each theorem is proved in the same way. First we define the appropriate space and a mapping. Each theorem is then proved using three lemmas. The first lemma shows that the mapping maps bounded sets into compact sets. The second lemma shows that the mapping is continuous. The third lemma establishes a priori bounds on the solution. The results then follow from Schaefer's theorem.

Existence theory for (1) usually rests on limiting arguments with ε-approximate solutions or on careful application of Schauder's fixed point theorem after constructing an appropriate set and a mapping of that set into itself; this is usually an intricate and tedious task. Schaefer's theorem is mainly Schauder's theorem followed by a simple retract argument. Its great advantage over Schauder's theorem is that a self-mapping set need not be found.

The reader will find standard developments of existence theory for (1) in [1], [2], [8], [10], [12], and [13]. Existence theory for (2) is found in [3] and [7], while theory for (3) is found in [1], [4], [5], [9], [12], and [17].

While this paper is clearly expository in nature, Theorems 3 and 4 are new. Hale [9; p. 142] has a form of Theorem 3 in the linear case. Theorem 4 allows for unbounded initial functions, as well as larger than linear growth of f.

2. Background and motivation. Classical existence theory for (1) begins with a local result. It is shown that for each $(t_0, x_0) \in [0, \infty) \times R^n$, there is at least one solution $x(t) = x(t, t_0, x_0)$ with $x(t_0, t_0, x_0) = x_0$ and satisfying (1) on an interval $[t_0, t_1]$, where t_1 is computed from a bound on f in a closed neighborhood of (t_0, x_0). This yields a new point $(t_1, x(t_1))$ and we begin once more computing bounds on f in a closed neighborhood of $(t_1, x(t_1))$ and obtain a continuation of the solution on $[t_1, t_2]$. These are the aforementioned local and piecemeal parts.

If we continue this process on intervals $\{[t_n, t_{n+1}]\}$, can we say that $t_n \to \infty$ as $n \to \infty$? It turns out that we can unless there is an α such that $|x(t)|$ tends to infinity as t tends to

α from the left. This is the awkward part. Either implicitly or explicitly (cf. Hale [9; p. 42]) we invoke Zorn's lemma to claim that there is a solution on $[t_0, \infty)$ or one on $[t_0, \alpha)$ which can not be continued to α. Some authors call a solution on $[t_0, \infty)$ noncontinuable. We do not; for our purposes, if a solution is defined on $[t_0, \alpha)$ with $\alpha < \infty$, and if it can not be extended to α, it is said to be noncontinuable. An example of the latter case is

$$x' = x^2, \quad (t_0, x_0) = (0, 1)$$

which has the solution $x(t) = \frac{1}{1-t}$ on $[0, 1)$ and is noncontinuable. In fact, two more examples complete the range of possibilities. Solutions of $x' = -x^3$ are all continuable to $+\infty$ because they are bounded, even though the right-hand-side grows faster than in the first example. Finally, solutions of $x' = t^3 x \ell n(1 + |x|)$ are unbounded, but continuable to $+\infty$ because the right-hand-side does not grow too fast.

We come then to the question of how to rule out noncontinuable solutions of the kind mentioned above. In principle, there is a fine way of doing so. Kato and Strauss [6] prove that it always works.

DEF. A continuous function $V : [0, \infty) \times R^n \to [0, \infty)$ which is locally Lipschitz in x is said to be mildly unbounded if for each $T > 0$, $\lim_{|x| \to \infty} V(t, x) = \infty$ uniformly for $0 \le t \le T$.

If there is a mildly unbounded V which is differentiable, then we invoke the local existence theory and consider a solution $x(t)$ of (1) on $[t_0, \alpha)$ so that $V(t, x(t))$ is an unknown but well-defined function. The chain rule than gives

$$\begin{aligned} \frac{dV}{dt}(t, x(t)) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial V}{\partial t} \\ &= \operatorname{grad} V \cdot f + \frac{\partial V}{\partial t}. \end{aligned}$$

We can also compute V' when V is only locally Lipschitz in x (cf. Yoshizawa [17; p. 3]) and we will display such an example in a moment; in that case, one uses the upper right-hand derivative.

If V is so shrewdly chosen that it is mildly unbounded and $V' \le 0$, then there can be no $\alpha < \infty$ with $\lim_{t \to \alpha-} |x(t)| = \infty$ because $V(t, x(t)) \le V(t_0, x_0)$.

There is a converse theorem; if f is continuous and locally Lipschitz in x for each fixed t, then Kato and Strauss [6] show that there is a mildly unbounded V with $V' \leq 0$ if and only if all solutions can be continued for all future time. Their result is not constructive, but investigators have constructed suitable V for many important systems without any growth condition on f; we offer examples following each of our theorems. In the example $x' = -x^3$ mentioned above, $V = x^2$ yields $V' = -2x^4 \leq 0$, showing global existence.

These remarks for (1) apply in large measure to (2) and (3). In those cases we require a functional $V(t, x(\cdot))$. More importantly, as mentioned above, for (1) the only way a solution can fail to be continuable to $+\infty$ is for there to exist an α with $\lim\limits_{t \to \alpha^-} |x(t)| = +\infty$; but for (3) we must take the limit supremum.

Wintner derived conditions on the growth of f to ensure that solutions of (1) could be continued to $+\infty$ and Conti used these to construct a suitable V. These results are most accessible in Hartman [10; pp. 29–30] for the Wintner condition and Sansone-Conti [13; p. 6] for V. A proof here will show how it works and will be a guide for an alternative proof of each of our Lemma 3 for each of our theorems.

Theorem (Conti-Wintner). *If there are continuous functions $\Gamma : [0, \infty) \to [0, \infty)$ and $W : [0, \infty) \to [1, \infty)$ with*

$$|f(t, x)| \leq \Gamma(t) W(|x|) \ \textit{and} \ \int_0^\infty \frac{ds}{W(s)} = \infty, \ \textit{then}$$

$$V(t, x) = \left\{ \int_0^{|x|} \frac{ds}{W(s)} + 1 \right\} \exp - \int_0^t \Gamma(s) ds$$

is mildly unbounded and $V'(t, x(t)) \leq 0$ along any solution of (1).

Proof. Let $x(t)$ be a noncontinuable solution of (1) on $[t_0, \alpha)$. By examining the difference quotient we see that $|x(t)|' \leq |x'(t)|$. Thus,

$$V'(t, x(t)) \leq \frac{|x(t)|'}{W(|x(t)|)} \exp - \int_0^t \Gamma(s) ds - \Gamma(t) V(t, x(t)) \leq 0$$

when we use $|x(t)|' \leq |x'(t)| \leq \Gamma(t) W(|x(t)|)$. This means that $V(t, x(t)) \leq V(t_0, x_0)$; since V is mildly unbounded, $\lim\limits_{t \to \alpha^-} |x(t)| \neq \infty$. This completes the proof.

In the next two sections we will prove a result which will give a global solution in one step. But we still want a local solution as a special case. This can be accomplished by extending f over a compact set to a continuous bounded function on R^{n+1}. There are classical extension theorems which we use in Theorems 3 and 4, but the following idea of a colleague, G. Makay, makes it elementary for Theorems 1 and 2.

Remark on extension. Suppose that f is continuous on $\Delta = \{(t,x)|t_0 \le t \le T, |x - x_0| \le J\}$ for $T > t_0$ and $J > 0$. We want to extend f to all of $R \times R^n$ in a bounded and continuous manner. Since Δ is convex, if we choose (t_1, y_1) as any interior point of Δ and if Q is any ray from (t_1, y_1) then Q intersects the boundary of Δ at exactly one point (t_Q, x_Q). Define $F : R \times R^n \to R^n$ by

(i) $F(t,x) = f(t,x)$ if $(t,x) \in \Delta$, and

(ii) $F(t,x) = f(t_Q, x_Q)$ if (t,x) is on Q and in the complement of Δ.

Clearly, F is bounded, continuous, and agrees with f on Δ.

Our results are based on the following theorem of Schafer [14] which is discussed and proved also in Smart [16; p. 29].

Theorem (Schaefer). *Let $(C, \|\cdot\|)$ be a normed space, H a continuous mapping of C into C which is compact on each bounded subset of C. Then either*

(i) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.

2. Existence theory for (1).

Let $0 \le \lambda \le 1$ and consider

$$x' = \lambda f(t,x), \quad x(t_0) = \lambda x_0 \tag{4}$$

or the equivalent integral equation

$$x(t) = \lambda \left[x_0 + \int_{t_0}^{t} f(s, x(s))ds \right] =: \lambda H(x)(t). \tag{5}$$

Theorem 1. *If either of the following conditions hold, then for each $(t_0, x_0) \in [0, \infty) \times R^n$, there is a solution of (1) on $[t_0, \infty)$.*

(I) There are continuous functions $\Gamma : [0,\infty) \to [0,\infty)$ *and* $W : [0,\infty) \to [1,\infty)$ *with*

$$|f(t,x)| \le \Gamma(t)W(|x|) \text{ and } \int_0^\infty \frac{ds}{W(s)} = \infty.$$

(II) There is a continuous function $V : [0,\infty) \times R^n \to [0,\infty)$ *which is locally Lipschitz in* x*, mildly unbounded, and* $V'(t,x(t)) \le 0$ *along any continuous solution of (4) defined on* $[t_0,\infty)$*.*

Proof. Let $T > t_0$ be given. We will show that there is a solution $x(t,t_0,x_0)$ of (1) on $[t_0,T]$.

Let $(C, \|\cdot\|)$ be the Banach space of continuous functions $\varphi : [t_0,T] \to R^n$ with the supremum norm. From (5) we consider $\varphi \in C$ and write

$$H(\varphi)(t) = x_0 + \int_{t_0}^t f(s,\varphi(s))ds.$$

The conditions of Schaefer's theorem will be verified by three simple lemmas.

Lemma 1. *$H : C \to C$ and H maps bounded sets into compact sets.*

Proof. For $\varphi \in C$ we have $f(t,\varphi(t))$ continuous and so $H(\varphi)(t)$ is continuous. Thus, $H : C \to C$.

For a given $J > 0$, if $\varphi \in C$ and $\|\varphi\| \le J$, then there is a $J^* > 0$ with $|f(t,\varphi(t))| \le J^*$ for $t_0 \le t \le T$. Thus, there is a $K > 0$ with $|H(\varphi)(t)| \le K$. Also, $|(H(\varphi)(t))'| = |f(t,\varphi(t))| \le J^*$. By Ascoli's theorem, this set of φ is mapped into a compact set.

Lemma 2. *H is continuous.*

Proof. Let $J > 0$ be given and let $\varphi_i \in C$ with $\|\varphi_i\| \le J$, $i = 1,2$. Now f is uniformly continuous for $|x| \le J$ and $t_0 \le t \le T$, so for each $\varepsilon > 0$ there is a $\delta > 0$ such that $[t_0 \le t \le T$ and $|\varphi_1(t) - \varphi_2(t)| \le \delta]$ imply that $|f(t,\varphi_1(t)) - f(t,\varphi_2(t))| < \varepsilon$. Thus, $\|\varphi_1 - \varphi_2\| \le \delta$ and $t_0 \le t \le T$ imply that

$$|H(\varphi_1)(t) - H(\varphi_2)(t)| = \left| \int_{t_0}^t \big[f(s,\varphi_1(s)) - f(s,\varphi_2(s))\big]ds \right| < \varepsilon[T - t_0]$$

and so $\|H(\varphi_1) - H(\varphi_2)\| < \varepsilon[T - t_0]$.

Lemma 3. *There is a $K > 0$ such that if $\varphi(t) = \lambda H(\varphi)(t)$ for $t_0 \le t \le T$, then $\|\varphi\| \le K$.*

Proof. If (I) holds, then a suitable V for (4) is constructed in the Conti-Wintner theorem since $\lambda \le 1$ which satisfies (II) and so (I) is a special case of (II). Thus, for the mildly unbounded V we always have $V(t_0, \varphi(t_0)) = V(t_0, \lambda x_0)$, a fixed constant for every φ; since V is continuous and x_0 is fixed, there is a $P > 0$ with $V(t_0, \lambda x_0) \le P$ if $0 \le \lambda \le 1$. But by the definition of V being mildly unbounded, $V(t, \varphi) \to \infty$ as $|\varphi| \to \infty$ uniformly for $t_0 \le t \le T$. Hence, there is a $K > 0$ with $\|\varphi\| \le K$ whenever φ satisfies (5).

All of the conditions of Schaefer's theorem are satisfied, his condition (ii) is ruled out by Lemma 3, and so (5) has a solution for $\lambda = 1$. That solution satisfies (1).

COR. *Let $f(t,x)$ be continuous for $|x - x_0| \le J$ and $t_0 \le t \le T$, and let $|f(t,x)| \le M$ on that set. Then (1) has a solution $x(t, t_0, x_0)$ defined for $t_0 \le t \le \alpha$ where $\alpha = \min[T, t_0 + J/M)]$.*

Proof. Let $\Omega = \{(t,x) | \quad |x - x_0| \le J,\ t_0 \le t \le T\}$. Since f is continuous on Ω, by the remark on extension in the previous section, we can extend f to a bounded and continuous function F on $R \times R^n$; hence, Condition (I) of Theorem 1 holds for F and there is a solution $x(t, t_0, x_0) = x(t)$ of $x' = F(t,x)$ for $t_0 \le t < \infty$. Certainly, $x(t)$ also satisfies (1) so long as $(t,x) \in \Omega$. If $(t, x(t))$ reaches the boundary of Ω at $t_1 < T$ and $t_1 < t_0 + (J/M)$, then

$$|x(t_1) - x_0| \le \int_{t_0}^{t} |f(s, x(s))| ds \le M(t_1 - t_0) < J,$$

a contradiction to $(t, x(t))$ reaching the boundary at t_1. This completes the proof.

Remark. Lakshmikantham and Leela [12 (vol. I); p. 46] have a global existence theorem partially in the spirit of Theorem 1(I). But the theorem and proof fall short of ours in four ways. First, they must work up differential inequality and maximal solution theory to find an upper bound on a solution set. Next, they require theory for a locally convex topological vector space and the Tychonov fixed point theorem. Thirdly, they must find a self-mapping set. Finally, they require that the Wintner function $W(|x|)$ be monotone, a condition we do not need until we consider (2).

Example 1. Consider the scalar equation

$$x'' + h(t, x, x')x' + g(x) = e(t)$$

where $h(t, x, y) \geq 0$, $xg(x) > 0$ if $x \neq 0$, h, g, and e are continuous. Write the equation as

$$x' = \lambda y$$
$$y' = \lambda[-h(t, x, y)y - g(x) + e(t)]$$

and define a mildly unbounded function by

$$V(t, x, y) = \left[y^2 + 2\int_0^x g(s)ds + ln(|x| + 1) + 1 \right] \exp - \int_0^t E(s)ds$$

where $E(t) = 2|e(t)| + 1$ so that

$$V'(t, x, y) \leq \left[-2\lambda h(t, x, y)y^2 + 2\lambda y e(t) + |y| - E(t)(y^2 + 1) \right] \exp - \int_0^t E(s)ds \leq 0.$$

Thus, by Theorem 1 all solutions exist on $[t_0, \infty)$.

3. An integral equation. Consider once more

(2) $$x(t) = a(t) + \int_0^t D(t, s, x(s))ds$$

with its continuity conditions.

Theorem 2. *If either of the following conditions hold, then (2) has a solution on $[0, \infty)$:*

(I) There are continuous increasing functions $\Gamma : [0, \infty) \to [0, \infty)$ and $W : [0, \infty) \to [1, \infty)$ with

(6) $$\int_0^\infty \frac{ds}{W(s)} = \infty \text{ and } |D(t, s, x)| \leq \Gamma(t)W(|x|) \text{ for } 0 \leq s \leq t.$$

(II) There is a differentiable scalar functional $V(t, x(\cdot))$ which is mildly unbounded along any solution of

$$x(t) = \lambda\left[a(t) + \int_0^t D(t, s, x(s))ds\right] =: \lambda H(x)(t), \quad 0 \le \lambda \le 1, \tag{7}$$

and which satisfies $V'(t, x(\cdot)) \le 0$ along such a solution.

Proof. Let $T > 0$ and $(C, \|\cdot\|)$ be the Banach space of continuous $\varphi : [0, T] \to R^n$ with the supremum norm. We will show that there is a solution $x(t)$ of (2) on $[0, T]$.

Lemma 1. *If H is defined by (7) then $H : C \to C$ and H maps bounded sets into compact sets.*

Proof. If $\varphi \in C$, then $D(t, s, \varphi(s))$ is continuous and so $H(\varphi)$ is a continuous function of t. Let $J > 0$ be given and let $B = \{\varphi \in C | \|\varphi\| \le J\}$. Now $a(t)$ is uniformly continuous on $[0, T]$ and $D(t, s, x)$ is uniformly continuous on $\Delta = \{(t, s, x) | 0 \le s \le t \le T, |x| \le J\}$. Thus, for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $(t_i, s_i, x_i) \in \Delta$, $i = 1, 2$, then $|(t_1, s_1, x_1) - (t_2, s_2, x_2)| < \delta$ implies that $|D(t_1, s_1, x_1) - D(t_2, s_2, x_2)| < \varepsilon$; a similar statement holds for $a(t)$. If $\varphi \in B$ then $0 \le t_i \le T$ and $|t_1 - t_2| < \delta$ imply that

$$\begin{aligned}
&|H(\varphi)(t_1) - H(\varphi)(t_2)| \le |a(t_1) - a(t_2)| \\
&+ \left| \int_0^{t_1} \Big[D\big(t_1, s, \varphi(s)\big) - D\big(t_2, s, \varphi(s)\big) \Big] ds \right| \\
&+ \left| \int_0^{t_1} D\big(t_2, s, \varphi(s)\big) ds - \int_0^{t_2} D\big(t_2, s, \varphi(s)\big) ds \right| \\
&\le \varepsilon + t_1\varepsilon + |t_1 - t_2| M \le \varepsilon(1 + T) + \delta M
\end{aligned}$$

where $M = \max_\Delta |D(t, s, x)|$. Hence, the set $A = \{H(\varphi) | \varphi \in B\}$ is equicontinuous. Moreover, $\varphi \in B$ implies that $\|H(\varphi)\| \le \|a\| + TM$. Thus, A is contained in a compact set by Ascoli's theorem.

Lemma 2. *H is continuous in φ.*

Proof. Let $J > 0$ be given, $\|\varphi_i\| \le J$ for $i = 1, 2$, and for a given $\varepsilon > 0$ find the δ of uniform continuity on the region Δ of the proof of Lemma 1 for D. If $\|\varphi_1 - \varphi_2\| < \delta$, then

$$\begin{aligned}|H(\varphi_1)(t) - H(\varphi_2)(t)| &\le \int_0^t \big|D\big(t,s,\varphi_1(s)\big) - D\big(t,s,\varphi_2(s)\big)\big|ds \\ &\le T\varepsilon\end{aligned}$$

so $\|H(\varphi_1) - H(\varphi_2)\| \le T\varepsilon$.

Lemma 3. *There is a $K > 0$ such that any solution of (7), for $0 < \lambda < 1$, satisfies $\|\varphi\| \le K$.*

Proof. Let (I) hold. If φ satisfies (7) on $[0, T]$, then

$$|\varphi(t)| \le \lambda\left[A(T) + \int_0^t \Gamma(T)W(|\varphi(s)|)ds\right] \text{ for } 0 \le t \le T$$

where $A(T) = \max\limits_{0\le t\le T}|a(t)|$. If we define $y(t)$ by

$$y(t) = \lambda\left[A(T) + 1 + \Gamma(T)\int_0^t W(|y(s)|)ds\right]$$

then $y(t) \ge |\varphi(t)|$ on $[0, T]$; clearly, $y(0) > |\varphi(0)|$ so if there is a first t_1 with $y(t_1) = |\varphi(t_1)|$, then a contradiction is clear. But the Conti-Wintner result gives a bound K on $\|y\|$ and so the lemma is true for (I). The argument when (II) holds is identical to the proof of Lemma 3 of Theorem 1.

The conditions of Schaefer's theorem hold and, by Lemma 3, H has a fixed point for $\lambda = 1$.

Remark. To appreciate the power of Schaefer's theorem, compare Theorem 2 with a standard treatment. For example, Corduneanu [3; pp. 95–109] arrives at Theorem 2(I) through several pages of analysis.

COR. Let $a(t)$ be continuous for $0 \le t \le T$ and $D(t, s, x)$ be continuous on $U = \{(t, s, x) \,|\, 0 \le s \le t \le T,\ |x - a(t)| \le J\}$. Then there is a solution of (2) on $[0, \alpha]$ where $\alpha = \min[T, J/M]$ and $M = \max\limits_U|D(t, s, x)|$.

We extend D to a bounded and continuous function on $R \times R \times R^n$ and apply Theorem 2, just as we did in the corollary to Theorem 1.

We now give an example which occupies a significant place in the literature (cf. Gripenberg et al [7; pp. 613–638]); and it also illustrates the change in language from Theorem 1(II) where we ask that $V(t,x)$ be mildly unbounded, and in Theorem 2(II) where we ask that $V(t,x(\cdot))$ be mildly unbounded along a solution of (7).

Example 2. Consider the scalar equation

$$x(t) = b(t) - \int_{-\infty}^{t} D(t,s)g(s,x(s))ds \tag{8}$$

where b, D, and g are continuous, while $xg(t,x) \geq 0$. To specify a solution of (8) we require a bounded and continuous initial function $\varphi : (-\infty, 0] \to R$ such that

$$a(t) := b(t) - \int_{-\infty}^{0} D(t,s)g(s,\varphi(s))ds \text{ is continuous} \tag{9}$$

so that

$$x(t) = a(t) - \int_{0}^{t} D(t,s)g(s,x(s))ds \tag{10}$$

is essentially of the form of (2) when φ is chosen so that $\varphi(0) = a(0)$. We also ask that there exist a continuous function M with

$$-2g(t,x)[x - \lambda a(t)] \leq M(t), g(t,x) \text{ bounded for } t \leq 0 \text{ if } x \text{ is bounded .} \tag{11}$$

But the defining property of this example is

$$D(t,s) \geq 0, \quad D_s(t,s) \geq 0, \quad D_{st}(t,s) \leq 0, \;\; D_t(t,0) \leq 0. \tag{12}$$

This is an infinite delay problem and the following convergence condition is required:

$$\int_{-\infty}^{t} D(t,s)ds \quad \textit{exists.} \tag{13}$$

It is not necessary that $\varphi(0) = a(0)$ but in that case x has a discontinuity:

$$x(0) \textit{ does not equal } \varphi(0). \tag{14}$$

For $0 \le \lambda \le 1$, consider the equation

$$x(t) = \lambda\left[a(t) - \int_0^t D(t,s)g(s,x(s))ds\right] \tag{15}$$

and define

$$V(t,x(\cdot)) = \tag{16}$$
$$\left\{\int_0^t D_s(t,s)\left(\int_s^t \lambda g\big(v,x(v)\big)dv\right)^2 ds + D(t,0)\left(\int_0^t \lambda g(v,x(v))dv\right)^2 + 1\right\}\exp - \int_0^t M(s)ds.$$

Proposition. *If (9) – (14) hold, then there is a solution $x(t,\varphi)$ of (8) on $[0,\infty)$.*

Proof. Let $T > 0$ be given. Because of the form of (10) and Theorem 2, we need only prove that there is a $K > 0$ such that any solution of (15) on $[0,T]$ for $0 < \lambda < 1$ satisfies $|x(t)| \le K$ on $[0,T]$. Here are the steps:

(i) Differentiate V.

(ii) Integrate by parts the term in V' obtained from differentiating the inner integral in the first integral in V.

(iii) Substitute $\lambda a(t) - x(t)$ from (15) into that last term obtained in (ii).

Since $D_{st} \le 0$ we will now have

$$V'(t,x(\cdot)) \le \{-2\lambda g(t,x(t))[x(t) - \lambda a(t)] - M(t)\}\exp - \int_0^t M(s)ds$$

and this is not positive by (11) since $xg(t,x) \ge 0$. Hence,

$$V(t,x(\cdot)) \le V(0,x(\cdot)). \tag{17}$$

Here is the surprising part. It does not appear that V is mildly unbounded; however, along a solution we have

$$(\lambda a(t) - x(t))^2 = \left(\int_0^t \lambda D(t,s)g(s,x(s))ds\right)^2$$

(from (15))

$$= \left(D(t,0)\int_0^t \lambda g(v,x(v))dv + \int_0^t D_s(t,s)\int_s^t \lambda g(v,x(v))dv\,ds\right)^2$$

(upon integration by parts)

$$\le 2\int_0^t D_s(t,s)ds\int_0^t D_s(t,s)\left(\int_s^t \lambda g(v,x(v))dv\right)^2 ds + 2D^2(t,0)\left(\int_0^t \lambda g(v,x(v))dv\right)^2$$

(by Schwarz's inequality) with $exp \int_0^T M(s)ds = U$

$$\le 2[D(t,0)+D(t,t)-D(t,0)]V(t,x(\cdot))U = 2D(t,t)V(t,x(\cdot))U.$$

Hence, $V(t,x(\cdot)) \to \infty$ as $|x(t)| \to \infty$ uniformly for $0 \le t \le T$. In particular, putting this together with (17) yields

$$(\lambda a(t) - x(t))^2 \le 2D(t,t)V(0,x(\cdot))U$$

for $0 \le t \le T$. Hence, there is a K with $|x(t)| \le K$ on $[0,T]$. This will satisfy Lemma 3 of Theorem 2 and the proposition is proved.

4. A finite delay equation. Let $(G, |\cdot|_h)$ denote the Banach space of continuous functions $\psi : [-h,0] \to R^n$ with the supremem norm and consider the system

$$x'(t) = f(t, x_t) \tag{18}$$

where $f : [0,\infty) \times G \to R^n$ is continuous. Here, if $x : [-h,A) \to R^n$ for $A > 0$, then $x_t(s) = x(t+s)$ for $-h \le s \le 0$.

To specify a solution of (18) we require a $t_0 \ge 0$ and a $\tilde{\psi} \in G$. We then seek a solution $x(t) = x(t, t_0, \tilde{\psi})$ with $x_{t_0} = \tilde{\psi}$ and $x(t)$ satisfying (18) on an interval $t_0 < t < \alpha$ for some $\alpha > 0$.

Yoshizawa [17; p. 184] shows that it is sufficient to ask that f be continuous in order to prove existence of such a solution. But if we do not at least ask that f take bounded sets into bounded sets, then a solution can have surprisingly bad behavior, as shown by Hale [9; p. 44]. On the other hand, if f is continuous and locally Lipschitz in x_t, then a contraction mapping argument will quickly lead to a unique local solution.

Hale [9; p. 142] has a version of the next result in case $W(r) = r$. His system is linear so he gets uniqueness as well.

Theorem 3. *Suppose that either:*

(I) there are continuous functions $\Gamma : [0, \infty) \to [0, \infty)$ *and* $W : [0, \infty) \to [1, \infty)$ *with* W *increasing and*

$$|f(t, \psi)| \le \Gamma(t) W(|\psi|_h) \text{ and } \int_0^\infty \frac{ds}{W(s)} = \infty; \text{ or}$$

(II) f *takes bounded sets into bounded sets and there is a continuous scalar functional* $V(t, x_t)$ *which is locally Lipschitz in* x_t*, mildly unbounded in* x_t*, and* $V' \le 0$ *along any solution of (19) (which is displayed in the following proof).*

Then for each $(t_0, \tilde{\psi}) \in [0, \infty) \times G$*, there is a solution* $x(t, t_0, \tilde{\psi})$ *of (18) on* $[t_0, \infty)$*.*

Proof. Let $(t_0, \tilde{\psi})$ be given and let $T > t_0$ be arbitrary. We will show that there is a solution of (18) on $[t_0, T]$ with $x_{t_0} = \tilde{\psi}$.

Let $(C, \|\cdot\|)$ be the Banach space of continuous functions $\varphi : [t_0 - h, T] \to R^n$ with the supremum norm. Consider the equations

$$\begin{aligned} x_{t_0} &= \lambda \tilde{\psi}, \\ x(t) &= \lambda \left[\tilde{\psi}(0) + \int_{t_0}^t f(s, x_s) ds \right] \end{aligned} \tag{19}$$

and define $H : [t_0 - h, T] \to R^n$ by $\varphi \in C$ implies that

$$\begin{aligned} H(\varphi)_{t_0} &= \tilde{\psi}, \\ H(\varphi)(t) &= \tilde{\psi}(0) + \int_{t_0}^t f(s, \varphi_s) ds \text{ for } t_0 \le t \le T. \end{aligned}$$

Lemma 1. *$H : C \to C$ and H maps bounded sets into compact sets.*

Proof. Let $\varphi \in C$. We first show that $f(t,\varphi_t)$ is a continuous function of t so that $H(\varphi)(t)$ will be continuous. Let $\varepsilon > 0$ and $t_1 \in [t_0,T]$ be given; we must find $\delta > 0$ such that $t_2 \in [t_0,T]$ and $|t_1 - t_2| < \delta$ imply that $|f(t_1,\varphi_{t_1}) - f(t_2,\varphi_{t_2})| < \varepsilon$. Now φ is uniformly continuous on $[t_0 - h, T]$ and so for each $\varepsilon_1 > 0$ there is a $\delta > 0$ such that $t_i \in [t_0 - h, T]$ and $|t_1 - t_2| < \delta$ implies that $|\varphi(t_1) - \varphi(t_2)| < \varepsilon_1$; hence, $[t_i \in [t_0,T]$, $-h \le s \le 0$, $|t_1 - t_2| < \delta]$ imply that $|\ \varphi(t_1+s) - \varphi(t_2+s)| < \varepsilon_1$ so $|\varphi_{t_1} - \varphi_{t_2}|_h < \varepsilon_1$. But f is continuous at (t_1,φ_{t_1}) so there is an $\varepsilon_1 > 0$ such that $|t_1 - t_2| < \varepsilon_1$ and $|\varphi_{t_1} - \varphi_{t_2}| < \varepsilon_1$ imply that $|f(t_1,\varphi_{t_1}) - f(t_2,\varphi_{t_2})| < \varepsilon$. Thus, $|t_1 - t_2| < \delta < \varepsilon_1$ implies $|\varphi_{t_1} - \varphi_{t_2}|_h < \varepsilon_1$ so $|f(t_1,\varphi_{t_1}) - f(t_2,\varphi_{t_2})| < \varepsilon$.

Let $J > 0$ be given, $B = \{\varphi \in C \,|\, \|\varphi\| \le J\}$. Then under either (I) or (II) f takes bounded sets into bounded sets so there is a $J^* > 0$ such that $|f(t,\varphi_t)| \le J^*$ for $\varphi \in B$ and $t_0 \le t \le T$. Since $H(\varphi)_{t_0} = \tilde{\psi}$, a fixed uniformly continuous function on $[-h, 0]$ and since $t_0 < t \le T$ implies that $|(H(\varphi)(t))'| = |f(t,\varphi_t)| \le J^*$, it follows that H maps B into an equicontinuous set. Also, $\|H(\varphi)\| \le |\tilde{\psi}|_h + J^*[T - t_0]$. The lemma now follows from Ascoli's theorem.

Lemma 2. *H is continuous.*

Proof. Let $\varphi \in C$ be given. We claim that for each $\varepsilon > 0$ there is a $\delta > 0$ such that $t \in [t_0,T]$, $\psi \in C$, $\|\varphi - \psi\| < \delta$ imply that $|f(t,\varphi_t) - f(t,\psi_t)| < \varepsilon$. If this is false, then there is an $\varepsilon > 0$, $\{t_n\} \subset [t_0,T]$, $\{\psi^{(n)}\} \subset C$ such that $\|\varphi - \psi^{(n)}\| \to 0$ as $n \to \infty$, but $|f(t_n,\varphi_{t_n}) - f(t_n,\psi^{(n)}_{t_n})| \ge \varepsilon$. Now there is a subsequence, say $\{t_n\}$ again, with $\{t_n\} \to t^*$; also, $\varphi_{t_n} \to \varphi_{t^*}$ as $n \to \infty$. Thus, for large n we have

$$|f(t_n,\varphi_{t_n}) - f(t_n,\psi^{(n)}_{t_n}| \le |f(t_n,\varphi_{t_n}) - f(t^*,\varphi_{t^*})| + |f(t^*,\varphi_{t^*}) - f(t_n,\psi^{(n)}_{t_n})| < \varepsilon.$$

This is because f is continuous and the following is small:

$$|\varphi_{t^*} - \psi^{(n)}_{t_n}|_h \le |\varphi_{t^*} - \varphi_{t_n}|_h + |\varphi_{t_n} - \psi^{(n)}_{t_n}|_h.$$

This is a contradiction, so there is a $\delta > 0$ such that $\|\psi - \varphi\| < \delta$ implies that

$$\|H(\varphi) - H(\psi)\| \le \int_{t_0}^{T} |f(s,\varphi_s) - f(s,\psi_s)|ds \le \varepsilon[T - t_0],$$

proving Lemma 2.

Lemma 3. *There is a $K > 0$ such that if $\varphi_{t_0} = \lambda\tilde{\psi}$ and $\varphi(t) = \lambda H(\varphi)(t)$ for $t_0 \le t \le T$ and for some $\lambda \in (0,1)$, then $\|\varphi\| \le K$.*

Proof. If (I) holds then we have

$$\big|\varphi(t)\big| \le \big|\tilde{\psi}\big|_h + \int_{t_0}^{t} \big|f(s,\varphi_s)\big|ds \le \big|\tilde{\psi}\big|_h + \int_{t_0}^{t} \Gamma(s)W(|\varphi_s|_h)ds$$

and, since the right-hand-side is increasing and $\varphi_{t_0} = \tilde{\psi}\lambda$, it follows that

$$\big|\varphi_t\big|_h \le \big|\tilde{\psi}\big|_h + \int_{t_0}^{t} \Gamma(s)W(|\varphi_s|_h)ds.$$

Since W is increasing, $|\varphi_t|_h$ is bounded by any solution $y(t)$ of

$$y(t) = |\tilde{\psi}|_h + 1 + \int_{t_0}^{t} \Gamma(s)W(y(s))ds$$

or of

$$y' = \Gamma(t)W(y), \quad y(t_0) = |\tilde{\psi}|_h + 1.$$

By the Conti-Wintner argument, those solutions are bounded by some K on $[t_0, T]$. Hence, $|\varphi_t|_h \le K$ or $\|\varphi\| \le K$. This proves Lemma 3 when (I) holds. If (II) holds then a parallel argument completes the proof.

By Schaefer's theorem there is a solution of $x = \lambda Hx$ for $\lambda = 1$. □

Cor. *Let $(t_0,\tilde{\psi}) \in [0,\infty) \times G, |\tilde{\psi}|_h = K$, and suppose there is a $J > 0$, $T > t_0$, and $M > 0$ such that $f(t,\psi)$ is continuous on*

$$\Omega = \{(t,\psi)\,|\,t_0 \le t \le T,\ |\psi|_h \le K + J\}$$

with $|f(t,\psi)| \le M$ on Ω. Then there is a solution $x(t) = x(t,t_0,\tilde{\psi})$ of (18) defined for $t_0 \le t \le \alpha$ where $\alpha = \min[T, t_0 + J/M]$.

To prove this corollary, we extend f to a bounded continuous function $F : R\times C \to R^n$ with $F(t,\psi) = f(t,\psi)$ on Ω, following Friedman [6; p. 111]. Then invoke Theorem 3 to get a global solution. The choice of α is verified exactly as in the proof of the corollary of Theorem 1.

Example 3. Consider the scalar equation

$$x' = a(t)W(x(t)) + \int_{t-h}^{t} b(s)W(x(s))ds$$

where a, b, and W are continuous, W is increasing, $|W(x)| \le W(|x|)$, $\int_0^\infty \frac{ds}{W(s)} = \infty$. Then

$$\begin{aligned} x' &= a(t)W(x(t)) + \int_{-h}^{0} b(t+s)W(x(t+s))ds \\ &= a(t)W(x(t)) + \int_{-h}^{0} b(t+s)W(x_t(s))ds \\ &=: f(t,x_t). \end{aligned}$$

Let $\Gamma(t) = |a(t)| + h \max_{t-h\le s\le t} |b(s)|$. Then $|f(t,x_t)| \le \Gamma(t)W(|x_t|_h)$ and the conditions of Theorem 3 are satisfied.

5. Infinite delay. Let $g : (-\infty,0] \to [1,\infty)$ be a continuous non-increasing function and let $(G, |\cdot|_g)$ be the Banach space of all continuous functions

$$\psi : (-\infty,0] \to R^n \text{ for which } |\psi|_g = \sup_{-\infty<s\le 0} |\psi(s)/g(s)|$$

exists as a finite number. If $A > 0$ and if $\varphi : (-\infty, A] \to R^n$ is continuous, then for $0 \le t \le A$ we define $\varphi_t(s) = \varphi(t+s)$ for $-\infty < s \le 0$.

Consider the system of functional differential equations

$$x'(t) = f(t,x_t) \tag{20}$$

and suppose that there is a space $(G, |\cdot|_g)$ for (20) so that $f : [0,\infty)\times G \to R^n$ is continuous in the sense that if $t_1 \ge 0$ and $\psi_1 \in G$, then for each $\varepsilon > 0$ there is a $\delta > 0$ such that $[t \ge 0,\ \psi \in G,\ |t-t_1| < \delta,\ |\psi_1 - \psi|_g < \delta]$ imply that $|f(t_1,\psi_1) - f(t,\psi)| < \varepsilon$.

Clearly, the space $(G, |\cdot|_g)$ is chosen in view of the properties of f. The prototype is

$$x'(t) = p(t, x(t)) + \int_{-\infty}^{t} q(t, s, x(s))ds$$

where p is continuous and $|q(t, s, x)| \le Ke^{-(t-s)}|x|^n$. In this case we can choose $g(t) = 1 + |t^j|$ for any $j > 0$, for example.

The space $(G, |\cdot|_g)$ admits $g(t) \equiv 1$ and then becomes the space of bounded continuous functions with the supremum norm, but it will not satisfy an important property encountered in many problems: when $x : (-\infty, A) \to R^n$ is bounded and continuous, then the mapping $t \to x_t$ need not be continuous, as the example $x(t) = \sin(t^2)$ shows. Virtually always we need to ask that $g(s) \to \infty$ as $s \to -\infty$ and that makes $(G, |\cdot|_g)$ a fading memory space and (20) a fading memory equation. Properties of this space are discussed extensively in Burton [1]. An example of Seifert [15]

$$x'(t) = -2x(t) + x(0)$$

with solutions

$$x(t, x(0)) = (1 + e^{-2t})x(0)/2$$

shows how disasterous the absence of a fading memory can be in asymptotic stability theory.

Notice the progression from Theorem 1 to Theorem 4 of the continuity of f. In Lemma 1 in the proofs of both Theorem 1 and 2 we point out that $f(t, x(t))$ and $D(t, s, x(s))$ are continuous functions of t and (t, s) (by the composite function theorem) when $x(t)$ is continuous. In Lemma 1 of Theorem 3 we prove that $f(t, x_t)$ is a continuous function of t when $x(t)$ is continuous. But that will not work in the infinite delay case and we are compelled to add that as a hypothesis.

For a given $t_0 \ge 0$, $\tilde{\psi}$ *in* G, and λ in $(0, 1)$, we also consider

$$x' = \lambda f(t, x_t), \; x_{t_0} = \lambda\tilde{\psi}. \tag{21}$$

Theorem 4. *Suppose that if $\varphi : R \to R^n$ with $\varphi_t \in G$ for $t \geq 0$, then $f(t, \varphi_t)$ is a continuous function of t. Let $(t_0, \tilde{\psi}) \in [0, \infty) \times G$ be given and suppose that either:*

(I) there are continuous functions $\Gamma : [0, \infty) \to [0, \infty)$ and $W : [0, \infty) \to [1, \infty)$ with W increasing, $|f(t, \psi)| \leq \Gamma(t) W(|\psi|_g)$, and $\int_0^\infty \frac{ds}{W(s)} = \infty$; or

(II) f takes bounded sets into bounded sets and there is a continuous functional $V : [0, \infty) \times G \to [0, \infty)$ which is locally Lipschitz in ψ with $V'(t, x_t) \leq 0$ along any solution of (21) and for each $T > 0$ then $V(t, \psi) \to \infty$ as $|\psi|_g \to \infty$ uniformly for $0 \leq t \leq T$.

Then there is a solution $x(t) = x(t, t_0, \tilde{\psi})$ of (20) on $[t_0, \infty)$ with $x_{t_0} = \tilde{\psi}$.

Proof. Let $T > 0$ be given and let $(C, \|\cdot\|)$ be the Banach space of continuous functions $\varphi : (-\infty, T] \to R^n$ for which $\|\varphi\| := \sup_{t_0 \leq s \leq T} |\varphi_s|_g$ exists as a finite number. Define $H : C \to C$ by $\varphi \in C$ implies that

$$
\begin{aligned}
H(\varphi)_{t_0} &= \tilde{\psi} \\
H(\varphi)(t) &= \tilde{\psi}(0) + \int_{t_0}^t f(s, \varphi_s) ds \text{ for } t \geq t_0.
\end{aligned}
$$

Lemma 1. *$H : C \to C$ and H maps bounded sets into compact sets.*

Proof. Since $\varphi \in C$ implies that $f(t, \varphi_t)$ is a continuous function of t and since $H(\varphi)(t) = \tilde{\psi}(t - t_0)$ for $t < t_0$ with $\tilde{\psi} \in G$, it follows that $H : C \to C$. For a given $J > 0$, if $t_0 \leq t \leq T$ and if $\{\varphi^{(n)}\} \subset C$ is any sequence satisfying $\|\varphi^{(n)}\| \leq J$, then there is an $M > 0$ with $|f(t, \varphi_t^{(n)})\| \leq M$ by either (I) or (II). Hence, by Ascoli's theorem there is a subsequence, say $\{\varphi^{(n)}\}$ again, such that $\{H(\varphi^{(n)})\}$ converges uniformly to a function φ for $t_0 \leq t \leq T$ and with $H(\varphi^{(n)})(t_0) = \tilde{\psi}(0)$, so that $\varphi(t_0) = \tilde{\psi}(0)$. Hence, $\{H(\varphi^{(n)})\}$ converges to φ on $[t_0, T]$ and $H(\varphi^{(n)})(t) = \tilde{\psi}(t - t_0)$ for $t < t_0$. Thus, the bounded set $\{\varphi \in C | \, \|\varphi\| \leq J\}$ is mapped into a compact set.

Lemma 2. *H is continuous.*

Proof. If we replace $|.|_h$ by $|.|_g$, since we assume that $f(t, \varphi_t)$ is a continuous function of t, the proof becomes identical to that of Lemma 2 for Theorem 3.

Lemma 3. *There is a constant K such that if $\varphi = \lambda H(\varphi)$ for $0 < \lambda < 1$, then $\|\varphi\| \leq K$.*

Proof. If $t_0 \leq t \leq T$, then for (I) we have

$$|\varphi(t)| \leq |\tilde{\psi}|_g + \int_{t_0}^{t} \Gamma(s) W(|\varphi_s|_g) ds$$

and the right-hand side is increasing, while $\varphi_{t_0} = \lambda\tilde{\psi}$; hence,

$$|\varphi_t|_g \leq |\tilde{\psi}|_g + \int_{t_0}^{t} \Gamma(s) W(|\varphi_s|_g) ds.$$

Thus, $|\varphi_t|_g$ is bounded by any solution of

$$y(t) = |\tilde{\psi}|_g + 1 + \int_{t_0}^{t} \Gamma(s) W(y(s)) ds.$$

By the Conti-Wintner argument, K exists. When (II) holds there is a parallel argument.

Theorem 4 now follows from Schaefer's result.

Cor. *Let $(t_0, \tilde{\psi}) \in [0, \infty) \times G$, $|\tilde{\psi}|_g = K$, and suppose there are $T > t_0$ and $J > 0$ such that for $(C, \|\cdot\|)$ defined in the proof of Theorem 4,*

(i) if $t_0 \leq t \leq T$ and $\varphi \in C$ with $|\varphi_t|_g \leq J + K$, then $f(t, \varphi_t)$ is a continuous function of t, and

(ii) if $t_0 \leq t \leq T$, $\varphi \in C$, $|\varphi_t|_g \leq K + J$ then $|f(t, \varphi_t)| \leq M$.

Then (19) has a solution $x(t, t_0, \tilde{\psi})$ for $t_0 \leq t \leq \alpha$ where $\alpha = \min[T, t_0 + J/M]$.

The corollary is proved by using the extension theorem [6; p. 111] again.

The most important idea that an example can convey is that of the fading memory and how it can make $f(t, \varphi_t)$ continuous when $\varphi_t \in G$ and $\varphi(t)$ is continuous. Thus, our example is a simple one.

Example 4. Consider the scalar equation

$$x'(t) = \int_{-\infty}^{t} D(t,s) x(s) ds = \int_{-\infty}^{0} D(t, u+t) x_t(u) du =: f(t, x_t)$$

where D is continuous and $|D(t,s)| \leq e^{-(t-s)}$. Select $g(t) = 1 + |t|$. If $\varphi : R \to R^n$ is continuous and if for each $t \geq 0$, $\varphi_t \in G$, then this means that $|\varphi_t|_g = \sup_{-\infty < s \leq 0} |\varphi(t +$

$s)/[1+|s|]|$ exists as a finite number, say k, so that $|\varphi(t+s)| \le k(1+|s|)$ for $s \le 0$. Now φ_t is certainly not continuous in the sense that $|\varphi_t - \varphi_s|_g$ is small for $|t-s|$ small and so we are not depending on a composite function theorem to make $f(t,\varphi_t)$ a continuous function of t. Instead, we rely on the fading memory. For

$$\int_{-\infty}^{0} D(t,u+t)\varphi_t(u)du = \int_{-\infty}^{-P} D(t,u+t)\varphi_t(u)du + \int_{-P}^{0} D(t,u+t)\varphi_t(u)du$$

and

$$\left| \int_{-\infty}^{-P} D(t,u+t)\varphi_t(u)du \right| \le \int_{-\infty}^{-P} e^u k(1+|u|)du \to 0 \text{ as}$$

$P \to \infty$, while $\int_{-P}^{0} D(t,u+t)\varphi_t(u)du$ is a continuous function of t. The tail of φ_t fades in importance. Clearly, the growth condition of (I) is satisfied.

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