BOUNDEDNESS OF SOLUTIONS OF INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT: The equation $x'(t) = -\int_{\alpha(t)}^{t} D(t,s)g(x(s))ds$ is studied with a view to giving conditions to ensure that all solutions are bounded when $\int_0^x g(s)ds$ is not necessarily unbounded with x. The cases $\alpha(t) = -\infty, \alpha(t) = 0,$ *and* $\alpha(t) = t - T$ are studied.

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1. Introduction. The parallel problems

(1) $$x'(t) = -\int_0^t D(t,s)g(x(s))ds$$

and

(2) $$x(t) = a(t) - \int_0^t D(t,s)g(x(s))ds$$

in which $a(t)$, $D(t,s)$, $g(x)$ are continuous scalar functions,

(3) $$D(t,s) \geq 0,\ D_s(t,s) \geq 0,\ D_{st}(t,s) \leq 0, D_t(t,0) \leq 0$$

and

(4) $$xg(x) > 0 \text{ if } x \neq 0$$

have been discussed extensively in the literature ([4], [5], [6; pp. 539–556, 614–631], [7], [10], [11], [12] [14]) with a view to showing that solutions are bounded and tend to zero as $t \to \infty$. Under the stated conditions there is a Liapunov function for each of them, patterned after the one of Levin [11] for (1).

The curious fact about the parallel results is that we seem to always require

$$\int_0^x g(s)ds \to \infty \text{ as } |x| \to \infty \tag{5}$$

in order to prove that solutions of (1) are bounded, but (5) never appears in boundedness results for (2). It is very easy to see why (5) is asked for (1) when we use the Liapunov function

$$\begin{aligned} V(t, x(\cdot)) = \int_0^x g(s)ds + \tfrac{1}{2}D(t,0)\Big(\int_0^t g(x(v))dv\Big)^2 \\ + \tfrac{1}{2}\int_0^t D_s(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds \end{aligned}$$

with $V'_{(1)} \leq 0$; but even when transform methods are used on (1), (5) is still required. Moreover, it is easy to see why (5) is not required for (2) when using its Liapunov function because one finds that under mild conditions that Liapunov function is bounded below by the quantity $(x(t) - a(t))^2$ along any solution, regardless of how small g may be.

Thus, the problem we address here is whether or not (5) can be removed for (1). We show that it frequently can be done. To motive the conditions we first treat the simple finite delay equation

$$x'(t) = -\int_{t-h}^t d(t-s)g(x(s))ds, \quad h > 0, \tag{6}$$

with

$$d(h) = 0,\ d'(t) < 0,\ d''(t) > 0,\ xg(x) > 0 \text{ if } x \neq 0, \tag{7}$$

and all functions in (7) continuous, h a positive constant. This equation has generated some interest, as may be seen in Levin and Nohel [13], Hale [9; pp. 120–3], and Burton

Hatvani [3], usually under the weaker assumption

$$(7^*) \qquad d(h)=0,\ d(t)\geq 0,\ d'(t)\leq 0,\ d''(t)\geq 0,\ xg(x)>0 \text{ if } x\neq 0$$

and, again,

$$(5) \qquad \int_0^x g(s)ds \to \infty \text{ as } |x|\to\infty.$$

In those references, (5) is used to show that all solutions are bounded. The weaker conditions in $(7)^*$ imply that solutions converge to a set satisfying

$$(x')^2 + 2\int_0^x g(s)ds = \text{ constant.}$$

In the next section we show that (5) is not needed at all for (6); (7) alone implies that all solutions are bounded and converge to zero. Condition (7) can be relaxed somewhat and d can be of nonconvolution type; details of such changes are easily gleaned from our subsequent work with (1). But that work is fairly technical and it seems well worth the simplification in (7) to show the process involved.

The last section contains results for infinite delay.

2. Constant delay. All of our results rest on the following theorem found in [1].

Let $H:[0,\infty)\times R^n \to R^n$ be continuous and

$$(i) \qquad x' = H(t,x).$$

Suppose that $V,P,U:[0,\infty)\times R^n\to[0,\infty)$ and $Q:[0,\infty)\times[0,\infty)\times R^n\to[0,\infty)$ are continuous with

$$(ii) \qquad V(t,x) = P(t,x) + U(t,z)$$

where $x=(x_1,\ldots,x_{n-1},z)$ and the derivative of V along a solution of (i) satisfies

$$(iii) \qquad V'(t,x) \leq -Q(P(t,x),t,x)$$

with $Q(P,t,x) > 0$ if $P > 0$ and with Q monotone increasing in P.

Theorem A. *Let (ii) and (iii) hold and suppose there is an $L > 0$ such that if $t_n \to \infty$ and $z_n \to \infty$ then*

(iv) $$U(t_n z_n) \to L, U(t,z) < L \text{ for all } (t,z) \text{ with } z > 0.$$

If $V_1(t,x) = V(t,x) - L$ and if $x(t)$ is a solution of (i) on $[t_0,\infty)$ with $\limsup_{t\to\infty} z(t) = \infty$, then

(v) $$V_1'(t,x) \le -Q(V_1(t,x)/2, t, x) \text{ for } z(t) > 0.$$

The result is stated for ordinary differential equations without a delay, but it is valid for delay equations, as is discussed in [1]. Also, in [1] the condition that Q be monotone increasing in P was inadvertantly left out, but was clearly used in the proof.

Theorem 1. *If (7) holds then every solution of (6) is bounded and converges to zero.*

Proof. To specify a solution of (6) we require a $t_0 \ge 0$ and a continuous initial function $\varphi : [t_0 - h, t_0] \to R$. There is then a solution $x(t,t_0,\varphi)$ with $x(t,t_0,\varphi) = \varphi(t)$ for $t_0 - h \le t \le t_0$, and satisfying (6) on an interval $[t_0,\alpha)$; if the solution remains bounded, then $\alpha = \infty$. Such theory is found in [9], for example.

Let $x(t)$ be a solution of (6) and define

$$V(t,x(\cdot)) = 2\int_0^x g(s)ds - \int_{t-h}^t d'(t-s)\Big(\int_s^t g(x(v))dv\Big)^2 ds$$

so that for $V = V(t,x(\cdot))$ we have

$$\begin{aligned} V' = &-2g(x)\int_{t-h}^t d(t-s)g(x(s))ds - \int_{t-h}^t d''(t-s)\Big(\int_s^t g(x(v))dv\Big)^2 ds \\ &+ d'(h)\Big(\int_{t-h}^t g(x(v))dv\Big)^2 - 2g(x)\int_{t-h}^t d'(t-s)\int_s^t g(x(v))dv\, ds. \end{aligned}$$

An integration by parts of the last term yields

$$\begin{aligned} &-2g(x)\Big[-d(t-s)\int_s^t g(x(v))dv\Big|_{s=t-h}^{s=t} - \int_{t-h}^t d(t-s)g(x(s))ds\Big] \\ &= 2g(x)\int_{t-h}^t d(t-s)g(x(s))ds \end{aligned}$$

so that

$$V' \le -\int_{t-h}^{t} d''(t-s)\left(\int_s^t g(x(v))dv\right)^2 ds \le 0.$$

Since $d'(t)$ and $d''(t)$ are continuous and not zero on $[0,h]$, there is a positive constant k_1 with

$$V' \le k_1 \int_{t-h}^{t} d'(t-s)\left(\int_s^t g(x(v))dv\right)^2 ds \tag{8a}$$

or

$$V' \le -k_1\left[V - 2\int_0^x g(s)ds\right]. \tag{8b}$$

Next, an integration by parts on (6) yields

$$\begin{aligned}(x')^2 &= \left(d(t-s)\int_s^t g(x(v))dv\Big|_{s=t-h}^{s=t} + \int_{t-h}^{t} d'(t-s)\int_s^t g(x(v))dv\,ds\right)^2 \\ &\le \int_{t-h}^{t}[d'(t-s)^2/d''(t-s)]ds \int_{t-h}^{t} d''(t-s)\left(\int_s^t g(x(v))dv\right)^2 ds,\end{aligned}$$

by the Schwarz inequality, so that

$$(x')^2 \le k_2 \int_{t-h}^{t} -d'(t-s)\left(\int_s^t g(x(v))dv\right)^2 ds \tag{9}$$

for some $k_2 > 0$, using once more that d' and d'' are not zero and continuous on $[0,h]$.

Thus, (8a) and (9) will yield

$$V' \le -\gamma(x')^2 \tag{8c}$$

for $\gamma = k_1/k_2$.

Lemma 1. *If $x(t)$ is a bounded solution of (6) then $x(t) \to 0$ as $t \to \infty$.*

Proof. If $x(t) \not\to 0$, then there is an $X \in R$ with $|X| = 3C$, $C > 0$, and a sequence $\{t_n\} \uparrow \infty$ such that $x(t_n) \to X$. Now $V' \le 0$ so for $V = V(t) = V(t, x(\cdot))$ we have $0 \le V(t_{n+1}) \le V(t_n)$. We may suppose, by renumbering, that $|x(t_n) - X| < C$ for all n. Also, for a given n, either:

(a) $|x(t) - X| \le 2C$ for $t_n \le t \le t_n + h$ or

(b) there is an $s_n \in [t_n, t_n + h)$ with $|x(s_n) - x(t_n)| \ge C$.

Moreover, there is a $y > 0$ with $|g(x)| \ge y$ if $|x - X| \le 2C$. If (a) holds then for $t_n \le t \le t_n + h$ we have from (8a) that

$$V' \le -k_1|d'(h)|y^2 \int_0^h s^2 ds =: -\mu < 0.$$

Hence, V decreases at least by μh on $[t_n, t_n + h]$. If (b) holds, then from (8c) we have by integration that

$$\begin{aligned} V(s_n) - V(t_n) &\le -\int_{t_n}^{s_n} \gamma(x'(s))^2 ds \\ &\le -(\gamma/h)\left(\int_{t_n}^{s_n} |x'(s)|ds\right)^2 \end{aligned}$$

(by the Schwarz inequality)

$$\begin{aligned} &\le -(\gamma/h)|x(s_n) - x(t_n)|^2 \\ &\le -\gamma C^2/h. \end{aligned}$$

In either case, $V(t_n) \to -\infty$, a contradiction proving the lemma.

Since $V \ge 2\int_0^x g(s)ds$ and $V' \le 0$, if (5) holds then all solutions are bounded and the proof is complete. Thus, we suppose that (5) fails and, to be definite, let $2\int_0^\infty g(s)ds = L < \infty$. We will show that $x(t)$ is bounded above. A parallel argument would show that $x(t)$ is bounded below if $2\int_0^{-\infty} g(s)ds = L < \infty$.

Thus, we turn to Theorem A and suppose there is a sequence $\{t_n\} \uparrow \infty$ such that $x(t_n) \to \infty$. In Theorem A we have $x = z$ and $n = 1$, while

$$U(t,z) = 2\int_0^x g(s)ds, \; P = P(t) = -\int_{t-h}^t d'(t-s)\left(\int_s^t g(x(v))dv\right)^2 ds,$$

$Q(P,t,x) = k_1 P(t)$ from (8a), so Q is monotone increasing in P. If $V_1 = V - L$, then (iv) holds and for $x(t) > 0$ from (v) and (8a), (8b) we have

$$V_1' \le -(k_1/2)V_1. \tag{8d}$$

We will not use (8d) here because this problem is so simple, but the corresponding step in (1) is crucial and we want the reader to see how Theorem A is used twice.

Now we apply Theorem A again, using (9), (8a), (8c), and the definition of P. We obtain a new Q by noting that

$$V' \le -\sqrt{k_1\gamma}\sqrt{P(t)}|x'| =: -Q(P(t),t) \tag{8e}$$

with Q again being monotone in P. Thus, by Theorem A when $x(t) > 0$ we have

$$V_1' \le -\sqrt{k_1\gamma/2}\sqrt{V_1}|x'|. \tag{8f}$$

If $x(s) > 0$ on an interval $[a,t]$ then

$$-V_1^{1/2}(t_0) \le V_1^{1/2}(t) - V_1^{1/2}(a) \le -\sqrt{k_1\gamma/2^3}|x(t) - x(a)|$$

where $x(t)$ is defined on $[t_0,\infty)$. This means that $x(t)$ is bounded above and the proof is complete.

3. Unbounded delay and necessary conditions. To see some of the essential differences between (1) and (6) we will derive a necessary condition on a simplified form of (1) to ensure that solutions tend to zero. Let

$$D(t,s) = E(s)/C(t) \tag{3a}$$

so that in view of (3) we ask

$$E'(t) \ge 0 \text{ and } C'(t) \ge 0. \tag{3b}$$

Proposition 1. *Let (3a) and (3b) hold for (1). Then (1) has a nonzero solution which tends to zero as $t\to\infty$ only if $C^2(t)E(t)\to\infty$ as $t\to\infty$.*

Proof. From (1) and (3a) we have

$$C(t)x' = -\int_0^t E(s)g(x(s))ds$$

so that

$$C'(t)x' + C(t)x'' + E(t)g(x) = 0$$

or

$$x'' + [C'(t)/C(t)]x' + [E(t)/C(t)]g(x) = 0.$$

Write this as the system

$$\begin{aligned} x' &= y \\ y' &= -[C'(t)/C(t)]y - [E(t)/C(t)]g(x). \end{aligned} \tag{10}$$

Define a function

$$V(t, x, y) = \frac{y^2}{2E(t)} + \frac{1}{C(t)} \int_0^x g(s)ds$$

so that if $(x(t), y(t))$ is a solution of (10) and if $V(t) = V(t, x(t), y(t))$, then

$$\begin{aligned} V'(t) &= -\frac{C'(t)}{C^2(t)} \int_0^x g(s)ds - \frac{E'(t)}{2E^2(t)} y^2 + \frac{g(x)y}{C(t)} \\ &- \frac{C'(t)y^2(t)}{C(t)E(t)} - \frac{E(t)yg(x)}{C(t)E(t)} \\ &\geq \left[\frac{-2C'(t)}{C(t)} - \frac{E'(t)}{E(t)} \right] V(t) \end{aligned}$$

so that if $C^2(t)E(t)$ is bounded, then

$$V(t) \geq V(0) \exp -[ln(C(t)/C(0))^2 + ln(E(t)/E(0))] \geq k$$

for some $k > 0$ since $V(0) > 0$. If $x(t) \to 0$, then for large t we have $y^2(t) \geq kE(t) \geq k_0$ for some $k_0 > 0$. But $x' = y$ implies then that $x(t)$ is unbounded. This completes the proof.

Remark. Much is known about (10). If, for example, (5) holds, $C(t)$ is constant, and $E(t) \to \infty$ in a fairly regular fashion, then it is well-known that $x(t) \to 0$. On the other hand, if $E(t)$ is constant, then $V'(t) \leq -[C'/C]V$; thus, if (5) holds then $x(t)$ is bounded and arguments of the type given in Burton-Grimmer [2] show that $x(t) \to 0$ provided that $C(t) \to \infty$ in a fairly regular fashion. Most of the results on (1) concern convolution type

D and so $E(t)$ and $C(t)$ increase at about the same rate; this example shows that E and C can increase at very different rates. Our up-coming Lemma 2 will show that solutions of (1) tend to zero when $D(t,s)$ decreases to zero as $t \to \infty$ for fixed s. And it would be very interesting to get a parallel result focusing on s.

We now give sufficient conditions for boundedness of solutions of (1) when (5) fails. Suppose there are $k_i > 0$, $T > 0$, a continuous function $f : [0, \infty) \to [0, \infty)$ such that $t \geq T$ and $0 \leq s \leq t$ imply that

$$D^2(t,0) \leq k_1 f(t) D(t,0), \tag{11}$$

$$\int_0^t -[D_s^2(t,s)/D_{st}(t,s)]ds \leq k_2, \quad k = 2\max[k_1, k_2], \tag{12}$$

$$|D_t(t,0)| \geq 2f(t)D(t,0)k_3, \quad k_3 \leq \tfrac{1}{2}, \tag{13}$$

$$-D_{ts}(t,s) \geq f(t)D_s(t,s). \tag{14}$$

Example 1. If $D(t,s) = (t-s+1)^{-n}$, $n > 1$, then (11) – (14) hold with $f(t) = n/(t+1)$, $k_1 = 1/n$, $k_2 = n/(n+1)(n-1)$, $k_3 = 1/2$.

Remark. We will see that f and k_3 are the crucial quantities.

Theorem 2. *If (11) – (14) hold, if $x(t)$ is a solution of (1), if $G(x) := \int_0^x g(s)ds \to L$ as $x \to \infty$, if*

$$V(t, x(\cdot)) = \int_0^x g(s)ds + \tfrac{1}{2}D(t,0)\Big(\int_0^t g(x(s))ds\Big)^2 + \tfrac{1}{2}\int_0^t D_s(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds,$$

then there is a $\gamma > 0$ with

$$V' \leq -\gamma (x')^2, \text{ and if } V_1 = V - L, \text{ then for } x(t) > 0$$

we have

$$V' \le -2k_3 f(t)(V - G(x)), \tag{15}$$

$$V_1' \le -k_3 f(t) V_1, \tag{16}$$

and

$$V_1' \le -\left[k_3/\sqrt{k}\right]\sqrt{f(t)V_1}\,|x'|. \tag{17}$$

Proof. An integration by parts of (1) yields

$$\begin{aligned}(x')^2 &= \left(D(t,s)\int_s^t g(x(v))dv\Big|_{s=0}^{s=t} - \int_0^t D_s(t,s)\int_s^t g(x(v))dv\,ds\right)^2 \\ &\le 2D^2(t,0)\left(\int_0^t g(x(v))dv\right)^2 + 2\left(\int_0^t D_s(t,s)\left[\sqrt{-D_{st}(t,s)}/\sqrt{-D_{st}(t,s)}\right]\int_s^t g(x(v))dv\,ds\right)^2\end{aligned}$$

so that by Schwarz's inequality and (12)

$$(x')^2 \le 2D^2(t,0)\left(\int_0^t g(x(v))dv\right)^2 + 2k_2\int_0^t -D_{st}(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds$$

while by (11)

$$(x')^2 \le k\left[f(t)D(t,0)\left(\int_0^t g(x(v))dv\right)^2 - \int_0^t D_{st}(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds\right]. \tag{18}$$

Next, along a solution of (1) we have

$$\begin{aligned}V' &= -g(x)\int_0^t D(t,s)g(x(s))ds + \tfrac{1}{2}D_t(t,0)\left(\int_0^t g(x(s))ds\right)^2 \\ &\quad + g(x)D(t,0)\int_0^t g(x(s))ds + \tfrac{1}{2}\int_0^t D_{st}(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \\ &\quad + g(x)\int_0^t D_s(t,s)\int_s^t g(x(v))dv\,ds.\end{aligned}$$

Integration by parts of the last term yields

$$g(x)\left[D(t,s)\int_s^t g(x(v))dv\Big|_{s=0}^{s=t} + \int_0^t D(t,s)g(x(s))ds\right]$$

so that

$$V' = \tfrac{1}{2}D_t(t,0)\Big(\int_0^t g(x(s))ds\Big)^2 + \tfrac{1}{2}\int_0^t D_{st}(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds$$

and by (13) we have

$$V' \le k_3\Big[-f(t)D(t,0)\Big(\int_0^t g(x(s))ds\Big)^2 + \int_0^t D_{st}(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds\Big]. \tag{19}$$

This, together with (18) proves that $V' \le -\gamma(x')^2$. Moreover,

$$V' \le -[k_3/\sqrt{k}]\Big[f(t)D(t,0)\Big(\int_0^t g(x(s))ds\Big)^2 - \int_0^t D_{st}(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds\Big]^{1/2}|x'|$$

and from (14) we then have

$$V' \le -\Big[k_3/\sqrt{k}\Big]\Big\{f(t)\Big[D(t,0)\Big(\int_0^t g(x(s))ds\Big)^2 + \int_0^t D_s(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds\Big]\Big\}^{1/2}|x'|$$

or

$$V' \le -\big[k_3/\sqrt{k}\big]\Big[2\Big(V - \int_0^x g(s)ds\Big)\Big]^{1/2}|x'|\sqrt{f(t)}. \tag{20}$$

In the same way, from (19) and (14) we have

$$V' \le k_3\Big[-f(t)D(t,0)\Big(\int_0^t g(x(s))ds\Big)^2 - f(t)\int_0^t D_s(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds\Big]$$

or

$$V' \le -2k_3 f(t)\Big[V - \int_0^x g(s)ds\Big]. \tag{15}$$

By Theorem A, and exactly as in the proof of Theorem 1, if $x(s) > 0$ on an interval $[a,t]$, it follows from (20) that (17) holds, while (15) yields (16). This proves the result.

Our main purpose here is to present boundedness results when (5) fails, but with Theorem 2 it is easy to obtain a result on asymptotic stability whenever $D(t,s)$ is a fading memory kernel. The next result is parallel to Lemma 1 in the proof of Theorem 1.

Lemma 2. *Let (11) – (14) hold. Suppose there is a $\delta > 0$ such that for each $\varepsilon > 0$ there are positive T_1 and T_2 such that if $t_n \to \infty$ and if $t_n + T_1 \le t \le t_n + T_1 + T_2$ then $\int_0^{t_n} D(t,s)ds < \varepsilon$, but $\int_{t_n+T_1}^{t} D(t,s)ds > \delta$ for $t_n + T_1 + T_2 \le t \le t_n + T_1 + T_2 + 1$. Then every bounded solution of (1) tends to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a bounded solution of (1) with $|g(x(t))| < M$ for all t and some $M > 0$. If $x(t) \nrightarrow 0$ then there is an X with $|X| = 3C$, for some $C > 0$, and a sequence $\{t_n\} \uparrow \infty$ such that $x(t_n) \to X$; we may suppose that $|x(t_n) - X| < C$. Now there is a $y > 0$ such that $|g(x)| \ge y$ if $|x - X| \ge 2C$. For this y, δ, and M, choose $\varepsilon > 0$ and T_1, T_2 so that $\varepsilon M < \delta y/2$. For each n either:

(a) $|x(t) - X| < 2C$ for $t_n \le t \le t_n + T_1 + T_2 + 1$ *or*

(b) there is an $s_n \in [t_n, t_n + T_1 + T_2 + 1]$ with $|x(s_n) - X| = 2C$.

From Theorem 2 we have $V' \le -\gamma(x')^2$. If (b) holds then

$$\begin{aligned} V(s_n) - V(t_n) &\le -[\gamma/(T_1 + T_2 + 1)]|x(s_n) - x(t_n)|^2 \\ &\le -[\gamma/(T_1 + T_2 + 1)]4C^2. \end{aligned}$$

If (a) holds then for $t_n + T_1 + T_2 \le t \le t_n + T_1 + T_2 + 1$ we have

$$\begin{aligned} V' \le &- \gamma\Big[-\int_0^t D(t,s)g(x(s))ds\Big]^2 \\ = &- \gamma\Big[-\int_0^{t_n} D(t,s)g(x(s))ds - \int_{t_n}^{t_n+T_1} D(t,s)g(x(s))ds \\ &- \int_{t_n+T_1}^{t} D(t,s)g(x(s))ds\Big]^2. \end{aligned}$$

The first integral on the right is bounded by εM; the second and third integrals have the same sign because $D \ge 0$, while $|x - X| \le 2C$; the third integral (in absolute value) is greater than δy. Hence,

$$V' \le -\gamma(\delta y/2)^2 \text{ on } [t_n + T_1 + T_2, t_n + T_1 + T_2 + 1].$$

Since $V' \le 0$, we then see that $V(t) \to -\infty$ as $t \to \infty$, a contradiction. This completes the proof.

Theorem 3. *Let (11) – (14) hold and let $V_1 = V - L$ where $G(x) := \int_0^x g(s)ds \to L$ as $x \to \infty$. Suppose there is a continuous function $h : [0,\infty) \to [0,\infty)$ with*

$$(21) \qquad [f(t)V_1(t)]^{1/2} \geq h(V_1(t)), \qquad \int_a^b [ds/h(s)] \text{ continuous for } 0 \leq a < b.$$

Then $x(t)$ is bounded above. If $G(x) \to L$ as $x \to -\infty$, then $x(t)$ is bounded below. In particular, if

$$[f(t)V_1(t)]^{1/2} \geq JV_1^{\alpha}(t) \text{ for } \alpha \in (0,1) \text{ and } J > 0,$$

then (21) holds.

Proof. From (17) and (21) we have

$$V_1' \leq -\left[k_3/\sqrt{k}\right] h(V_1(t))|x'|$$

and so if $x(s) > 0$ on an interval $[a,t]$, then

$$\int_a^t [V_1'(s)/h(V_1(s))]ds = \int_{V_1(s)}^{V_1(t)} [ds/h(s)] \leq -\left[k_3/\sqrt{k}\right]|x(t) - x(a)|.$$

Since the integral is continuous and $V_1(a)$ is fixed, while $0 \leq V_1(t) \leq V_1(a)$, the left side is bounded. Thus, $x(t)$ is bounded.

Remark. Condition (21) can be verified by (16). We now give a typical comparison pair illustrating this and the fact that if $f(t)$ satisfies (21), so does a larger one.

Example 2. If (11) – (14) hold, if $k_3 f(t) = \alpha/(t+1)$ for $\alpha > 1$, then (21) is satisfied.

Proof. We have $V_1' \leq -k_3 f(t)V_1 = -\alpha V_1/(t+1)$ and so if $x(s) > 0$ on $[a,t]$ then

$$V_1(t) \leq V_1(a) \exp -\alpha ln[(t+1)/(a+1)] = V_1(a)[(a+1)/(t+1)]^{\alpha}$$

or

$$f(t)^{\alpha} \geq \beta^{2\alpha} V_1(t) \text{ for some } \beta > 0. \text{ Thus, } f(t) \geq \beta^2 V_1^{1/\alpha}(t)$$

and

$$[f(t)V_1(t)]^{1/2} \geq \beta[V_1^{1+(1/\alpha)}]^{1/2} = \beta V_1^{\frac{\alpha+1}{2\alpha}} = \beta V_1^{\gamma}$$

where $\gamma < 1$. This completes the proof.

Theorem 4. *If (11) – (14) hold and if $k_3 f(t) \geq \alpha/(t+1)$ for $\alpha > 1$, then (21) is satisfied and $x(t)$ is bounded.*

Proof. Under the stated conditions, $V_1(t) \leq m(t+1)^{-\alpha}$ for some $m > 0$ and so

$$\left[f(t)V_1(t)\right]^{1/2} \geq \left[\alpha(t+1)^{-1}V_1(t)\right]^{1/2} \geq \left[\alpha(t+1)^{\frac{-\alpha}{\alpha}}V_1(t)\right]^{1/2}$$
$$= \left\{\alpha\left[(t+1)^{-\alpha}\right]^{1/\alpha}V_1(t)\right\}^{1/2} \geq \beta\left[V_1^{1/\alpha}V_1\right]^{1/2} = \beta V_1^{\gamma}$$

where $0 < \gamma < 1$. This completes the proof.

4. Infinite delay. When $D(t,s)$ decays sufficiently fast then (1) can be studied as a limiting equation

(22) $$x' = -\int_{-\infty}^{t} D(t,s)g(x(s))ds$$

and it has also been studied in its present form (cf. Hale [8]) under the assumptions

(3) $$D(t,s) \geq 0, \quad D_s(t,s) \geq 0, \quad D_{st}(t,s) \leq 0,$$

(4) $$xg(x) > 0 \text{ if } x \neq 0,$$

and with (5) holding to ensure boundedness. There are also strong conditions needed in the form of

(23) $\int_{-\infty}^{t}\left[D(t,s) + D_s(t,s)(t-s)^2 + D_{st}(t,s)(t-s)^2\right]ds$ is continuous for $t \geq 0$ *and*,

(24) $$\lim_{s\to-\infty}(t-s)D(t,s) = 0.$$

To specify a solution of (22) we require a $t_0 \geq 0$ and a bounded continuous function $\varphi : (-\infty, t_0] \to R$. There is then a solution $x(t, t_0, \varphi) = \varphi(t)$ if $t \leq t_0$; if the solution remains bounded, then $\alpha = \infty$. Thus, for $t_0 = 0$, (22) can always be considered as a perturbed form of (1):

$$(1^*) \qquad x' = -\int_0^t D(t,s)g(x(s))ds - \int_{-\infty}^0 D(t,s)g(\varphi(s))ds$$

with initial condition $x(0) = \varphi(0)$.

If we try to extend (11) – (14) to (22), consideration of Example 1 quickly indicates difficulties. But if we are willing to ask more than (23) and (24), then it turns out that there is a simple alternative to (5) yielding boundedness. Suppose there are numbers $K > 0$, $p > 1$, $1 < q < 3/2$, such that

$$(25) \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad \int_{-\infty}^t \left[D_s^p(t,s)(t-s)^2/(-D_{st}(t,s))^{p/q}\right]ds \le K.$$

In Example 1, (25) would require $n > 4$, while (24) asks $n \ge 2$. Thus, (25) is a nontrivial increase in requirements.

Theorem 5. *Let $D(t,t) \le k$ for some $k > 0$ and let (3), (4), (23), (24), and (25) hold. Suppose also that there is an $M > 0$ such that if $\int_0^\infty g(s)ds < \infty$ then $|g(x)| < M$ on $[0,\infty)$ and a similar condition on $(-\infty, 0]$. Then all solutions of (22) are bounded.*

Proof. We define

$$(26) \qquad V(t,x(\cdot)) = \int_0^x g(s)ds + \frac{1}{2}\int_{-\infty}^t D_s(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds$$

so that if $x(t)$ is any solution of (22) on $[t_0,\alpha)$ with $x(t) = \varphi(t)$ on $(-\infty,0]$ and φ is bounded, then for $V = V(t) = V(t,x(\cdot))$ we have

$$\begin{aligned} V' = &-g(x)\int_{-\infty}^t D(t,s)g(x(s))ds + \tfrac{1}{2}\int_{-\infty}^t D_{st}(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \\ &+ g(x)\int_{-\infty}^t D_s(t,s)\int_s^t g(x(v))dv\,ds. \end{aligned}$$

Integration by parts of the last term yields

$$\begin{aligned} g(x)&\left[D(t,s)\int_s^t g(x(v))dv\Big|_{s=-\infty}^{s=t} + \int_{-\infty}^t D(t,s)g(x(s))ds\right] \\ &= g(x)\int_{-\infty}^t D(t,s)g(x(s))ds \end{aligned}$$

using (24) and the fact that $x(t)=\varphi(t)$, a bounded initial function on $(-\infty, t_0]$. Thus,

$$V' = \tfrac{1}{2}\int_{-\infty}^{t} D_{st}(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds \le 0. \tag{27}$$

If (5) holds, then (26) and (27) imply that $x(t)$ is bounded. Thus, we suppose (5) fails and, to be definite, let $\int_0^\infty g(s)ds = L < \infty$. We can now say that if $x(t)$ satisfies (22) then there is an $M \ge 1$ with $|g(x(t))| \le M$ by the form of (26), $V' \le 0$, and the assumption that g is bounded if its integral is.

Next, for p and q in (25) and for $C = -D_{st}(t,s)$ for typographical reasons we write

$$\begin{aligned}
&\int_{-\infty}^{t} D_s(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds =: \int_{-\infty}^{t} AB^2 ds \\
&= \int_{-\infty}^{t} AC^{-1/q}B^{2/p}C^{1/q}B^{2/q} ds \\
&\le \Big[\int_{-\infty}^{t} A^p C^{-p/q} B^2 ds\Big]^{1/p}\Big[\int_{-\infty}^{t} CB^2 ds\Big]^{1/q} \\
&\le \Big(KM^2\Big)^{1/p}\Big[\int_{-\infty}^{t} -D_{st}(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds\Big]^{1/q}.
\end{aligned}$$

Thus, with (27) we now have

$$V' \le -\tfrac{1}{2}(KM^2)^{-q/p}\Big[\int_{-\infty}^{t} D_s(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds\Big]^q. \tag{28}$$

Now integration by parts of (22) yields

$$\begin{aligned}
(x')^2 &= \Big(D(t,s)\int_s^t g(x(v))dv\Big|_{s=-\infty}^{s=t} - \int_{-\infty}^{t} D_s(t,s)\int_s^t g(x(v))dv\, ds\Big)^2 \\
&\le \int_{-\infty}^{t} D_s(t,s)ds \int_{-\infty}^{t} D_s(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds
\end{aligned}$$

since the first term in the integration by parts is zero by (24) and then use of Schwarz's inequality. Thus,

$$(x')^2 \le k\int_{-\infty}^{t} D_s(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds. \tag{29}$$

From (28), (29), and V we then have

$$V' \le -\big(1/2\sqrt{k}\big)(KM^2)^{-q/p}|x'|\Big[2(V - \int_0^x g(s)ds)\Big]^{q-(1/2)}.$$

If $\int_0^\infty g(s)ds = L < \infty$, if $x(s) > 0$ on $[a, t]$, if $V_1 = V - L$, then by Theorem A we have

$$V_1' \leq -\beta |x'| V_1^{q-(1/2)}, \quad \beta > 0. \tag{30}$$

Since $q < 3/2$, an integration will yield $x(t)$ bounded above. Similar analysis will yield $x(t)$ bounded below and complete the proof.

Remark. A result for (22) parallel to Lemma 2 can be formulated and proved. In the same way, if we write (22) as (1)* and if $D(t, s)$ is a fading memory kernel such that for each bounded initial function φ, then

$$p(t) := -\int_{-\infty}^{0} D(t, s) g(\varphi(s)) ds \in L^1[0, \infty),$$

we can treat (22) as (1) using

$$V_2(t, x(\cdot)) = [V(t, x(\cdot)) + M] \exp - \int_0^t p(s) ds$$

where $|g(x(t))| \leq M$ and V is the Liapunov function used for (1).

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