BOUNDEDNESS AND PERIODICITY IN INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

T.A. Burton

Department of Mathematics
Southern Illinois University
Carbondale, Illinois 62901

AMS(1980)Subject Classification: 45D05, 45J05, 45M99

Key words and phrases: Integral equation, integrodifferential equation, periodic solution

1. Introduction. The first four sections of this paper are concerned with an integral equation

$$(1) \qquad x(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(s,x(s))ds$$

where $a : R \to R$, $D : R \times R \to R$, $g : R \times R \to R$ are all continuous, g is bounded for x bounded, if $0 \le \lambda \le 1$

$$(2) \qquad \text{there is an } M > 0 \text{ with } 2g(t,x)[\lambda a(t) - x] \le -|xg(t,x)| + M$$

(this condition can be weakened using (20) as our final remark shows),

$$(3) \qquad \text{there is a } K > 0 \text{ with } |g(t,x)| \le K + |xg(t,x)|,$$

(this is notation, rather than an assumption)

$$(4) \qquad D_s(t,s) \ge 0, D_{st}(t,s) \le 0, \text{ both continuous,}$$

and D satisfies conditions to be stated later.

The last section is concerned with an equation

$$(*) \qquad x'(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(x(s))ds$$

under similar conditions.

The object is to give conditions to ensure that there is a periodic solution; in the process we also give conditions to ensure boundedness and asymptotic behavior of solutions.

Both of these equations have been studied intensively under (4). Hale [5] considers

$$x'(t) = -\int_{-\infty}^{t} d(t-s)g(x(s))ds$$

with $d(t) > 0$, $d'(t) < 0$, $d''(t) \ge 0$ and obtains information on limit sets of solutions. Frequently the equation studied is

$$(L) \qquad x(t) = a(t) - \int_{0}^{t} D(t,s)g(s,x(s))ds$$

or

$$(L^*) \qquad x'(t) = a(t) - \int_0^t D(t,s)g(x(s))ds$$

where $D(t,s) \geq 0$ and (4) holds. Such work may cover (1) and (*) since they are written this way as follows. To specify a solution of (1) or (*) we require a bounded continuous function $\varphi : (-\infty, 0] \to R$ and ask that D be so small that

$$U(t,\varphi) := \int_{-\infty}^0 D(t,s)g(s,\varphi(s))ds$$

and

$$R(t,\varphi) := \int_{-\infty}^0 D(t,s)g(\varphi(s))ds$$

are continuous for $t \geq 0$. Then U and R are treated as part of $a(t)$ so that we have

$$(1) \qquad x(t) = a(t) - U(t,\varphi) - \int_0^t D(t,s)g(s,x(s))ds$$

and

$$(*) \qquad x'(t) = a(t) - R(t,\varphi) - \int_0^t D(t,s)g(x(s))ds.$$

These equations then have solutions $x(t,\varphi)$ which agree with φ on $(-\infty, 0]$ and satisfy the equations on an interval $[0,\alpha)$, provided in (1) that $\varphi(0) = a(0) - U(0,\varphi)$; if the solutions remain bounded then $\alpha = \infty$ (cf. [1; pp. 66–79], for example). Theorems 1 and 3 treat only solutions which are continuous in that they match up with their initial functions. A continuity argument shows that such solutions do exist, while Theorem 2 explicitly shows such existence.

Clearly, convergence conditions on the integral of D are required for (1) and (*) which may not be needed in (L) and (L^*). But (1) and (*) are set up for periodic theory, whereas (L) and (L^*) give rise to asymptotically periodic solutions, as is seen in [3; p. 631], and is to be expected from the theory of limiting equations.

The pioneering work on (L) and (L^*) was done by Levin ([6], [7], [8]). Halanay [4] followed Levin and treated (*) in the convolution case using transform theory. That work was corrected and extended by MacCamy and Wong [9], starting a long line of investigation into positive kernel theory which is further discussed by Corduneanu [2] and is summarized both for (L) and (L^*) by Gripenberg et al [3; pp. 539–556, 614–631].

That last reference contains some very good work on boundedness for (L) (and, hence, for (1)), but their boundedness results for (*) seem to always ask that $\int_0^x g(s)ds \to \infty$ as $|x| \to \infty$, a condition we avoid in our last section on a priori bounds for (*). But the existence of periodic solutions is far more challenging than boundedness, as may be seen from the result in [3; p. 631] which requires stringent conditions including that D be of convolution type and that g be strictly monotone.

Periodic solutions of integral equations with infinite delay, and more generally for neutral functional differential equations, have been obtained recently by several authors when it is assumed that solutions are uniformly bounded and uniformly ultimately bounded. References may be found in Wu and Xia [11]. Those methods, conditions, and conclusions are far different from those found here. In particular, equations considered need to have continuous solutions for a given initial function. Our periodic results do not have that requirement. Our Theorems 1 and 3 deal with solutions having initial functions which make the resulting solution continuous. But the proof of Theorem 2 deals with a priori bounds on periodic solutions which would automatically be continuous on the whole real line. Moreover, our equations may have periodic solutions as well as unbounded solutions.

We generally ask less on a sign condition on $g(t,x)$ than is asked in [3], but the conditions are fairly close. Our main results here are contained in Theorems 2 and 4 on the existence of periodic solutions under conditions much weaker than those in [3;p.631].

Our proofs are by means of Liapunov functions. It is to be noted in Theorems 1 and 2 that the technique introduced allows us to prove that V is bounded without ever asking a growth condition on g that makes the derivative of V negative in any region. Equations

(5) and (6) are required in order that (1), V, and V' will make sense for bounded solutions. And that will be enough to make V itself bounded along a solution.

Boundedness. We require that

$$\int_{-\infty}^{t} \left[|D(t,s)| + D_s(t,s)(t-s)^2 + |D_{st}(t,s)|(t-s)^2\right] ds \tag{5}$$

is continuous,

$$\lim_{s\to-\infty} (t-s)D(t,s) = 0 \text{ for fixed } t, \tag{6}$$

(7) there are constants A and B with

(i) $\int_{-\infty}^{t} D_s(t,s)(t-s)^2 ds \le A$,

(ii) $\int_{-\infty}^{t} D_s(t,s) ds \le B$.

Theorem 1. *If (2) – (7) hold then $(a(t)-x(t))^2$ is bounded for any continuous solution $x(t)$ of (1) on $[0,\infty)$.*

Proof. Let us modify Levin's [7] Liapunov functional and define

$$V(t,x(\cdot)) = \int_{-\infty}^{t} D_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds.$$

If $x(t)$ is any continuous solution of (1) with bounded initial function then V is defined and

$$\begin{aligned} V'(t,x(\cdot)) = &\int_{-\infty}^{t} D_{st}(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds \\ &+ 2g(t,x(t))\int_{-\infty}^{t} D_s(t,s)\int_s^t g(v,x(v))dv\, ds. \end{aligned}$$

If we integrate the last term by parts we have

$$2g(t,x(t))\left[D(t,s)\int_s^t g(v,x(v))dv\Big|_{-\infty}^{t} + \int_{-\infty}^{t} D(t,s)g(s,x(s))ds\right].$$

The first term vanishes at both limits by (6); the first term of V' is not positive; and if we use (1) on our last term then we obtain

$$\begin{aligned} V'(t,x(\cdot)) &\le 2g(t,x(t))[a(t)-x(t)] \\ &\le -|xg(t,x)| + M \end{aligned}$$

by (2). By the Schwarz inequality

$$\begin{aligned} &\left(\int_{-\infty}^{t} D_s(t,s)\int_s^t g(v,x(v))dv\,ds\right)^2 \le \\ &\int_{-\infty}^{t} D_s(t,s)ds \int_{-\infty}^{t} D_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds \\ &\le BV(t,x(\cdot)) \end{aligned}$$

by (7) (ii). We have just integrated the left side by parts, obtaining

$$\left(\int_{-\infty}^{t} D(t,s)g(s,x(s))ds\right)^2$$

so that by (1) we now have $(a(t)-x(t))^2 \le BV(t,x(\cdot))$. We will complete the proof by showing that V is bounded.

If V is not bounded, then there is a sequence

$$\{t_n\} \uparrow \infty \text{ with } V(t_n,x(\cdot)) \ge V(s,x(\cdot)) \text{ for } 0 \le s \le t_n.$$

Thus,

$$0 \le V(t_n,x(\cdot)) - V(s,x(\cdot)) \le -\int_s^{t_n} |x(v)g(v,x(v))|dv + M(t_n - s)$$

and so

$$\begin{aligned} V(t_n,x(\cdot)) &= \int_{-\infty}^{0} D_s(t_n,s)\left(\int_s^{t_n} g(v,x(v))dv\right)^2 ds \\ &\quad + \int_0^{t_n} D_s(t_n,s)\left(\int_s^{t_n} g(v,x(v))dv\right)^2 ds \\ &\le 2\int_{-\infty}^{0} D_s(t_n,s)\left[\left(\int_s^0 g(v,x(v))dv\right)^2 + \left(\int_0^{t_n} g(v,x(v))dv\right)^2\right] ds \\ &\quad + \int_0^{t_n} D_s(t_n,s)\left[\int_s^{t_n}\Big(K + |x(v)g(v,x(v))|\Big)dv\right]^2 ds \end{aligned}$$

by (3). If we let φ be the initial function for x and denote by $\|g(.,\varphi)\|$ the supremum of $|g(s,\varphi(s))|$ for $-\infty < s \le 0$, then we have

$$\begin{aligned}
V(t_n, x(\cdot)) \le 2\|g(.,\varphi)\|^2 &\int_{-\infty}^{0} D_s(t_n,s)s^2 ds \\
&+ 2\int_{-\infty}^{0} D_s(t_n,s)\big[Kt_n + Mt_n\big]^2 ds \\
&+ \int_{0}^{t_n} D_s(t_n,s)\big[K(t_n-s) + M(t_n-s)\big]^2 ds \\
&\le 2\|g(.,\varphi)\|^2 \int_{-\infty}^{0} D_s(t_n,s)s^2 ds + 2(K+M)^2 \int_{-\infty}^{0} D_s(t_n,s)t_n^2 ds \\
&+ [K+M]^2 \int_{0}^{t_n} D_s(t_n,s)(t_n-s)^2 ds \\
&\le 2\|g(.,\varphi)\|^2 \int_{-\infty}^{0} D_s(t_n,s)s^2 ds + 2(K+M)^2 A + (K+M)^2 A
\end{aligned}$$

by (7)(i) since $\int_{-\infty}^{0} D_s(t_n,s)t_n^2 ds \le \int_{-\infty}^{0} D_s(t_n,s)(t_n-s)^2 ds$ and $\int_{0}^{t_n} D_s(t_n,s)(t_n-s)^2 ds \le \int_{-\infty}^{t_n} D_s(t_n,s)(t_n-s)^2 ds$. We obtain

$$V(t_n, x(\cdot)) \le 3A(K+M)^2 + 2A\|g(.,\varphi)\|.$$

$\square$

3. Periodicity. We now suppose that there is a $T > 0$ such that

$$D(t+T, s+T) = D(t,s) \tag{8}$$

and that

$$a(t+T) = a(t), g(t+T, x) = g(t,x). \tag{9}$$

We also suppose that there is a $Q > 0$ such that

$$\int_{-\infty}^{t} |D(t_1,s) - D(t_2,s)| ds \le Q|t_1 - t_2|, \quad \text{if} \quad 0 \le t_1 \le t_2 \le T. \tag{10}$$

It will follow from (5) and (8) that (7) holds and also that

$$\text{for each fixed } p \ge 0 \text{ then } \int_{-\infty}^{p} D_s(t,s)(t-s)^2 ds \to 0 \text{ as } t \to \infty. \tag{11}$$

Theorem 2. *If (2) – (10) hold, then (1) has a T-periodic solution.*

Proof. Let $0 \le \lambda \le 1$ and write

$$(1_\lambda) \qquad x(t) = \lambda a(t) - \int_{-\infty}^{t} D(t,s)\lambda g(s,x(s))ds.$$

Lemma 1. *There is a $J > 0$ such that any T-periodic solution x of (1_λ) satisfies $\|x\| \le J$ where $\|\cdot\|$ is the supremum norm.*

Proof. Suppose that x is a T-periodic solution of (1_λ) with $\|x\| = X$. Since x is T-periodic, so is $V(t,x(\cdot))$ where V is now defined by

$$V(t,x(\cdot)) = \lambda^2 \int_{-\infty}^{t} D_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds.$$

That is, in the proof of Theorem 1 we now identify g as λg and $a(t)$ as $\lambda a(t)$ so that we will have

$$V'(t,x(\cdot)) \le -\lambda g(t,x)[x - \lambda a(t)] \le -\lambda |xg(t,x)| + \lambda M.$$

Since V is T-periodic there is a sequence $\{t_n\} \uparrow \infty$ with $V(t_n,x(\cdot)) \ge V(s,x(\cdot))$ for $s \le t_n$. Thus,

$$0 \le V(t_n,x(\cdot)) - V(s,x(\cdot)) \le -\lambda \int_s^{t_n} |x(v)g(v,x(v))|dv + \lambda M(t_n - s).$$

Therefore,

$$\begin{aligned} V(t_n,x(\cdot)) &\le \lambda^2 \int_{-\infty}^{t_n} D_s(t_n,s)\left(\int_s^{t_n} \big[K + |x(v)g(v,x(v))|\big]dv\right)^2 ds \\ &\le \lambda^2 \int_{-\infty}^{t_n} D_s(t_n,s)(K(t_n - s) + M(t_n - s))^2 ds \\ &\le \lambda^2 (K+M)^2 A. \end{aligned}$$

Since $(\lambda a(t) - x(t))^2 \le BV(t,x(\cdot))$ there is a $J > 0$ with $\|x\| \le J$ and Lemma 1 is true.

Now, let $(P, \|\cdot\|)$ be the Banach space of continuous T-periodic functions with the supremum norm and define a homotopy H by $\varphi \in P$ and $0 \le \lambda \le 1$ implies that

$$H(\lambda,\varphi)(t) = \lambda\Big[a(t) - \int_{-\infty}^{t} D(t,s)g(s,\varphi(s))ds\Big]. \tag{12}$$

Lemma 2. *$H : P \to P$ and H maps bounded sets into compact sets.*

Proof. A change of variable shows that if $\varphi \in P$ then $H(\lambda,\varphi)(t+T) = H(\lambda,\varphi)$. Since $\varphi \in P$, there is a J with $\|\varphi\| = J$ and a Y with $\|g(t,\varphi)\| = Y$. Thus, if $0 \le t_1 \le t_2 \le T$, then

$$\begin{aligned}
&\big|H(\lambda,\varphi)(t_1) - H(\lambda,\varphi)(t_2)\big| \le \lambda\Big[\big|a(t_1) - a(t_2)\big| \\
&+ \Big|\int_{-\infty}^{t_1}\Big[D(t_1,s) - D(t_2,s)\Big]g(s,\varphi(s))ds\Big| \\
&+ \Big|\int_{t_1}^{t_2} D(t_2,s)g(s,\varphi(s))ds\Big|\Big] \\
&\le |a(t_1) - a(t_2)| + (YQ + YE)|t_1 - t_2|
\end{aligned}$$

where $E = \sup\limits_{\substack{0\le s\le T\\ 0\le t_2\le T}} |D(t_2,s)|$. Thus, H is equi-continuous, while boundedness is clear. So, by the Ascoli theorem, it lies in a compact set. □

Lemma 3. *$H(\lambda,\varphi)$ is continuous in φ.*

Proof. Let $\varphi_1, \varphi_2 \in P$ so that $\|\varphi_i\| < J$ for some $J > 0$. Then by the uniform continuity of $g(t,x)$ and by (5) and (8) we can make

$$|H(\lambda,\varphi_1)(t) - H(\lambda,\varphi_2)(t)| = \lambda\Big|\int_{-\infty}^{t} D(t,s)\big[g(s,\varphi_1(s)) - g(s,\varphi_2(s))\big]ds\Big|$$

as small as we please. □

Theorem (Schaefer [10]). *Let $(P, \|\cdot\|)$ be a normed space, G a continuous mapping of P into P which is compact on each bounded subset of P. Then either*

(i) the equation $\varphi = \lambda G\varphi$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions φ, for $0 < \lambda < 1$, is unbounded.

Theorem 2 now follows from Schaefer's result with $\lambda G = H(\lambda, \varphi)$ since a solution of $\varphi = H(\lambda, \varphi)$ solves (1_λ) and those solutions are bounded by Lemma l.

4. Asymptotic behavior. We now suppose that

$$a(t) \to 0 \text{ as } t \to \infty, \int_0^t D_s(t,s)(t-s)ds \leq A, \tag{13}$$

$$2g(t,x)[x - a(t)] \leq -|xg(t,x)| + M(t), M(t) \downarrow 0 \text{ as } t \to \infty, \tag{14}$$

for each $J > 0$ and $\delta > 0$ there is an $S > 0$ such that

$$|x| \leq J \text{ implies } |g(t,x)| \leq \delta + S|x|. \tag{15}$$

Theorem 3. *Let (4) – (7), (11), and (13) – (15) hold. Then every continuous solution of (1) with bounded initial function tends to zero as $t \to \infty$.*

Proof. Since (14) and (15) imply (2) and (3), by Theorem 1 all solutions are bounded. Let $x(t)$ be a fixed solution so that $|x(t)| \leq J$ and $|g(t, x(t))| \leq Y$ for J and Y positive numbers. From the proof of Theorem 1, if $x(t) \nrightarrow 0$ then $V(t, x(\cdot)) \nrightarrow 0$ since $a(t) \to 0$ and $(a(t) - x(t))^2 \leq BV(t, x(\cdot))$. Thus, we let $\limsup_{t\to\infty} V(t, x(\cdot)) = P > 0$. Then for each $\varepsilon > 0$ there is a $K > 0$ and a sequence $\{t_n\} \uparrow \infty$ with $V(t_n, x(\cdot)) \geq V(s, x(\cdot)) - \epsilon$ for $K \leq s \leq t_n$. Thus

$$\begin{aligned} -\varepsilon \leq V(t_n, x(\cdot)) - V(s, x(\cdot)) &\leq M(K)(t_n - s) \\ &\quad - \int_s^{t_n} |x(v)g(v, x(v))|dv \end{aligned}$$

or

$$\int_s^{t_n} |x(v)g(v, x(v))|dv \leq \varepsilon + M(K)(t_n - s). \tag{16}$$

Let δ and ε be small positive numbers and find K from the preceeding paragraph. Then

$$V(t_n, x(\cdot)) \le \int_{-\infty}^{K} D_s(t_n, s)(t_n - s)^2 ds Y^2 \\ + \int_K^{t_n} D_s(t_n, s)(t_n - s) \int_s^{t_n} g^2(v, x(v)) dv ds$$

Denote the first term on the right by $L(n)$ and have

$$\begin{aligned} V(t_n, x(\cdot)) &\le L(n) \\ &+ \int_K^{t_n} D_s(t_n, s)(t_n - s) \int_s^{t_n} |g(v, x(v))| \big[\delta + S|x(v)|\big] dv\, ds \\ &\le L(n) + \int_K^{t_n} D_s(t_n, s)(t_n - s) \left[Y\delta(t_n - s) + \int_s^{t_n} S|x(v) g(v, x(v))| dv \right] ds \\ &\le L(n) + \delta Y \int_K^{t_n} D_s(t_n, s)(t_n - s)^2 ds \\ &+ S \int_K^{t_n} D_s(t_n, s)(t_n - s)\big[\varepsilon + M(K)(t_n - s)\big] ds \\ &= L(n) + \delta Y \int_K^{t_n} D_s(t_n, s)(t_n - s)^2 ds \\ &+ \varepsilon S \int_K^{t_n} D_s(t_n, s)(t_n - s) ds + SM(K) \int_K^{t_n} D_s(t_n, s)(t_n - s)^2 ds \\ &\le L(n) + \delta Y A + \varepsilon S A + SM(K) A \end{aligned}$$

Now, for a given $\delta > 0$, find S in (15). Then choose ε so small that $\varepsilon S A < \delta$. Next, choose K so large that $SM(K)A < \delta$. Now $L(n) \to 0$ as $t_n \to \infty$, so that as $\delta \to 0$, we see that $V(t_n, x(\cdot)) \to 0$ as $t_n \to \infty$. Hence, $x(t) \to 0$. □

5. Periodic solutions of an integrodifferential equation. Consider the equation

$$x' = a(t) - \int_{-\infty}^{t} D(t, s) g(x(s)) ds \tag{17}$$

with $g : R \to R$, g, a, and D continuous,

$$D(t, s) \ge 0, D_s(t, s) \ge 0, D_{st}(t, s) \le 0, \tag{18}$$

and

$$a(t + T) = a(t), D(t + T, s + T) = D(t, s). \tag{19}$$

Levin introduced a Liapunov functional whose derivative was essentially $\int_{-\infty}^{t} D_{st}(t,s)(\int_{s}^{t} g(x(v))dv)^2 ds$ and it was then a great struggle to show that $x(t) \to 0$ even when $a(t)$ is very small. Our work here is motivated by the idea that this integral can be used very effectively whenever

(20) $$-\int_{-\infty}^{t} \left[D_s^2(t,s)/D_{st}(t,s)\right] ds \le L \text{ for some } L > 0.$$

This is a mild condition as can be seen from $D(t,s) = (t-s+1)^{-3}$, in which case that integrand is $\frac{3}{4}(t-s+1)^{-3}$ and $L = \frac{3}{8}$.

We repeat some of our earlier conditions for easy reference here and we add two more conditions. Let

(21) $$g(x)/x^2 \to 0 \text{ as } |x| \to \infty, xg(x) > 0 \text{ if } |x| > U \text{ for some } U > 0,$$

(22) $$\int_{-\infty}^{t} \left[D(t,s) + D_s(t,s)(t-s)^2 + |D_{st}(t,s)|(t-s)^2\right] ds \text{ be continuous}$$

(23) $$\lim_{s \to -\infty} (t-s)D(t,s) = 0 \text{ for fixed } t,$$

(24) $$\int_{0}^{T} a(s)ds = 0$$

and recall that by (19) and (22) we have

(25) $$\text{there is a } B > 0 \text{ with } \int_{-\infty}^{t} D_s(t,s)ds \le B.$$

Theorem 4. *Let D satisfy (10), and let (18) – (24) hold. Then (17) has a T-periodic solution.*

Proof. To construct a homotopy for (17), let $0 \le \lambda \le 1$ and write

(17_λ) $$x' + x = \lambda x - \lambda \int_{-\infty}^{t} D(t,s)g(x(s))ds + \lambda a(t).$$

If x is a T-periodic solution of (17_λ), then multiply (17_λ) by e^t and integrate from $-\infty$ to t obtaining

$$x(t) = \lambda \int_{-\infty}^{t} e^{-(t-v)} \left[x(v) - \int_{-\infty}^{v} D(v,s)g(x(s))ds + a(v) \right] dv. \tag{26}$$

Lemma 1. *There is a $K > 0$ such that any T-periodic solution x of (26) satisfies $\|x\| \le K$.*

Proof. Notice that if x is a T-periodic solution of (17_λ), then $x'(t)$ and $a(t)$ both have mean value zero, as does the remaining term

$$-(1-\lambda)x(t) - \lambda \int_{-\infty}^{t} D(t,s)g(x(s))ds.$$

Since $D(t,s) \ge 0$ and $xg(x) \ge 0$ if $|x| \ge U$, that remaining term can not have mean value zero for $0 < \lambda < 1$ if $|x(t)| > U$ for all t. Hence,

$$\text{there is a } t_1 \in [0,T] \text{ with } |x(t_1)| \le U. \tag{27}$$

Define

$$V(t,x(\cdot)) = 2\int_0^x g(s)ds + \lambda \int_{-\infty}^{t} D_s(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \tag{28}$$

so that along the T-periodic solution $x(t)$ of (17_λ) we have

$$\begin{aligned} V'(t,x(\cdot)) &= \lambda \int_{-\infty}^{t} D_{st}(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \\ &\quad + 2g(x)\left[(\lambda-1)x - \lambda\int_{-\infty}^{t} D(t,s)g(x(s))ds + \lambda a(t)\right] \\ &\quad + 2\lambda g(x)\int_{-\infty}^{t} D_{st}(t,s)\int_s^t g(x(v))dv\,ds. \end{aligned}$$

If we integrate the last term by parts we get

$$2\lambda g(x)\int_{-\infty}^{t} D(t,s)g(x(s))ds \text{ so that}$$

$$V'(t,x(\cdot)) = \lambda \int_{-\infty}^{t} D_{st}(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds$$

$$(29) \qquad + 2g(x)\big[(\lambda-1)x + \lambda a(t)\big].$$

Now for $\lambda > 0$ we have

$$\begin{aligned}(\lambda a(t) + (\lambda-1)x(t) - x'(t))^2 &= \Big(\lambda \int_{-\infty}^{t} D(t,s)g(x(s))ds\Big)^2 \\ &= \Big(\lambda \int_{-\infty}^{t} D_s(t,s)\int_s^t g(x(v))dv\, ds\Big)^2 \\ &= \Big[\lambda \int_{-\infty}^{t} \big[D_s(t,s)(-D_{st}(t,s))^{1/2}(-D_{st}(t,s))^{-1/2}\int_s^t g(x(v))dv\, ds\Big]^2 \\ &\le -\lambda^2 \int_{-\infty}^{t} \big[D_s^2(t,s)/D_{st}(t,s)\big]ds \int_{-\infty}^{t} -D_{st}(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds \\ &\le -L\lambda^2 \int_{-\infty}^{t} D_{st}(t,s)\Big(\int_s^t g(x(v))dv\Big)^2 ds.\end{aligned}$$

Using this in (29) yields

$$(30) \qquad V'(t,x(\cdot)) \le -\frac{1}{\lambda L}\Big((\lambda-1)x(t) - x'(t) + \lambda a(t)\Big)^2 + 2\lambda a(t)g(x) + Q$$

where

$$(31) \qquad Q = \max 2g(x)(\lambda-1)x \text{ for } |x| \le U.$$

Now $1-\lambda = d \ge 0$ so we write (31) as

$$V'(t,x(\cdot)) \le Q + 2\lambda\|a\|\|g(x)\| - \frac{e^{-2dt}}{\lambda L}\Big(|(xe^{dt})'| - |\lambda a(t)e^{dt}|\Big)^2.$$

Since x is T-periodic, so is V, and an integration yields

$$\begin{aligned}0 = V(T,x(\cdot)) - V(0,x(\cdot)) &\le 2\lambda\|a\|\|g(x)\|T + QT \\ &\quad - \frac{e^{-2dT}}{\lambda LT}\Big(\int_0^T |(xe^{dt})'|dt - \int_0^T |\lambda a(t)e^{dt}|dt\Big)^2.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T |(xe^{dt})'|dt &\le \int_0^T |\lambda a(t)e^{dt}|dt \\ &\quad + [\lambda LTe^{2dT}(QT + 2\lambda\|a\|\|g(x)\|T)]^{1/2}\end{aligned}$$

and so by (27) since $|x(t_1)| = U$ we have $t_2 \in [0,T]$ with

$$\|x\| = |x(t_2)| \le Ue^{dT} + \lambda T\|a\|e^{dT} + [\lambda LT^2 e^{2dT}(Q + 2\lambda\|a\|\|g(x)\|)]^{1/2}.$$

If $\|g(x)\| = |g(x(t_3))|$ for some $t_3 \in [0,T]$ then $|x(t_2)| \ge |x(t_3)|$ and so

$$|x(t_3)| \le |x(t_2)| \le Ue^{dT} + \lambda\|a\|Te^{dT} + \left[2\lambda T^2 Le^{2dT}(Q + 2\lambda\|a\||g(x(t_3))|\right]^{1/2}.$$

Now this x is simply a representative T-periodic solution and we want an a priori bound on the whole class. If $|x(t_3)|$ is bounded over this class, but $\|x\| \to \infty$, then we divide the above inequality by $|x(t_2)|$ and get a contradiction for large $|x(t_2)|$. If $|x(t_3)| \to \infty$, then we divide the above inequality by $|x(t_3)|$ and get a contradiction to $g(x)/x^2 \to 0$ as $|x| \to \infty$. Hence, there is a $K > 0$ with $\|x\| \le K$ and Lemma 1 is true.

Let $(P, \|\cdot\|)$ be the Banach space of continuous T-periodic functions and for $\varphi \in P$ define

$$H(\lambda,\varphi)(t) = \lambda\int_{-\infty}^{t} e^{-(t-v)}\left[\varphi(v) - \int_{-\infty}^{v} D(v,s)g(\varphi(s))ds + a(v)\right]dv. \tag{33}$$

Now $H : P \to P$ since a change of variable shows that $H(\lambda,\varphi)$ is T-periodic and, since φ is continuous, $H(\lambda,\varphi)$ is differentiable (and, thus, continuous).

Also, H maps bounded sets into compact sets. To see this, if $\|\varphi\| \le J$, then there is a bound in terms of J on H and on the derivative of $H(\lambda,\varphi)$. The result now follows by Ascoli's theorem.

It can be proved that H is continuous in φ just as was done in the proof of Theorem 2.

By Schaefer's theorem, H has a fixed point for $\lambda = 1$. □

Remark. We pointed out after (2) that it could be weakened. We can do so by using the technique of this proof with (20) on (1) to obtain an additional term $-\beta(a(t) - x(t))^2$ in V'.

REFERENCES

1. Burton, T.A., *Volterra Integral and Differential Equations*, Academic Press, Orlando, Florida, 1983.

2. Corduneanu, C., *Integral Equations and Stability of Feedback Systems*, Academic Press, Orlando, Florida, 1973.

3. Gripenberg, G., Londen, S.-O., Staffans, O., *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.

4. Halanay, A., On the asymptotic behavior of the solutions of an integro-differential equation, J. Math. Anal. Appl. **10** (1965), 319–324.

5. Hale, J., Dynamical systems and stability, J. Math. Anal. Appl. **26** (1969), 39–59.

6. Levin, J., The asmptotic behavior of a Volterra equation, Proc. Amer. Math. Soc. **14** (1963), 534–541.

7. Levin, J., A nonlinear Volterra equation not of convolution type, J. Differential Equations **4** (1968), 176–186.

8. Levin, J., On a nonlinear Volterra equation, J. Math. Anal. Appl. **39** (1972), 458–476.

9. MacCamy, R.C. and Wong, J.S.W., Stability theorems for functional equations, Trans. Amer. Math. Soc. **164** (1972), 1–37.

10. Schaefer, H., Über die Methode der a priori Schranken, Math. Ann. **129** (1955), 415–416.

11. Wu, J. and Xia, H., Existence of periodic solutions to integro-differential equations of neutral type via limiting equations, Math. Proc. Cambridge Phil. Soc. **112** (1992), 403-418.