Asymptotic Stability of Second Order Ordinary, Functional, and Partial Differential Equations

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1. INTRODUCTION

One of the oldest problems in differential equations may be described as follows: If **a** is a positive constant, then every solution $u(t)$ of

$$\text{(C)} \qquad u'' + au' + u = 0, \quad (\cdot)' = (d/dt)(\cdot),$$

satisfies

$$\text{(S)} \qquad |u(t)| + |u'(t)| \to 0 \text{ as } t \to \infty.$$

If $a : [0, \infty) \to (0, \infty)$ is continuous, will (S) still hold?

The answer is "sometimes" and the literature on the question is large. Briefly, (S) will hold only if $\int_0^\infty a(t)dt = \infty$; very roughly, if, in addition, $\int_0^\infty [dt/(a(t) + 1)] = \infty$, then (S) will hold.

We will discuss the literature more fully in the next section, but to focus on the problem one result will be mentioned now. If $a(t) \geq a_0 > 0$, Smith [10] shows that a necessary and sufficient condition for (S) to hold is that

$$\text{(N)} \qquad \int_0^\infty e^{-A(t)} \int_0^t e^{A(s)} ds\, dt = \infty \text{ where } A(t) = \int_0^t a(s)ds.$$

This is a fine result, but for a given function the condition is not easy to verify and the technique rests heavily on the linearity of the equation. Ballieu and Peiffer[1] show that when $1/a(t)$ is well behaved and when $a(t) \geq a_0 > 0$, then (N) is equivalent to $\int_0^\infty [ds/a(s)] = +\infty$.

More recently investigators have become increasingly interested in properties and techniques that are valid for wider classes of problems including ordinary, functional, and partial differential equations of both linear and nonlinear type. The book by Nicolaenko, et al [9], for example, is devoted to such questions.

In this paper we develop a transformation which allows us to treat

$$u'' + a(t)u' + u = 0, \tag{1a}$$

$$u_{tt} = u_{xx} - a(t)u_t, \quad u(t,0) = u(t,\pi) = 0, \tag{1b}$$

and

$$u'' + a(t)u' + u(t-r) = 0 \tag{1c}$$

in such a unified way that there is no distinction at all between the results for (1a) and (1b), while the results for (1c) agree with those for (1a) and (1b) as $r \to 0$.

The transformation makes the proofs for (S) to hold essentially trivial; yet, the results themselves compare favorably with some of the best classical ones for (1a). For simplicity, brevity, and unity we treat the linear problems and then show how to proceed with the nonlinear ones.

2. A BRIEF SURVEY

Most results on (1a) or its nonlinear analogs rely on the Liapunov function

$$V(u,u') = u^2 + (u')^2$$

whose derivative along a solution of (1a) satisfies

$$V' = -2a(t)(u')^2 \le 0.$$

The difficulty with this technique is that V' is merely semi-definite. If $V \to 0$, then (S) holds. If $V \to C > 0$, then in the (u,u')-plane, the solution approaches a circle. Investigators then give conditions to ensure that the solution oscillates (and, hence, spirally approaches that circle) and that V' is negative enough of the time that an integration of V' sends V to $-\infty$, a contradiction. Such arguments yield many results of the following type.

THEOREM (Smith [10]) Condition (S) holds for (1a) if there is a sequence $I_1, I_2, \ldots$ of disjoint open intervals on $(0, \infty)$ such that

$$\infty = \sum_{n=1}^{\infty} m_n T_n \delta_n^2$$

where m_n and M_n are the minimum and maximum values of $a(t)$ in I_n, T_n is the length of I_n, and δ_n is the smaller of the two numbers T_n and $(1+M_n)^{-1}$.

The result is obviously tedious in application, but it is very effective and implies the following result.

If $a(t)$ is monotonically decreasing on $(0, \infty)$ and if $\int_0^\infty a(t)dt = \infty$, then (S) holds.

One of our results will prove this result for both (1a) and (1b).

A recent preprint of Hatvani and Totik [3] continues the type of analysis used in proving Smith's result and substantially sharpen it.

In the paper by Ballieu and Peiffer [1] mentioned in Section 1 in which (N) is elucidated, several results of much interest are obtained. The full statement concerning (N) is as follows.

THEOREM (Ballieu and Peiffer). Let $a(t) > \epsilon > 0$ with $a(t)$ continuous. Then the integrals

$$\int_0^\infty [dt/a(t)] \text{ and } \int_0^t e^{-A(t)} \int_0^t e^{A(u)} du\, dt$$

converge or diverge simultaneously provided that one of the following conditions hold:

(a) $1/a(t)$ is of bounded variation (in particular if $a(t)$ is monotone),

(b) $a(t)$ is differentiable and $-1 < -\eta < a'(t)/a^2(t) < K$.

Here $A(t) = \int_0^t a(s)ds$ and K is an arbitrary constant. One of our subsequent results will replace the lower bound in (b) with an arbitrary constant and show that (S) will then hold for both (1a) and (1b).

Now this result is coupled with Smith's statement for (N) as follows.

THEOREM (Ballieu and Peiffer). Let $a(t) > \epsilon > 0$ be a continuous function such that $1/a(t)$ is of bounded variation. If, in addition, $\int_0^\infty [dt/a(t)] = \infty$, then (S) holds.

Recently, Karsai [4] obtained a number of interesting results on (1a) including the following one which we cannot prove with our techniques.

THEOREM (Karsai). Let $0 \leq a(t) \leq 2$ and let $0 < \tau < \pi$. If $\int_0^\infty a(t)dt = \infty$ then every solution of

$$u'' + [a(t) + a(t + \tau)]u' + u = 0$$

satisfies (S).

The recent preprint of Hatvani and Totik relates to both of the results mentioned above by Smith. The one concerning (N) can be stated as follows. It will be mentioned again when we discuss separation of variables for (1b).

THEOREM (Hatvani and Totik). Suppose that $\alpha < \pi/k$ and there is a constant δ such that $\int_t^{t+\alpha} a(s)ds \geq \delta > 0$ for large t. Then (S) holds for $u'' + a(t)u' + ku = 0$ if and only if (N) holds.

Klincsik [5] discusses (1b) and proves existence theory for it. He also shows that if $a(t)$ is bounded and integrally positive then $\int_0^\pi [u_t^2 + u_x^2]dx \to 0$ as $t \to \infty$. He deals with a nonlinear form.

In Section 4 we will mention some results known for (1c) and nonlinear generalizations. Nonlinear forms of (1a) are treated in some depth in the papers of Levin and Nohel [7] and Thurston and Wong [11].

3. A TRANSFORMATION

Given $a(t) \geq 0$, we select a bounded differentiable function

$$\lambda : [0, \infty) \to [0, \infty)$$

and then define

$$b : [0, \infty) \to [0, \infty)$$

by

$$a(t) = \lambda(t) + b(t) \tag{2}$$

and require that

$$\int_0^\infty \lambda(t)dt = \infty. \tag{3}$$

We begin with (1a) and define $u' = z - \lambda u$ so that $u'' = z' - \lambda' u - \lambda u' = -(\lambda + b)u' - u$ and $z' = \lambda' u - b(z - \lambda u) - u$ or $z' = (\lambda' + b\lambda - 1)u - bz$ yielding the system

$$\begin{cases} u' = z - \lambda(t)u \\ z' = [\lambda'(t) + b(t)\lambda(t) - 1]u - b(t)z \end{cases} \tag{4a}$$

which is equivalent to (1a). Define a Liapunov function

$$V(u, z) = u^2 + z^2 \tag{5a}$$

so that if (u, z) is a solution of (4a) then

$$\begin{aligned} V' &= 2u[z - \lambda u] + 2z[\lambda' + b\lambda - 1]u - 2bz^2 \\ &= -2\lambda u^2 - 2bz^2 + 2zu(\lambda' + b\lambda). \end{aligned}$$

Now if we were interested only in (1a) then we would, at this point, invoke a condition for this quadratic form to be negative definite; however, to handle the PDEs in the same way we will need to use Wirtinger's inequality and proceed otherwise. The following method will yield unity.

We will seek a continuous function $f : [0, \infty) \to (0, \infty)$ and replace $zu = (fz)(u/f)$ in V' to obtain

$$V' \le -2\lambda u^2 - 2bz^2 + |\lambda' + b\lambda|(f^2 z^2 + (u^2/f^2))$$

or

$$V' \le [-2\lambda + (|\lambda' + b\lambda|/f^2)]u^2 + [-2b + f^2|\lambda' + b\lambda|]z^2. \tag{6a}$$

Our problem will be solved if we can find continuous functions $f : [0,\infty) \to [0,\infty)$ and $\mu : R \to R$ such that

$$(7) \qquad \begin{cases} A(t) := -2\lambda(t) + |\lambda'(t) + b(t)\lambda(t)|/f^2(t) \le \mu(t), \\ B(t) := -2b(t) + f^2(t)|\lambda'(t) + b(t)\lambda(t)| \le \mu(t), \\ \int_0^\infty \mu(t)dt = -\infty. \end{cases}$$

It was intentional that this was denoted by (7) and not by (7a).

THEOREM 1a. Let (7) hold. Then

$$(8a) \qquad |u(t)| + |u'(t)| \to 0 \text{ as } t \to \infty$$

for every solution $u(t)$ of (1a).

PROOF. From (5a), (6a), and (7) we have $V' \le \mu(t)V$ and so $V \to 0$ as $t \to \infty$. Recall that λ is bounded and that $V \to 0$ implies that $u(t) \to 0$. Thus, when $V = u^2 + (u' + \lambda u)^2$ tends to zero we conclude that $|u'|$ also tends to zero since $\lambda u \to 0$. This completes the proof.

REMARK. From the proof we can estimate the rate of decay of solutions by

$$|u(t)|, |u'(t)| = O\left(\exp\left[\frac{1}{2}\int_0^t \mu(s)ds\right]\right) \text{ as } t \to \infty.$$

In the next section we will extensively discuss conditions under which (7) holds, but now we turn our attention to (2b). Existence theory for (1a) and (1c) is so well known that it may be ignored here. Adequate existence theory for (1b) can be found in Klincsik [5].

Let (2) and (3) hold, let $u(t,x)$ be a solution of (2b), and define

$$u_t = z - \lambda(t)u$$

to obtain the system

$$(4b) \qquad \begin{cases} u_t = z - \lambda(t)u \\ z_t = u_{xx} - b(t)z + (\lambda'(t) + \lambda(t)b(t))u. \end{cases}$$

Notice that the boundary conditions $u(t,0) = u(t,\pi) = 0$ induce $u_t(t,0) = u_t(t,\pi) = 0$ and this will be used subsequently several times. Define a Liapunov function

$$V(t) = \int_0^\pi [u_x^2 + z^2]dx \tag{5b}$$

so that if $u(t,x)$ is a solution of (1b) on $[0,\infty)$ then

$$\begin{aligned}
V'(t) &= \int_0^\pi [2u_x u_{xt} + 2zz_t]dx = 2u_x u_t|_0^\pi \\
&\quad + \int_0^\pi \{-2u_{xx}u_t + 2z[u_{xx} - bz + (\lambda' + b\lambda)u]\}dx \\
&= \int_0^\pi [-2u_{xx}(z - \lambda u) + 2zu_{xx} - 2bz^2 + 2(\lambda' + b\lambda)uz]dx \\
&= \int_0^\pi [2\lambda u u_{xx} - 2bz^2 + 2(\lambda' + b\lambda)uz]dx \\
&= 2uu_x|_0^\pi \\
&\quad + \int_0^\pi [-2\lambda u_x^2 - 2bz^2 + 2(\lambda' + b\lambda)uz]dx
\end{aligned}$$

(introducing f as before)

$$\leq \int_0^\pi [-2\lambda u_x^2 - 2bz^2 + (|\lambda' + b\lambda|/f^2)u^2 + f^2|\lambda' + b\lambda|z^2]dx$$

(but $\int_0^\pi u^2 dx \leq \int_0^\pi u_x^2 dx$ under $u(t,0) = u(t,\pi) = 0$)

$$\leq \int_0^\pi \{[-2\lambda + (|\lambda' + b\lambda|/f^2)]u_x^2 + [-2b + f^2|\lambda' + b\lambda|]z^2\}dx.$$

THEOREM 1b. Let (7) hold. Then

$$|u|_\infty^2 + \int_0^\pi (u_x^2 + u_t^2)dx \to 0 \text{ as } t \to \infty \tag{8b}$$

for every solution $u(t,x)$ of (1b) defined on $[0,\infty)$. (Here, $|u|_\infty$ is the supremum norm in x for fixed t.)

PROOF. If (7) holds then $V' \leq \mu(t)V$ so $V \to 0$ as $t \to \infty$. Moreover, the boundary condition implies $|u|_\infty^2 \leq k\int_0^\pi u_x^2\, dx$ for some $k > 0$. The remainder of the proof is almost identical to that of Theorem 1a.

The delay case is very straightforward when the delay r is a positive constant and that is the assumption made here. But the interested reader can study the work of Krasovskii [6; p. 173] and Yoshizawa [12] to see appropriate changes when $r = r(t)$. In particular, interesting connections between $r(t)$ and $a(t)$ emerge when we try to satisfy the counterpart of (7).

Let (2) and (3) hold and define $u' = z - \lambda(t)u$ so that $u'' = z' - \lambda' u - \lambda u' = -(\lambda + b)u' - u(t-r)$ or

$$\begin{aligned} z' &= \lambda' u - b(z - \lambda u) - u(t-r) = (\lambda' + b\lambda)u - bz - u(t-r) \\ &= (\lambda' + b\lambda - 1)u - bz + \int_{t-r}^{t} (z(s) - \lambda(s)u(s))ds \end{aligned}$$

yielding the system

$$\text{(4c)} \qquad \begin{cases} u' = z - \lambda(t)u \\ z' = (\lambda'(t) + b(t)\lambda(t) - 1)u - bz + \int_{t-r}^{t} (z(s) - \lambda(s)u(s))ds. \end{cases}$$

We seek a function $\mu_1 : [0, \infty) \to [0, \infty)$, $\mu_1'(t) \le 0$, and define

$$\text{(5c)} \qquad V = u^2 + z^2 + [\mu_1(t) + 1] \int_{-r}^{0} \int_{t+s}^{t} (z(v) - \lambda(v)u(v))^2 dv\, ds$$

so that along a solution of (4c) we have

$$\begin{aligned} V' \le\; & 2u(z - \lambda u) + 2z(\lambda' + b\lambda - 1)u - 2bz^2 \\ & + 2z \int_{t-r}^{t} (z(s) - \lambda(s)u(s))ds \\ & + [\mu_1 + 1] \int_{-r}^{0} [(z(t) - \lambda(t)u(t))^2 - (z(t+s) - \lambda(t+s)u(t+s))^2] ds \\ \le\; & -2\lambda u^2 - 2bz^2 + 2(\lambda' + b\lambda)uz + rz^2 \\ & + \int_{t-r}^{t} (z(s) - \lambda(s)u(s))^2 ds + r(\mu_1 + 1)(z - \lambda(u))^2 \\ & - (\mu_1 + 1) \int_{t-r}^{t} (z(s) - \lambda(s)u(s))^2 ds \\ \le\; & -2\lambda u^2 - 2bz^2 + 2(\lambda' + b\lambda)uz + rz^2 \\ & + r(\mu_1 + 1)(z^2 - 2\lambda uz + \lambda^2 u^2) - \mu_1 \int_{t-r}^{t} (z(s) - \lambda(s)u(s))^2 ds \end{aligned}$$

or

$$V' \le [-2\lambda + r(\mu_1+1)\lambda^2 + (|\lambda' + b\lambda - r(\mu_1+1)\lambda|/f^2)]u^2$$

(6c)
$$+ [-2b + r + r(\mu_1+1) + f^2|\lambda' + b\lambda - r(\mu_1+1)\lambda|]z^2$$
$$- \mu_1(t)\int_{t-r}^{t}(z(s) - \lambda(s)u(s))^2 ds.$$

Note also that

$$\int_{-r}^{0}\int_{t+s}^{t}(z(v) - \lambda(v)u(v))^2 dv\, ds$$
$$\le \int_{-r}^{0}\int_{t-r}^{t}(z(v) - \lambda(v)u(v))^2 dv\, ds \le r\int_{t-r}^{t}(z(s) - \lambda(s)u(s))^2 ds.$$

Our task is to find functions $\mu_1 : [0,\infty) \to [0,\infty)$, $\mu_2 : [0,\infty) \to R$, and $f : [0,\infty) \to (0,\infty)$ with

(7c)
$$\begin{cases} \text{i) } \mu_1' \le 0 \\ \text{ii) } A(t) := [-2\lambda + r(\mu_1+1)\lambda^2 + (|\lambda' + b\lambda - r(\mu_1+1)\lambda|/f^2] \le -\mu_2(t), \\ \quad B(t) := [-2b + r + r(\mu_1+1) + f^2|\lambda' + b\lambda - r(\mu_1+1)\lambda|] \le -\mu_2(t), \\ \text{iii) } \int_0^\infty \min\{[\mu_1(t)/r(\mu_1(t)+1)]; \quad \mu_2(t)\}dt = \infty. \end{cases}$$

We will then have

$$V' \le -\mu_2(u^2 + z^2) - [\mu_1(\mu_1+1)/(r\mu_1+1)]\int_{-r}^{0}\int_{t+s}^{t}(z(v) - \lambda(v)u(v))^2 dv.$$

Taking $\gamma(t) = \min\{[\mu_1(t)/r(\mu_1(t+1))]; \mu_2(t)\}$, we will obtain $V' \le -\gamma(t)V$.

THEOREM 1c. Let (7c) hold. Then

(8c)
$$|u(t)| + |u'(t)| \to 0 \text{ as } t \to \infty$$

for every solution of (1c).

REMARK. Notice that (7cii) can be satisfied only if $b(t) > r$, a condition noted by other investigators and discussed in our examples. Also, as $r \to 0$, then (7c) approaches (7). We also remark that the delay in (1c) is not necessarily restricted to being constant or even a bounded function. If $C : R \to R$ is small enough, then one may consider the equation

$$u'' + a(t)u' + \int_{-\infty}^{t} C(t-s)u(s)ds = 0 \tag{1d}$$

where $G(t) = \int_t^\infty C((u)du$ is also small. Then a solution of

$$u'' + a(t)u' + G(0)u - \int_{-\infty}^{t} G(t-s)u'(s)ds = 0 \tag{1d*}$$

is also a solution of (1d). Using the Liapunov functional

$$V_1 = G(0)u^2 + (u')^2 + \int_{-\infty}^{t}\int_{t-s}^{\infty} |G(u)|du(u'(s))^2 ds$$

it is readily shown that if $a(t) \geq \int_0^\infty |G(t)|dt$ for all $t \geq 0$ then $|u(t)| + |u'(t)|$ is bounded for each solution of (1d*). If there is a $\delta > 0$ with

$$a(t) - \delta \geq \int_0^\infty |G(t)|dt, \text{ then } \int_0^\infty |u'(t)|^2 dt < \infty.$$

The transformation $u' = z - \lambda u$ in (1d) yields the system

$$\begin{cases} u' = z - \lambda u \\ z' = (\lambda' + b\lambda - G(0))u - bz + \int_{-\infty}^{t} G(t-s)(z(s) - \lambda(s)u(s))ds \end{cases}$$

and the Liapunov function

$$V = G(0)u^2 + z^2 + [\mu_1(t) + 1]\int_{-\infty}^{t}\int_{t-s}^{\infty} |G(u)|du(z(s) - \lambda(s)u(s))^2 ds$$

will lead us to an analog of (7c) and a Theorem 1d.

4. EXAMPLES

We begin with two propositions giving conditions under which (7) holds. They are, so to speak, the two extreme cases; first, $a(t)$ becomes very small, and then very large, at least along a sequence.

PROPOSITION 1. Let (2) and (3) hold with

$$\int_0^\infty [-a(t) + |\frac{1}{2}a'(t) + \frac{1}{4}a^2(t)|]dt = -\infty.$$

Then (7) is satisfied with $\lambda(t) = b(t) = a(t)/2$, $f(t) = 1$, and $\mu(t) = -a(t)+|\frac{1}{2}a'(t)+\frac{1}{4}a^2(t)|$.

PROOF. In (7) we have

$$A(t) = -a(t) + |\frac{1}{2}a'(t) + \frac{1}{4}a^2(t)| = B(t).$$

EXAMPLE 2. Let $a(t) = [\sin^2 \ln(t+1)]/(t+1)$ so that $\int_0^\infty a(t)dt = \int_0^\infty \sin^2 u\, du = \infty$ and $a'(t) = \{2[\sin \ln(t+1)][\cos \ln(t+1)] - \sin^2 \ln(t+1)\}/(t+1)^2$ so both a' and $a^2 \in L^1[0,\infty)$.

In preparation for the next example, note that

$$a_1(t) = |\sin t| - \sin t$$

is zero on $0 \le t \le \pi$ with $a(t) = 2\sin t$ on $\pi \le t \le 2\pi$. But it is not differentiable at $t = n\pi$. The function

$$a_2(t) = a_1(t)\sin^2 t$$

has very much the same behavior, but it has a continuous derivative. The function

$$a_3(t) = a_2(\ln(t+1))$$

has a continuous derivative, while the intervals on which it is alternately zero and nonzero tend to infinity in length.

EXAMPLE 3. Let $\lambda(t) = b(t) = a(t)/2 = \{|\sin[\ln(t+1)]| - \sin[\ln(t+1)]\}(\sin^2[\ln(t+1)])/(t+1)$. At nonzero points $\lambda(t) = -2(\sin^3[\ln(t+1)])/(t+1)$ and $(t+1)^2\lambda'(t) = -6(\sin^2[\ln(t+1)])\cos[\ln(t+1)] + 2\sin^3[\ln(t+1)]$ so that $\lambda'(t)$ is $L^1[0,\infty)$ and is zero when $a(t) = 0$. Thus, λ has a continuous derivative. Moreover, $\lambda b = \lambda^2 \in L^1[0,\infty)$, while if t_1, t_2 are consecutive values with $\ln(t_1+1) = (2n+1)\pi$ and $\ln(t_2+1) = (2n+2)\pi$ then $\int_{t_1}^{t_2} \lambda(t)dt = -\int_{(2n+1)\pi}^{2(n+1)\pi} 2\sin^3 s ds$. Thus, if we take $\mu(t) = -2\lambda(t) + |\lambda'(t)| + \lambda^2(t)$

then $\int_0^\infty \mu(t)dt = \infty$. Thus, the solutions of (1a) are 2π-periodic on arbitrarily long time intervals but do tend to zero. Equation (1b) behaves similarly.

COROLLARY 4. If $a(t) \searrow 0$ with $\int_0^\infty a(s)ds = \infty$, then (7) holds with $\lambda(t) = b(t) = a(t)/2$, $f^2 = 1$, and $\mu(t) = -2\lambda(t) + |\lambda'(t)| + \lambda^2(t)$.

PROOF. Clearly, $\int_0^\infty |\lambda'(t)|dt < \infty$, while $\lambda^2(t) < \lambda(t)/2$ for large t so that the conditions of Proposition 1 hold.

In preparation for the next result we recall that if $a(t) > 0$ with $a'(t) \geq \alpha a^2(t)$ for some $\alpha > 0$, then $a(t)$ is very badly behaved. In fact, if $a'(t) \geq 0$ and if there is a positive constant α with $a'(t) \geq \alpha a^2(t)$ on a set of intervals $[t_n, t_n + k_n]$, $t_n \to \infty$, then $k_n \to 0$ as $n \to \infty$; otherwise, $a(t)$ has finite escape time.

PROPOSITION 5. Suppose that

(i) $a(t) \geq a_0 > 0$ for $t \geq 0$,

(ii) $\int_0^\infty (dt/a(t)) = \infty$,

and

(iii) $a'(t)/a^2(t)$ is bounded for $t \geq 0$.

Then (7) holds with $\lambda(t) = c_1/a(t)$ and $f^2(t) = c_2 a(t)$, where c_1 and c_2 are appropriate positive constants.

PROOF. We have $\lambda' = -c_1 a' a^{-2}$, $b(t) = a - c_1 a^{-1}$,

$$\begin{aligned} A(t) &= -2c_1 a^{-1} + |-c_1 a' a^{-2} + c_1 - c_1^2 a^{-2}|/c_2 a \\ &\leq -c_1 a^{-1}\{2 - (K + 1 + c_1 a_0^{-2})c_2^{-1}\} \end{aligned}$$

where K is the bound for $|a' a^{-2}|$. On the other hand

$$\begin{aligned} B(t) &= -2a - 2c_1 a^{-1} + |-c_1 a' a^{-2} + c_1 - c_1 a^{-2}|c_2 a \\ &\leq -a_0\{2 - (K + 1 + c_1 a_0^{-2})c_1 c_2\}. \end{aligned}$$

Fix $c_2 > 0$ large enough so that $(K + 1 + a_0^{-2})/c_2 < 1$. Then choose $c_1 \in (0, 1)$ so small that

$$c_1(K + 1 + a_0^{-2})/c_2 < 1 \text{ and } c_1/a_0 < a_0.$$

With these constants we have the estimates $A(t) \le -c_1/a(t)$, $B(t) \le -a_0$, so that $\int_0^\infty \min\{c_1/a(t); a_0\}dt = \int_0^\infty c_1 a^{-1}(t)dt = \infty$ and (7) holds with $\mu(t) = c_1 a^{-1}(t)$.

REMARK. In [8] Murakami used the following conditions:

(ii′) There are sequences $\{s_n\}$ and a constant $d > 0$ such that $s_{n+1} \ge s_n + d$ and either $a(t) \equiv 0$ on the set $\bigcup\limits_{n=1}^{\infty} [s_n, s_n + d]$ or $\sum\limits_{n=1}^{\infty} \left[\int_{s_n}^{s_n+d} a(t)dt\right]^{-1} = \infty$. By Schwarz's inequality

$$d^2 = \left(\int_{s_n}^{s_n+d} 1\, dt\right)^2 \le \int_{s_n}^{s_n+d} a(t)dt \int_{s_n}^{s_n+d} [dt/a(t)].$$

Murakami's condition implies that $\int_0^\infty [dt/a(t)] = \infty$, but not conversely, while $\int_0^\infty [dt/a(t)] = \infty$ and $a(t)$ non-decreasing implies Murakami's condition; however it is unknown if $\int_0^\infty [dt/a(t)] = \infty$ and $a'(t)/a^2(t)$ bounded imply Murakami's condition. At any rate, we note that Murakami's work does not apply to the PDE, as ours does.

The next result shows that some of the smoothness and sign conditions which we are assuming can be relaxed. We do the work for (1a) only.

PROPOSITION 6. Consider the equations

$$(\alpha) \qquad x'' + (\lambda(t) + b_1(t))x' + x = 0$$

and

$$(\beta) \qquad x'' + (\lambda(t) + b_1(t) + b_2(t))x' + x = 0.$$

Let $V = u^2 + z^2$ and suppose that there is a $\mu_1 : [0, \infty) \to R$ with $\int_0^\infty \mu_1 = -\infty$ and such that $V'_{(\alpha)}(t, u, z) \le \mu_1(t)V$. If $\int_0^\infty |\lambda(t)b_2(t)|dt < \infty$, then there is a $\mu_2 : [0, \infty) \to R$ with $\int_0^\infty \mu_2(t)dt = -\infty$ and such that $V'_{(\beta)}(t, u, z) \le \mu_2(t)V$.

PROOF. We have

$$\begin{aligned}
V'_{(\beta)} &= -2\lambda u^2 - 2b_1 z^2 + 2uz(\lambda' + b_1\lambda) - 2b_2 z^2 + 2uz(b_2\lambda) \\
&= V'_{(\alpha)} - 2b_2 z^2 + 2uz(b_2\lambda) \\
&\leq (\mu_1(t) + |b_2(t)\lambda(t)|)(u^2 + z^2) = \mu_2(t)(u^2 + z^2).
\end{aligned}$$

This completes the proof.

As a corollary to Prop. 5 and Prop. 6, we have that if $a = a_1 + a_2$ where a_1 satisfies the conditions of Prop. 5 and $\int_0^\infty [a_2(t)/a_1(t)]dt < \infty$, then all solutions of (1a) tend to zero as $t \to \infty$.

EXAMPLE 7. Let $a_1(t) = (t+2)\ln(t+2)$ and $a_2(t)$ be an arbitrary continuous function with $\int_0^\infty [a_2(t)/(t+2)\ln(t+2)]dt < \infty$ and $a_2(t) \geq -a_1(t)$. Then $a_1'(t) = 1 + \ln(t+2)$ and the conditions of Prop. 6 are satisfied so that all solutions of (1a) and their derivatives tend to zero as $t \to \infty$.

PROPOSITION 8. Let (2) and (3) hold with $\lambda(t) \searrow 0$. If $b(t)\lambda(t) \in L^1[0,\infty)$ and $\phi(t) := \min[\lambda(t), b(t)]$ satisfies $\int_0^\infty \phi(t)dt = \infty$, then $f^2(t) = 1$ and $\mu(t) = -\phi(t) + |\lambda'(t)| + b(t)\lambda(t)$ will satisfy (7).

PROOF. In (7) we have

$$A(t) \leq -2\lambda(t) + |\lambda'(t)| + b(t)\lambda(t)$$

and

$$B(t) \leq -2b(t) + |\lambda'(t)| + b(t)\lambda(t).$$

Since λ' and $b\lambda$ are both $L^1[0,\infty)$, the indicated μ will suffice.

PROPOSITION 9. Let (2) and (3) hold with $\lambda' \in L^1[0,\infty)$. If

$$\limsup_{t\to\infty}[\lambda(t)(1 + b(t)) + |\lambda'(t)|] < 2$$

and

$$\phi(t) := \min[\lambda(t), b(t)] \notin L^1[0,\infty)$$

then $f^2(t) = 1 + b(t)$ and $\mu(t) = -\phi(t)$ will satisfy (7).

PROOF. We have

$$A(t) \leq -2\lambda(t) + |\lambda'(t)| + \lambda(t) = -\lambda(t) + |\lambda'(t)|$$

and

$$\begin{aligned} B(t) &\leq -2b(t) + (1+b(t))|\lambda'(t)| + b(t)\lambda(t) + b(t)\lambda(t)b(t) \\ &= -b(t)[2 - |\lambda'(t)| - \lambda(t)(1+b(t))] + |\lambda'(t)| \end{aligned}$$

from which the statement follows.

EXAMPLE 10. Suppose that c and C are positive constants with

$$C(t+1)^{-1} \leq a(t) \leq C(t+1)$$

holding for t large. Then (7) can be satisfied.

PROOF. Let $c_1 = \min(c/2, 1/C)$ and define $\lambda(t) = c_1(t+1)^{-1}$, $\psi(t) = a(t)(t+1)^{-1} - c_1(t+1)^{-2}$, and $b(t) = (t+1)\psi(t)$. Then $a(t) = \lambda(t) + b(t)$ and $(c/2)(t+1)^{-2} \leq (c - c_1)(t+1)^{-2} \leq \psi(t) \leq C - c_1(t+1)^{-2} \leq C$. Thus, all conditions of Proposition 7 are satisfied; in fact

$$\limsup_{t\to\infty}[c_1(t+1)^{-1}(1 + \psi(t)(t+1)) + c_1(t+1)^{-2}] \leq c_1 C \leq 1$$

and

$$\begin{aligned} \phi(t) = \min\{c_1(t+1)^{-1}, \psi(t)(t+1)\} &\geq \min\{c_1(t+1)^{-1}, c(t+1)^{-1}/2\} \\ &= c_1/(t+1). \end{aligned}$$

That is, $\phi \notin L^1[0,\infty)$.

REMARK. It is well-known that the quadratic form $Ax^2 + By^2 + 2Cxy$ can be estimated by $Ax^2 + By^2 + 2Cxy \geq \gamma(x^2 + y^2)$ where γ is the smallest eigenvalue of the symmetric matrix

$$\begin{pmatrix} A & C \\ C & B \end{pmatrix}.$$

If we use this fact to estimate the derivative V' in (1a) (this fails for the PDE) then we get the inequality

$$V' \leq -c\{(b+\lambda) - [(b-\lambda)^2 + (\lambda' + b\lambda)^2]^{1/2}\}V.$$

This allows us to ask that

$$\text{(7*)} \qquad \int_0^\infty \{(b+\lambda) - [(b-\lambda)^2 + (\lambda' + b\lambda)^2]^{1/2}\}dt = \infty$$

instead of (7) in the case of (1a). Sharper results hold in some cases.

EXAMPLE 11. Suppose that

(i) $a(t) \geq 2(t+1)^{-1}$ for $t \geq 0$,

(ii) $\int_0^\infty [2(t+1)^{-1} - a(t)(t+1)^{-2}]dt = \infty$.

Then (7*) is satisfied with $\lambda(t) = (t+1)^{-1}$ and the zero solution of (1a) is asymptotically stable.

PROOF. For the integrand in (7*) we have the estimate

$$\begin{aligned} a(t) &- \{[a(t) - 2(t+1)^{-1}]^2 + [a(t)(t+1)^{-1} - 2(t+1)^{-2}]^2\}^{1/2} \\ &= a(t) - (a(t) - 2(t+1)^{-1})[1 + (t+1)^{-2}]^{1/2} \\ &= a(t)\{1 - [1 + (t+1)^{-2}]^{1/2}\} + 2(t+1)^{-1}[1 + (t+1)^{-2}]^{1/2} \\ &= -a(t)\{(t+1)^2 + (t+1)[(t+1)^2 + 1]^{1/2}\}^{-1} + 2[1 + (t+1)^{-2}]^{1/2}(t+1)^{-1} \\ &\geq -a(t)(t+1)^{-2} + 2(t+1)^{-1} \end{aligned}$$

and the condition of the remark is satisfied and (7*) holds.

PROPOSITION 12. Let (2) and (3) hold with

(i) $\lambda\lambda' \in L^1[0,\infty)$,

(ii) $\limsup_{t\to\infty} b(t)\lambda(t) < 2$,

(iii) $-b(t) + |\lambda'(t)/\lambda(t)| \le -\alpha\lambda(t)$

for some $\alpha > 0$ and large t. Then (7) can be satisfied with $f^2(t) = 1/\lambda(t)$ and $\mu(t) = -\gamma\lambda(t) + |\lambda'(t)\lambda(t)|$ for some $\gamma > 0$.

REMARK. The condition $b(t)\lambda(t) \le k$ and $\lambda \notin L^1[0,\infty)$ means that $\lambda(t) \le k/b(t) \notin L^1[0,\infty)$. These indicate the growth and decay requirements on $a(t) = b(t) + \lambda(t)$ which are approximated throughout the literature. For if $b(t) = \lambda(t)$, then $a(t)$ can not become too small: $\int_0^\infty a(t)dt = \infty$; on the other hand, $a(t)$ cannot be too large: $\int_0^\infty [dt/a(t)] = \infty$.

Many more examples and propositions can be presented, but these illustrate that simple choices for λ, f, and μ produce results which compare favorably with the classical ones for (1a) alone. We now consider the delay equation (1c) and the inequalities of (7c), and give analogs of Proposition 9, Example 10, and Proposition 5.

PROPOSITION 13. Suppose that (2) holds and that there is a nonincreasing function $\mu_1 : [0,\infty) \to (0,\infty)$ such that

(i) $\limsup_{t\to\infty} [|\lambda'| + \lambda(1 + (b-r) + r\mu_1) + r\mu_1(b-r)^{-1}] < 2$,

(ii) $\int_0^t \min\{\lambda(t); b(t) - r; \mu_1(t)\}dt = \infty$,

(iii) $\lambda^2, \mu_1\lambda, \lambda' \in L^1[0,\infty)$.

Then (7c) holds.

PROOF. Let $f(t) = 1 + (b(t) - r)$. Then

$$A(t) \le -\lambda[1 - r\mu_1 - r(\mu_1 + 1)\lambda] + |\lambda'|$$

and

$$B(t) \le -(b-r)\{2 - |\lambda'| - \lambda(1 + (b-r) + r\mu_1) - r\mu_1(b-r)^{-1}\} + |\lambda'| + r\mu_1\lambda$$

and the assertion follows.

EXAMPLE 14. Suppose that there are positive constants c and C such that $r + c(t+1)^{-1} \leq a(t) \leq C(t+1)$ for large t. Then (7c) is satisfied.

PROOF. Let $c_1 := \min(c/2; 1/2C)$, $c_2 = c/4r$, and define $\lambda(t) := c_1(t+1)^{-1}$, $\psi(t) := (a(t) - r)(t+1)^{-1} - c_1(t+1)^{-2}$, $b(t) := r + (t+1)\psi(t)$, and $\mu_1(t) = c_2(t+1)^{-1}$. Then $a(t) = \lambda(t) + b(t)$ and by the conditions we have $(c/2)(t+1)^{-2} \leq (c - c_1)(t+1)^{-2} \leq \psi(t) \leq C$ and all conditions of Proposition 9 are satisfied. In fact,

$$\limsup_{t\to\infty} \{c_1(t+1)^{-2} + c_1(t+1)^{-1}[1 + (t+1)\psi(t) + rc_2(t+1)^{-1}] \\ + rc_2[\psi(t)(t+1)^2]^{-1}\} \\ \leq c_1 C + rc_2(2/c) \leq (\frac{1}{2}) + (2rc_2)/c = \frac{1}{2} + \frac{1}{2} = 1$$

and the function

$$\min\{c_1(t+1)^{-1}; \psi(t)(t+1); c_2(t+1)^{-1}\} \\ \geq \min\{c_1(t+1)^{-1}; (c/2)(t+1)^{-1}; c_2(t+1)^{-1}\} \\ = \min(c_1, c_2)(t+1)^{-1}$$

is not integrable over $[0, \infty)$.

REMARK. Krasovskii [6; p. 173] and Yoshizawa [12; p. 1150] investigated equation (1c) and its nonlinear generalizations. Both of them assumed $a(t)$ to be bounded below by a constant greater than r, so Example 14 improves these results in the linear case.

REMARK. When $a(t) \equiv p$, a positive constant, then lengthy calculations with the characteristic quasi-polynomial establishes a certain transcendental number $p_*(r) \in (0, r)$ with the property that $p > p_*$ implies that the zero solution of (1c) is asymptotically stable and $p < p_*$ implies instability (see [2; pp. 131-138].). The analog of Example 10 would be: If there are positive constants c, C such that $p_*(r) + c(t+1)^{-1} \leq a(t) \leq C(t+1)$ for large t, then every solution $u(t)$ of (1c) will satisfy $|u(t)| + |u'(t)| \to 0$ as $t \to \infty$.

It is an open question whether this assertion is true.

PROPOSITION 15. Consider (1a) and suppose that the following conditions are satisfied:

(i) $a(t) \geq r + \epsilon$ for $t \geq 0$ and ϵ some positive constant;

(ii) $\int_0^\infty (dt/a(t)) = \infty$;

(iii) $a'(t)a^{-2}(t)$ is bounded for $t \geq 0$.

Then there are positive constants c_1, c_2, and c_3 such that (7c) holds with $\lambda = c_1 a^{-1}$, $f^2 = c_2 a$, and $\mu_1 = c_3$.

PROOF. Let K be a bound for $|a'a^{-2}|$. Then we have the following estimate:

$$\begin{aligned} A(t) &= -2(c_1/a) + r(c_3+1)c_1^2 a^{-2} \\ &\qquad + |-c_1 a' a^{-2} + (a - r - c_1 a^{-1})c_1 a^{-1} - r c_1 a^{-1} c_3|/ac_2 \\ &\leq -c_1 a^{-1}\{2 - (1+c_3)c_1 - [K + 2 + c_1 r^{-2} + c_3]/c_2\}. \end{aligned}$$

We can suppose that $0 < c_3 < c_1 < 1/4$. Now fix $c_2 > 0$ so large that

$$[K + 2 + r^{-2} + 1]/c_2 \leq 1/2.$$

Then

$$A(t) \leq -c_1 a^{-1}\{2 - (\frac{1}{2}) - [K + 2 + r^{-2} + 1]/c_2\} \leq -c_1 a^{-1}.$$

On the other hand,

$$\begin{aligned} B(t) &\leq -2(a - r - c_1 a^{-1}) + r c_3 \\ &\qquad + \{|c_1 a'/a^2| + (1 + r a^{-1} + c_1 a^{-2})c_1 + r c_1 a^{-1} c_3\} a c_2 \\ &\leq -2(a - r) + 2c_1 r^{-1} + r c_1 \\ &\qquad + \{c_1(K + 2 + c_1 r^{-2} + c_3)\} c_2 a (a - r)^{-1}(a - r) \\ &\leq -2(a - r) \\ &\qquad + c_1\{2r^{-1}c_2^{-1} + r c_2^{-1} + K + 2 + r^{-2} + 1\} c_2 (1 + r\epsilon^{-1})(a - r). \end{aligned}$$

Let $c_1 > 0$ be so small that

$$c_1\{2r^{-1}c_2^{-1} + rc_2^{-1} + K + 2 + r^{-2} + r\}c_2(1 + r\epsilon^{-1}) < 1.$$

Then $B(t) \leq -(a-r) \leq -\epsilon$. Take $\mu(t) := \min\{\epsilon, c_3, c_1/a(t)\}$. It remains to prove that

$$\int_0^\infty [dt/a(t)] = \infty \text{ implies that } \int_0^\infty \mu(t)dt = \infty.$$

We can assume that $c_3 < \epsilon$; that is,

$$\mu(t) = \min(c_3, c_1/a).$$

Then for $D = \{t | c_3 \leq c_1/a(t)\}$ and $E = \{t | c_3 > c_1/a(t)\}$ we have

$$\begin{aligned}\int_0^\infty \mu(t)dt &= \int_D c_3 dt + \int_E (c_1/a(t))dt \\ &\geq (c_3 r/c_1)\int_D (c_1/a(t))dt + \int_E (c_1/a(t))dt \\ &\geq \min(1, c_3 r/c_1)\int_0^\infty (c_1/a(t))dt = \infty.\end{aligned}$$

This completes the proof.

REMARK. Using the method of location of limit sets, Yoshizawa [12] proved asymptotic stability assuming the following condition (ii′) instead of (ii):

(ii′) there is a sequence $\{s_n\}$ and a constant $d > 0$ such that $s_{n+1} \geq s_n + d$ and either $a(t) \equiv 0$ on the intervals $[s_n, s_n + d]$ or $\sum_{n=1}^{\infty}\left[\int_{s_n}^{s_n+d} a(t)dt\right]^{-1} = \infty$.

It may be remarked that examples similar to those previously given can also be constructed for delay equations since (7) and (7c) are very similar.

5. NONLINEAR CONSIDERATIONS

Nonlinearities can be introduced into the restoring force in (1a) without any significant changes being required in what has been presented so far. We simply look at $x'' + a(t)x' + g(x) = 0$ with $xg(x) > 0$ if $x \neq 0$ and use the Liapunov function $V = 2\int_0^x g(s)ds + z^2$.

When $\int_0^x g(s)ds \to \infty$ as $|x| \to \infty$ and when g' is continuous, the nonlinear case is very similar to the linear one. Thus, we focus on (1b) and (1c).

Consider the equation

$$u_{tt} = g(u_x)_x - a(t)u_t, \quad u(t,0) = u(t,\pi) = 0. \tag{9}$$

There is good reason to require $g'(u) \geq g_0 > 0$, and to make our results parallel to the linear case we take $g_0 = 1$ and ask that there exist $\alpha > 0$ such that

$$ug(u) > 0 \text{ if } u \neq 0, \quad 1 \leq g'(u) \leq \alpha, \quad g' \text{ continuous.} \tag{10}$$

Our results here will be in the way of *a priori* bounds; any solution $u(t,x)$ existing on $[0,\infty)$ will satisfy these bounds. If (2) and (3) hold then the transformation (4) will yield

$$\begin{cases} u_t = z - \lambda(t)u \\ z_t = g(u_x)_x - bz + (\lambda' + b\lambda)u. \end{cases} \tag{11}$$

Let $G(u) = \int_0^u g(s)ds$ and define

$$V(t) = \int_0^\pi [2G(u_x) + z^2]dx \tag{12}$$

and obtain

$$V'(t) \leq \int_0^\pi [-2\lambda u_x^2 - 2bz^2 + 2(\lambda' + b\lambda)uz]dx.$$

We then have

$$\alpha \int_0^\pi [u_x^2 + z^2]dx \geq V(t) \geq \int_0^\pi [u_x^2 + z^2]dx;$$

if (7) is satisfied then $V' \leq -[\mu(t)/\alpha]V$ so that Theorem 1b will hold for (9) as well as for (1b).

REMARK. If we drop the condition $\alpha \geq g'(r)$, then we can still obtain local asymptotic stability by using convex function theory and Jensen's inequality.

Next, consider

$$u'' + a(t)u' + g(u(t-r)) = 0 \tag{13}$$

with (2) and (3) holding. Assume that there are constants $0 < \alpha < \beta$ with

$$\alpha \le g'(s) \le \beta \text{ for all } s \in R, \quad g' \text{ continuous.} \tag{14}$$

We obtain the system

$$\begin{cases} u' = z - \lambda(t)u \\ z' = (\lambda' + b\lambda)u - bz - g(u) + \int_{t-r}^{t} g'(u(s))(z(s) - \lambda(s)u(s))ds. \end{cases} \tag{15}$$

Let $G(u) = \int_0^u g(s)ds$ and define

$$V = 2G(u) + z^2 + (\mu_1(t) + 1)\int_{-r}^{0}\int_{t+s}^{t} \beta(z(v) - \lambda(v)u(v))^2 dv. \tag{16}$$

One proceeds in a straightforward way to an analog of (7c) and Theorem 1c.

6. SEPARATION OF VARIABLES

It is not possible to separate variables in our nonlinear equation (9) for a general solution, but variables are readily separated in (1b) yielding the pair

$$T''(t) + a(t)T'(t) + n^2 T(t) = 0 \tag{17}$$

and

$$X''(x) + n^2 X(x) = 0. \tag{18}$$

It is natural to argue that we need only solve (1a) to also solve (1b); but that is not necessarily true. There are several pitfalls along the way. First, we may give conditions on $a(t)$ so that solutions of (17) tend to zero; but the solution of (1b) will be an infinite series of such terms, and this will require great care. But what is even worse, n^2 will enter (7); in fact, n^2 will enter the computations of the other investigators. For example the Hatvani-Totik equation in Section 2 will require $\int_t^{t+\alpha} a(t)dt \ge \delta > 0$ for all large t and for $\alpha < \pi/n$; they conclude that (S) holds if and only if (N) holds. Thus, if we try for a series solution, then $n \to \infty$ and the condition will ultimately reduce to Smith's requirement that $a(t) \ge a_0 > 0$. When we handle (1b) directly, n does not enter the picture.

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