ALMOST PERIODIC SOLUTIONS OF VOLTERRA EQUATIONS AND ATTRACTIVITY

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1. Introduction. This paper is concerned with almost periodic and asymptotically almost periodic solutions of the integral equations

$$x(t) = a(t) - \int_0^t D(t, s, x(s))ds, \tag{1}$$

$$x(t) = a(t) - \int_{-\infty}^t D(t, s, x(s))ds, \tag{2}$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t, s, x(s))ds \tag{3}$$

where $a(t)$ and $D(t, s, x)$ converges to $p(t)$ and $P(t, s, x)$ in some sense. Equation (3) is a limiting equation of (1), while Equation (2) is a perturbed form of (3).

An equation of the form

$$x(t) = a(t) - \int_\alpha^t D(t, s, x(s))ds, \qquad \alpha \geq -\infty, \tag{E}$$

has two types of solutions. It may have a solution which satisfies (E) on (α, ∞); the existence of such a solution is frequently shown by a limiting equations argument or by a fixed point theorem. A more direct solution is obtained from a specified continuous initial function $\phi : (\alpha, t_0] \to R^n$; we then write (E) as

$$\begin{aligned} x(t) &= \left\{ a(t) - \int_{\alpha}^{t_0} D(t,s,\phi(s))ds \right\} + \int_{t_0}^{t} D(t,s,x(s))ds \\ &=: b(t) + \int_{t_0}^{t} D(t,s,x(s))ds \end{aligned}$$

so that ϕ becomes part of the inhomogeneous term b. Under mild continuity conditions (cf., Corduneanu [5], Gripenberg-Londen-Staffans [8], or Burton [1; p. 79]), there is then a solution, say $x(t, t_0, \phi)$ on an interval $[t_0, t_0 + \beta)$ and, if the solution remains bounded, $\beta = \infty$. Translation of $x(t, t_0, \phi)$ to the left may yield a solution of (2) or (3).

In this paper we begin with statements of basic results on almost periodic functions, including old results and some new ones. We then obtain several theorems, corollaries, and examples. The following brief summary will help the reader to understand the direction of the paper.

In Theorem 1 we show that if (3) and a related equation have unique R-bounded solutions, then those solutions are almost periodic. Moreover, bounded solutions of those equations with initial functions converge to those almost periodic solutions as $t \to \infty$. The proof is by means of limiting equations theory.

In Theorem 2 we show that if (1) has an asymptotically almost periodic solution with initial function, then its almost periodic part is an almost periodic solution of (3).

Theorem 3 lists five equivalent conditions about the existence of almost periodic solutions.

Theorem 4 uses a growth and Lipschitz condition to obtain an almost periodic solution of an equation related to (3).

Theorem 5 shows that a linear form of (3) has an almost periodic solution which is globally attractive.

Theorem 6 uses a Liapunov functional to show uniqueness of R-bounded solutions.

For other results on almost periodic and asymptotically almost periodic solutions of integral equations, including transform techniques not used here, see Corduneanu [4; p. 212], Gripenberg [7], Gripenberg-Londen-Staffans [8; p. 10], and Miller ([9], [10]).

2. Preliminaries. Consider the systems of Volterra equations

$$x(t) = a(t) - \int_0^t D(t, s, x(s))ds, \qquad t \in R^+, \tag{1}$$

$$x(t) = a(t) - \int_{-\infty}^t D(t, s, x(s))ds, \qquad t \in R \tag{2}$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t, s, x(s))ds, \qquad t \in R \tag{3}$$

where $R^+ := [0, \infty)$, $R := (-\infty, \infty)$, $a, p : R \to R^n$ and $D, P : \Delta \times R^n \to R^n$ are continuous, and where $\Delta := \{(t, s) : s \leqq t\}$.

In the following, we denote a real sequence by a greek letter as $\alpha = \{s_k\}$, and $\alpha \subset \beta$ means that α is a subsequence of β. For $\alpha = \{s_k\}$ and $\beta = \{t_k\}$, $\alpha + \beta$ denotes the sequence $\{s_k + t_k\}$. Next, $T_\alpha p$ and $T_\alpha P$ denote $\lim_{k\to\infty} p(t + s_k)$ and $\lim_{k\to\infty} P(t + s_k, s + s_k, x)$ respectively, where $\alpha = \{s_k\}$ and the limits exist for each t, s and x. Moreover, $H(p)$ denotes the hull of p, that is, a set of e such that there is an α with $T_\alpha p = e$ uniformly, and similarly $H(p, P)$ denotes the hull of (p, P).

We suppose that

(4) $\quad q(t) := a(t) - p(t) \to 0$ as $t \to \infty$ and $a(t)$ is almost periodic,

(5) $\quad Q(t, s, x) := D(t, s, x) - P(t, s, x),$ and $P(t, s, x)$ is almost periodic in t,

that is $\Pi(t, s, x) := P(t, s + t, x)$ is almost periodic in t uniformly for $(s, x) \in (-\infty, 0] \times R^n$, and for any $J > 0$ there are continuous functions $P_J : \Delta \to R^+$ and $Q_J : \Delta \to R^+$ such that

$$P_J(t,s) \text{ is almost periodic in } t,$$

$$|P(t,s,x)| \leqq P_J(t,s) \text{ if } (t,s,x) \in \Delta \times X_J,$$

where $|\cdot|$ denotes a norm of R^n and $X_J := \{x \in R^n : |x| \leqq J\}$,

$$|Q(t,s,x)| \leqq Q_J(t,s) \text{ if } (t,s,x) \in \Delta \times X_J,$$

$$\int_{-\infty}^{t} P_J(t+T,s)ds \to 0 \text{ uniformly for } t \in R \text{ as } T \to \infty, \tag{6}$$

$$\int_{0}^{t} Q_J(t,s)ds \to 0 \text{ as } t \to \infty, \tag{7-1}$$

or

$$\int_{-\infty}^{t} Q_J(t,s)ds \to 0 \text{ as } t \to \infty, \text{ and } \int_{-\infty}^{t} Q_J(t+T,s)ds \to 0$$
$$\text{uniformly for } t \in R \text{ as } T \to \infty.$$

First we state some well-known results concerning almost periodic functions without proofs. For the proofs, see, for example, [4] of Corduneanu, [6] of Fink, or [12] of Yoshizawa.

Theorem I. Let $x : R \to R^n$ be continuous, and assume that for any real sequence α', there is a subsequence $\alpha \subset \alpha'$ such that $T_\alpha x$ converges uniformly on R. Then $x(t)$ is almost periodic.

Theorem II. Let $x : R \to R^n$ be continuous and almost periodic. Then, for any real sequence α', there is a subsequence $\alpha \subset \alpha'$ and a continuous almost periodic function $y(t)$ such that $T_\alpha x = y$ uniformly on R.

Theorem III. Let $P(t,s,x) : \Delta \times R^n \to R^n$ be continuous and almost periodic in t. Then,

(i) for any real sequence α', there is a subsequence $\alpha \subset \alpha'$ and a continuous almost periodic function E such that $T_\alpha P = E$ uniformly on any $\Delta_T \times X_J$, where $\Delta_T := \{(t,s) : 0 \leqq t - s \leqq T\}$,

(ii) if $T_\alpha P$ exists uniformly on any $\Delta_T \times X_J$ implies that $T_\alpha x$ exists uniformly on R, where $x : R \to R^n$ is a continuous almost periodic function, then $\mathrm{mod}(x)$ is contained in $\mathrm{mod}(P)$.

Let $(C, \|\cdot\|)$ be a Banach space of bounded and continuous functions $\xi : R \to R^n$ with the supremum norm $\|\cdot\|$. For any $t_0 \in R^+$, let $C_1(t_0)$ be a set of bounded functions $\xi : R^+ \to R^n$ such that $\xi(t)$ is continuous on R^+ except at t_0 and $\xi(t_0) = \xi(t_0+)$. Similarly, for any $t_0 \in R$, let $C_2(t_0)$ be a set of bounded functions $\xi : R \to R^n$ such that $\xi(t)$ is continuous on R except at t_0 and $\xi(t_0) = \xi(t_0+)$. For any $\xi \in C$, define a map M on C by

$$(M\xi)(t) := p(t) - \int_{-\infty}^{t} P(t,s,\xi(s))ds, \qquad t \in R.$$

Next for any $\xi \in C_1(t_0)$, define a map M_1 on $C_1(t_0)$ by

$$(M_1\xi)(t) := a(t) - \int_{0}^{t} D(t,s,\xi(s))ds, \qquad t \geqq t_0.$$

Similarly for any $\xi \in C_2(t_0)$, define a map M_2 on $C_2(t_0)$ by

$$(M_2\xi)(t) := a(t) - \int_{-\infty}^{t} D(t,s,\xi(s))ds, \qquad t \geqq t_0.$$

Moreover, for any $J > 0$ let $C_J := \{\xi \in C : \|\xi\| \leqq J\}$, $C_{1,J}(t_0) := \{\xi \in C_1(t_0) : \|\xi\|_+ \leqq J\}$, and $C_{2,J}(t_0) := \{\xi \in C_2(t_0) : \|\xi\| \leqq J\}$, where $\|\cdot\|_+$ denotes the supremum norm on R^+.

3. Basic lemmas. In this section we prepare three basic lemmas. First we have the following basic lemma.

Lemma 1. Under the assumptions (4)–(6), the following hold.

(i) For any $J > 0$ there is a continuous increasing function $\delta = \delta_J(\epsilon) : (0, \infty) \to (0, \infty)$ with

$$|(M\xi)(t_1) - (M\xi)(t_2)| < \epsilon \text{ if } \xi \in C_J \text{ and } |t_1 - t_2| < \delta. \tag{8}$$

(ii) If (7-k) with $k = 1$ (or 2) holds, then for any t_0 in R^+ (or R) and any $J > 0$ there is a continuous increasing function $\delta^{(k)} = \delta^{(k)}_{t_0,J}(\epsilon) : (0, \infty) \to (0, \infty)$ with

$$|(M_k\xi)(t_1) - (M_k\xi)(t_2)| < \epsilon \text{ if } \xi \in C_{k,J}(t_0) \text{ and } t_0 \leqq t_1 < t_2 < t_1 + \delta^{(k)}. \tag{9-k}$$

Since this lemma can be proved easily by a similar method to the one used in the proof of Lemma 1 in [3], we omit the proof.

Now corresponding to Equation (3), for any (e, E) in $H(p, P)$ we consider the equation

$$x(t) = e(t) - \int_{-\infty}^{t} E(t, s, x(s))ds, \qquad t \in R. \tag{3_H}$$

Then we have the following lemma.

Lemma 2. If (4)–(6) and (7-1) (or (7-2)) hold, and if (1) (or (2)) has an R^+ (or R)-bounded solution $x(t)$ with an initial time in R^+ or (R), then for any sequence $\{s_k\}$ of nonnegative numbers with $s_k \to \infty$ as $k \to \infty$, the sequence of functions $\{x_k(t)\}$ contains a subsequence which converges to an R-bounded solution $y(t)$ of (3_H) with some (e, E) in $H(p, P)$ uniformly on $[r, \infty)$ for any $r \in R$, where $x_k(t)$ is defined by

$$x_k(t) := \begin{cases} x(0), & t < -s_k, \\ x(t + s_k), & t \geqq -s_k, \end{cases} \qquad t \in R$$

(or $x_k(t) := x(t + s_k)$, $t \in R$), and $y(t)$ satisfies (3_H) on R. (In particular, if (3) has an R-bounded solution $x(t)$ which satisfies (3) on R, then the same conclusion holds for any sequence $\{s_k\}$ without (7-2)).

Proof. Suppose that (1) has an R^+-bounded solution $x(t)$ with an initial time $t_0 \in R^+$. Let $x(t)$ denote again the R-extension of the function $x(t)$ obtained by defining $x(t) := x(0)$

for $t < 0$. Clearly the set $\{x_k(t)\}$ is uniformly bounded on R. Taking a subsequence if necessary, we may assume that the sequence $\{s_k\}$ is non-decreasing. From Lemma 1(ii) with $k = 1$, $x(t)$ is uniformly continuous on $[t_0, \infty)$, and since $x_k(t)$ is an s_k-translation of $x(t)$ to the left, for any $j \in N$, the set $\{x_k(t)\}_{k \geqq j}$ is equicontinuous on $[t_0 - s_j, \infty)$, where N denotes the set of positive integers. Thus, taking a subsequence if necessary, we may assume that the sequence $\{x_k(t)\}$ converges to a bounded continuous function $y(t)$ uniformly on any compact subset of R.

Now we show that $y(t)$ satisfies (3_H) with some (e, E) in $H(p, P)$. From Theorem II and Theorem III(i), $\{s_k\}$ contains a subsequence, say $\alpha = \{s_k\}$ again for simplicity, and there are continuous almost periodic functions e and E such that $T_\alpha p = e$ uniformly on R, and for any $J > 0$ and $K > 0$, $T_\alpha P = E$ uniformly on $\Delta_K \times X_J$. From (1) we have

$$x_k(t) = p(t+s_k)+q(t+s_k)-\int_{-s_k}^{t} P(t+s_k, s+s_k, x_k(s))ds-\int_{0}^{t+s_k} Q(t+s_k, s, x(s))ds. \tag{10}$$

Let $J > 0$ be a number with $\|x\| \leqq J$. From (6), for any $\epsilon > 0$ there is a $T > 0$ with

$$\int_{-\infty}^{t} P_J(t+T, s)ds < \epsilon \text{ if } t \in R. \tag{11}$$

From (4) and (7-1), for any $t \in R$ we obtain

$$\lim_{k\to\infty} q(t+s_k) = 0$$

and

$$\limsup_{k\to\infty}\left|\int_{0}^{t+s_k} Q(t+s_k, s, x(s))ds\right| \leqq \limsup_{k\to\infty}\int_{0}^{t+s_k} Q_J(t+s_k, s)ds = 0.$$

Moreover, from (11), for any $t \in R$ we have

$$\begin{aligned}
&\limsup_{k\to\infty}\left|\int_{-s_k}^{t} P(t+s_k, s+s_k, x_k(s))ds - \int_{-\infty}^{t} E(t, s, y(s))ds\right| \\
&\leqq \limsup_{k\to\infty}\left|\int_{t-T}^{t} \big(P(t+s_k, s+s_k, x_k(s)) - E(t, s, y(s))\big)ds\right| \\
&\quad + \limsup_{k\to\infty}\int_{-\infty}^{t-T} P_J(t+s_k, s+s_k)ds + \int_{-\infty}^{t-T} |E(t, s, y(s))|ds \\
&\leqq \int_{-\infty}^{t-T} |E(t, s, y(s))|ds + \epsilon.
\end{aligned} \tag{12}$$

On the other hand, since $E = T_\alpha P$ uniformly on $\Delta_K \times X_J$ for any $K > 0$, from (6) we can easily obtain

$$\int_{-\infty}^{t-T} |E(t,s,y(s))|ds \leqq \sup\left\{\int_{-\infty}^{t-T} P_J(t,s)ds : t \in R\right\} \leqq \epsilon,$$

which together with (12) implies that $\int\limits_{-s_k}^{t} P(t+s_k, s+s_k, x_k(s))ds$ converges to $\int\limits_{-\infty}^{t} E(t,s,y(s))ds$ uniformly on R. Thus, letting $k \to \infty$ in (10), we obtain

$$y(t) = e(t) - \int_{-\infty}^{t} E(t,s,y(s))ds, \qquad t \in R,$$

which shows that $y(t)$ is an R-bounded solution of (3_H) with (e, E) in $H(p, P)$, and $y(t)$ satisfies (3_H) on R. Moreover it is easy to see that $x_k(t)$ converges to $y(t)$ uniformly on $[r, \infty)$ for any $r \in R$ as $k \to \infty$.

By similar arguments, other parts can be easily proved.

Now a function $\xi : R^+$ (or R) $\to R^n$ is said to be asymptotically almost periodic if $\xi = \psi + \mu$ such that $\psi : R \to R^n$ is continuous almost periodic, $\mu \in C_1(t_0)$ (or $C_2(t_0)$) for some $t_0 \in R^+$ (or R) and $\mu(t) \to 0$ as $t \to \infty$. Then we have the following lemma.

Lemma 3. Under the assumptions (4)–(6), the following hold.

(i) If (7-1) holds, then for any asymptotically almost periodic function ξ on R^+ such that $\xi = \psi + \mu$, $\psi : R \to R^n$ is continuous almost periodic, $\|\xi\|_+ \leqq J$ for some $J > 0$, $\mu \in C_1(t_0)$ for some $t_0 \in R^+$, and $\mu(t) \to 0$ as $t \to \infty$,

$$d_1(t) := \int_0^t D(t,s,\xi(s))ds, \qquad t \in R^+$$

is continuous asymptotically almost periodic, and its almost periodic part is given by $\pi(t) := \int\limits_{-\infty}^{t} P(t,s,\psi(s))ds$, $t \in R$, and $\mathrm{mod}(\pi) \subset \mathrm{mod}(\psi, P(X_J))$, where $P(X_J)$ denotes the restrictions of P on $\Delta \times X_J$.

(ii) If (7-2) holds, then for any asymptotically almost periodic function ξ on R such that $\xi = \psi + \mu$, $\psi : R \to R^n$ is continuous almost periodic, $\|\xi\| \leqq J$ for some $J > 0$, $\mu \in C_2(t_0)$ for some $t_0 \in R$, and $\mu(t) \to 0$ as $t \to \infty$,

$$d_2(t) := \int_{-\infty}^{t} D(t, s, \xi(s))ds, \qquad t \in R$$

is continuous asymptotically almost periodic, and its almost periodic part is given by the $\pi(t)$ in (i). In particular, if $Q(t, s, x) \equiv 0$, then

$$d(t) := \int_{-\infty}^{t} P(t, s, \xi(s))ds, \qquad t \in R$$

is continuous asymptotically almost periodic, and its almost periodic part is given by the $\pi(t)$ in (i) without (7-2).

Proof. (i) From $\|\xi\|_+ \leqq J$ we have $\|\psi\| \leqq J$. From (9-1) and the uniform continuity of $a(t)$ on R^+, $d_1(t)$ is uniformly continuous on $[t_0, \infty)$. On the other hand, if $t_0 > 0$ then from the continuity of D, it is easy to see that $d_1(t)$ is uniformly continuous on $[0, t_0]$. Thus $d_1(t)$ is uniformly continuous on R^+.

Next we prove that $\pi(t)$ is continuous and almost periodic on R. From (8), $\pi(t)$ is uniformly continuous on R. In order to prove that $\pi(t)$ is almost periodic on R, by Theorem I, it is sufficient to show that for any real sequence α', there is a subsequence $\alpha \subset \alpha'$ such that $T_\alpha \pi$ converges uniformly on R. Since P and ψ are almost periodic, taking a subsequence $\alpha = \{t_k\} \subset \alpha'$ if necessary, we may assume that $T_\alpha P$ converges to some E uniformly on $\Delta_K \times X_J$ for any $K > 0$, and $T_\alpha \psi$ converges to some η uniformly on R. Now we show that $\int_{-\infty}^{t+t_k} P(t + t_k, s, \psi(s))ds$ converges to $\int_{-\infty}^{t} P(t, s, \eta(s))ds$ uniformly on R. From (6), for any $\epsilon > 0$ there is a $T_1 > 0$ with

$$(13) \qquad \int_{-\infty}^{t-T_1} P_J(t, s)ds < \epsilon \quad \text{if} \quad t \in R.$$

Since $T_\alpha D$ converges to E uniformly on $\Delta_{T_1} \times X_J$, there is a $K_1 > 0$ such that $k \geqq K_1$ implies

$$|P(t+t_k, s+t_k, x) - E(t,s,x)| < \frac{\epsilon}{T_1} \text{ on } \Delta_{T_1} \times X_J. \tag{14}$$

Moreover, since $E(t,s,x)$ is uniformly continuous on $\Delta_{T_1} \times X_J$, there is a $\delta > 0$ with

$$|E(t,s,x) - E(t,s,y)| < \frac{\epsilon}{T_1} \text{ if } |x-y| < \delta \text{ on } \Delta_{T_1} \times X_J. \tag{15}$$

For this δ, there is a $K_2 > 0$ such that $k \geqq K_2$ implies

$$|\psi(t+t_k) - \eta(t)| < \delta \text{ on } R. \tag{16}$$

Thus from (13)–(16), for $k \geqq \max(K_1, K_2)$ and any $t \in R$ we have

$$\begin{aligned}
&\left| \int_{-\infty}^{t+t_k} P(t+t_k, s, \psi(s))ds - \int_{-\infty}^{t} E(t,s,\eta(s))ds \right| \\
\leqq& \int_{t-T}^{t} |P(t+t_k, s+t_k, \psi(s+t_k)) - E(t,s,\eta(s))|ds \\
&+ \int_{-\infty}^{t+t_k-T} P_J(t+t_k, s)ds + \int_{-\infty}^{t-T} |E(t,s,\eta(s))|ds \\
<& \int_{t-T}^{t} |P(t+t_k, s+t_k, \psi(s+t_k)) - E(t,s,\psi(s+t_k))|ds \\
&+ \int_{t-T}^{t} |E(t,s,\psi(s+t_k)) - E(t,s,\eta(s))|ds + \int_{-\infty}^{t-T} |E(t,s,\eta(s))|ds + \epsilon \\
<& \int_{-\infty}^{t-T} |E(t,s,\eta(s))|ds + 3\epsilon.
\end{aligned} \tag{17}$$

From (13) we have

$$\int_{-\infty}^{t-T_1} |E(t,s,\eta(s))|ds \leqq \sup\left\{ \int_{-\infty}^{t-T_1} P_J(t,s)ds : t \in R \right\} \leqq \epsilon,$$

which together with (17) yields that $\int\limits_{-\infty}^{t+t_k} P(t+t_k, s, \xi(s))ds$ converges to $\int\limits_{-\infty}^{t} E(t,s,\eta(s))ds$ uniformly on R. Thus, $\pi(t)$ is almost periodic on R. Moreover, by

Theorem III(ii) and similar arguments as in the above, it is easy to see that $\mathrm{mod}(\pi) \subset \mathrm{mod}(\psi, P(X_J))$.

In order to prove that $d_1(t)$ is asymptotically almost periodic and its almost periodic part is given by $\pi(t)$, we need only prove that

$$d_1(t) - \pi(t) \to 0 \text{ as } t \to \infty. \tag{18}$$

For any $t \in R^+$, $d_1(t) - \pi(t)$ is expressed as

$$\int_0^t (P(t,s,\xi(s)) - P(t,s,\psi(s)))\,ds - \int_{-\infty}^0 P(t,s,\psi(s))ds + \int_0^t Q(t,s,\xi(s))ds. \tag{19}$$

For any $\epsilon > 0$ let $T_1 > 0$ be a number as in (13). Since $P(t,s,x)$ is uniformly continuous on $U := \Delta_{T_1} \times X_J$, for the ϵ there is a $\delta > 0$ with

$$|P(t,s,x) - P(t,s,\eta)| < \frac{\epsilon}{T_1} \text{ if } (t,s,x),\ (t,s,y) \in U \text{ and } |x-y| < \delta.$$

Moreover, since $\mu(t) \to 0$ as $t \to \infty$, for the δ there is a $T_2 > 0$ with

$$|\mu(t)| = |\xi(t) - \psi(t)| < \delta \text{ if } t \geqq T_2.$$

For $t \geqq T_1 + T_2$ the first term of (19) is estimated as

$$\begin{aligned}
&\left|\int_0^t (P(t,s,\xi(s)) - P(t,s,\psi(s)))\,ds\right| \\
&\leqq 2\int_0^{t-T_1} P_J(t,s)ds + \int_{t-T_1}^t |P(t,s,\xi(s)) - P(t,s,\psi(s))|ds < 3\epsilon,
\end{aligned}$$

which implies

$$\int_0^t (P(t,s,\xi(s)) - P(t,s,\psi(s)))\,ds \to 0 \text{ as } t \to \infty. \tag{20}$$

Next from (13), for $t \geqq T_1$ we obtain

$$\left|\int_{-\infty}^0 P(t,s,\psi(s))ds\right| \leqq \int_{-\infty}^0 P_J(t,s)ds < \epsilon,$$

which yields

$$\int_{-\infty}^{0} P(t,s,\psi(s))ds \to 0 \text{ as } t \to \infty. \tag{21}$$

Finally the third term of (19) is estimated as

$$\left|\int_{0}^{t} Q(t,s,\xi(s))ds\right| \leqq \int_{0}^{t} Q_J(t,s)ds, \qquad t \in R^+,$$

which together with (7-1) implies

$$\int_{0}^{t} Q(t,s,\xi(s))ds \to 0 \text{ as } t \to \infty. \tag{22}$$

Thus, from (19) and (20)–(22) we can conclude that (18) holds.

(ii) From (9-2) and the uniform continuity of $a(t)$ on $[t_0,\infty)$, $d_2(t)$ is uniformly continuous on $[t_0,\infty)$. On the other hand, from (6), (7-2) and the continuity of D, it is easy to see that $d_2(t)$ is continuous on $(-\infty, t_0]$. Thus $d_2(t)$ is continuous on R.

In order to prove that $d_2(t)$ is asymptotically almost periodic and its almost periodic part is given by $\pi(t)$, we need only prove that

$$d_2(t) - \pi(t) \to 0 \text{ as } t \to \infty. \tag{23}$$

For any $t \in R$ we have

$$d_2(t) - \pi(t) = \int_{-\infty}^{t} \left(P(t,s,\xi(s)) - P(t,s,\psi(s))\right) ds + \int_{-\infty}^{t} Q(t,s,\xi(s))ds,$$

which together with similar arguments to those in the proof of (i) easily yields (23). The latter part is a direct consequence of the former part.

4. Relations among (1)–(3). In this section we discuss relations among solutions of (1)–(3). First we have the following theorem.

Theorem 1. In addition to (4)–(6), suppose that (3) has a unique R-bounded solution $x_0(t)$ with $\|x_0\| \leqq J$ for some $J > 0$ which satisfies (3) on R, and that for any (e, E) in $H(p, P)$, (3_H) with (e, E) has a unique R-bounded solution which satisfies (3_H) on R.

Then, $x_0(t)$ and the R-bounded solution of (3_H) with any (e, E) in $H(p, P)$ are almost periodic, $\mathrm{mod}(x_0) \subset \mathrm{mod}(p, P(X_J))$, and any R-bounded solution of (3) with an initial time in R is asymptotically almost periodic and approaches $x_0(t)$ as $t \to \infty$. Moreover, if (7-1) (or (7-2)) holds, then any R^+ (or R)-bounded solution of (1) (or (2)) with an initial time in R^+ (or R) is asymptotically almost periodic and approaches $x_0(t)$ as $t \to \infty$.

Proof. First we prove that $x_0(t)$ is almost periodic. By Theorem I, it is sufficient to prove that any real sequence $\rho' = \{r_k\}$ contains a subsequence ρ such that $T_\rho x_0$ converges uniformly on R. For any $k \in N$, let $x_k(t) := x_0(t+r_k)$, $t \in R$. Then clearly the set $\{x_k(t)\}$ is uniformly bounded on R. Moreover from Lemma 1(i), the set $\{x_k(t)\}$ is equicontinuous on R. Thus, taking a subsequence if necessary, we may assume that the sequence $\{x_k(t)\}$ converges to a bounded continuous function $y(t)$ uniformly on any compact subset of R. Since $p(t)$ and $P(t, s, x)$ are almost periodic, taking a subsequence if necessary, we may assume that for some (e, E) in $H(p, P)$, $T_\rho p = e$ uniformly on R and $T_\rho P = E$ uniformly on $\Delta_K \times X_J$ for any $K > 0$. From (3) we have

$$x_k(t) = p(t + r_k) - \int_{-\infty}^{t} P(t + r_k, s + r_k, x_k(s))ds, \qquad t \in R.$$

Thus, by similar arguments as in the proof of Lemma 2, it is easily seen that $\{x_k(t)\}$ converges to $e(t) - \int_{-\infty}^{t} E(t, s, y(s))ds$ uniformly on R, and $y(t)$ satisfies (3_H) with the (e, E) on R. Hence $x(t)$ is almost periodic. The almost periodicity of each R-bounded solution of (3_H) can be proved similarly. Moreover, by Theorem III(ii), the uniqueness of R-bounded solutions of (3_H) for any (e, E) in $H(p, P)$ satisfying (3_H) on R, and similar arguments as in the above, it is easy to see that $\mathrm{mod}(x_0) \subset \mathrm{mod}(p, P(X_J))$.

Next we prove that any R-bounded solution $x(t) = x(t, t_0, \phi)$ with $t_0 \in R$ and a bounded continuous initial function ϕ is asymptotically almost periodic and approaches $x_0(t)$ as $t \to \infty$. Replacing the J by a greater one if necessary, we may assume that $\|x\| \leqq J$. Let $\alpha = \{t_k\}$ be a sequence such that $t_k \geqq t_0$, $t_k \to \infty$ as $k \to \infty$, $T_\alpha p = p$ uniformly on R, and $T_\alpha P = P$ uniformly on $\Delta_K \times X_J$ for any $K > 0$. Suppose that $x(t)$ does not approach $x_0(t)$ as $t \to \infty$. Then there is an $\epsilon_0 > 0$ with

$$\limsup_{t\to\infty} |x(t) - x_0(t)| > 2\epsilon_0. \tag{24}$$

For any $k \in N$ let $x_k(t) := x_0(t + t_k)$ and $\xi_k(t) := x(t + t_k)$, $t \in R$. From Lemma 2 and the assumption that $x_0(t)$ is a unique R-bounded solution of (3) which satisfies (3) on R, taking a subsequence if necessary, we may assume that each of $\{x_k(t)\}$ and $\{\xi_k(t)\}$ converges to $x_0(t)$ uniformly on R^+. Thus for the ϵ_0 there is a $k \in N$ with

$$\sup\big\{\max\big(|x_k(t) - x_0(t)|,\ |\xi_k(t) - x_0(t)|\big) : t \in R^+\big\} \leqq \epsilon_0,$$

which contradicts (24). Another case can be proved similarly as in the above using the arguments in the proof of Lemma 2.

Remark 1. (i) By assuming that (3) satisfies a separation condition instead of the assumption that for any (e, E) in $H(p, P)$, (3_H) has a unique R-bounded solution which satisfies (3_H) on R, we can obtain a similar theorem to Theorem 1.

(ii) Without the uniqueness assumption of R-bounded solutions, we can conclude that for any (e, E) in $H(p, P)$, any R-bounded solution of (3_H) with (e, E) satisfying (3_H) on R is almost periodic, though the module containment is not necessarily obtained.

Corresponding to (1) (or (2)) for any $J > 0$ and $t_0 \in R^+$ (or R) consider the assumption

$$|a(t)| + \int_0^t P_J(t, s)ds + \int_0^t Q_J(t, s)ds \leqq J \text{ if } t \geqq t_0 \tag{25-1}$$

or

$$|a(t)| + \int_{-\infty}^{t} P_J(t,s)ds + \int_{-\infty}^{t} Q_J(t,s)ds \leqq J \text{ if } t \geqq t_0. \tag{25-2}$$

Then, from Theorem 1 we have the following corollary.

Corollary 1. Under the assumptions of Theorem 1, the following hold.

(i) If (25-1) (or (25-2)) holds for some $J > 0$ and $t_0 \in R^+$ (or R), then for any initial function ϕ taking values in X_J, any solution $x(t, t_0, \phi)$ of (1) (or (2)) is asymptotically almost periodic and approaches $x_0(t)$ as $t \to \infty$.

(ii) If for any $K > 0$ there is a $J > K$ such that (25-1) (or (25-2)) holds for the J and any $t_0 \in R^+$ (or R), then any solution of (1) (or (2)) is asymptotically almost periodic and approaches $x_0(t)$ as $t \to \infty$.

Remark 2. (i) It is easy to see that if $\|a\| < \infty$, $P(t,s,x) = P(t,s)x$, and if for some $\lambda < 1$, $\sup\left\{\int\limits_{-\infty}^{t} |P(t,s)|ds : t \in R\right\} \leqq \lambda$ with $|P| = \sup\{|Px| : |x| = 1\}$, then for any $K > 0$ there is a $J > K$ such that (25-1) (or (25-2)) with $Q_J(t,s) \equiv 0$ holds for the J and any $t_0 \in R$.

(ii) In [3], we showed the existence of an asymptotically periodic solution of an asymptotically periodic equation using Schauder's first theorem. But a similar method is not applicable to (1) unless (1) is an asymptotically periodic equation (see 8.1 in [6]).

Next, from Lemma 3 we have the following theorem.

Theorem 2. If (4)–(6) and (7-1) (or (7-2)) hold, and if (1) (or (2)) has an asymptotically almost periodic solution with an initial time in R^+ (or R), then its almost periodic part is an almost periodic solution of (3).

Proof. Let $x(t)$ be an asymptotically almost periodic solution of (1) with an initial time $t_0 \in R^+$ such that $x(t) = y(t) + z(t)$, $y(t)$ is continuous almost periodic on R,

$z \in C_1(t_0)$ and $z(t) \to 0$ as $t \to \infty$. Then we have

$$y(t) + z(t) = p(t) + q(t) - \int_0^t D(t, s, x(s))ds, \quad t \geqq t_0. \tag{26}$$

From Lemma 3, taking the almost periodic part of both sides of (26) we obtain

$$y(t) = p(t) - \int_{-\infty}^t P(t, s, y(s))ds, \qquad t \geqq t_0.$$

From this, it is easy to see that $y(t)$ is an almost periodic solution of (3).

Another part can be proved similarly.

Concerning a relation between (1) and (3), we have the following theorem.

Theorem 3. Under the assumptions (4)–(6) and (7-1), the following five conditions are equivalent.

(i) Equation (3) has an almost periodic solution.

(ii) For some $q(t)$ and $Q(t, s, x) \equiv 0$, (1) has an almost periodic solution which satisfies (1) on R^+.

(iii) For some $q(t)$ and $Q(t, s, x) \equiv 0$, (1) has an asymptotically almost periodic solution with an initial time in R^+.

(iv) For some $q(t)$ and $Q(t, s, x)$, (1) has an almost periodic solution which satisfies (1) on R^+.

(v) For some $q(t)$ and $Q(t, s, x)$, (1) has an asymptotically almost periodic solution with an initial time in R^+.

Proof. First we prove that (i) implies (ii). Let $\psi(t)$ be an almost periodic solution of (3) and let

$$q(t) := -\int_{-\infty}^0 P(t, s, \psi(s))ds, \qquad t \in R^+.$$

Then clearly $q(t)$ is continuous and $q(t) \to 0$ as $t \to \infty$. Thus it is easy to see that for the $q(t)$ and $Q(t, s, x) \equiv 0$, (1) has an almost periodic solution $\psi(t)$, which satisfies (1) on R^+.

Next, it is clear that (ii) and (iii) imply (iii) and (v) respectively. Moreover, from Theorem 2, (v) yields (i).

Finally, since it is trivial that (ii) implies (iv), we prove that (iv) yields (ii). Let $\pi(t)$ be an almost periodic solution of (1) with some $q(t)$ and $Q(t,s,x)$ which satisfies (1) on R^+, and let

$$\gamma(t) := -\int_0^t Q(t,s,\pi(s))ds, \qquad t \in R^+.$$

Then clearly $\gamma(t)$ is continuous and $\gamma(t) \to 0$ as $t \to \infty$. Thus it is easy to see that for $a(t) = p(t) + q(t) + \gamma(t)$ and $Q(t,s,x) \equiv 0$, (1) has an almost periodic solution $\pi(t)$ which satisfies (1) on R^+.

5. Almost periodic solutions. Among the assumptions of Theorem 1, the uniqueness of R-bounded solutions of (3) which satisfy (3) on R seems to be most important. In this section we give two different conditions for the uniqueness of R-bounded solutions of (3) which satisfy (3) on R, and show the existence of almost periodic solutions.

First we give a condition of a contraction type. Suppose that for any $J > 0$ there is a continuous function $L_J : \Delta \to R^+$ such that $L_J(t,s)$ is almost periodic in t and that

$$(27) \qquad |P(t,s,x) - P(t,s,y)| \leqq L_J(t,s)|x-y| \text{ if } t,s \in R,\ |x| \leqq J \text{ and } |y| \leqq J.$$

Then we have the following theorem.

Theorem 4. Under the assumptions (4)–(6) and (27), the following hold.

(i) If for some $\lambda < 1$

$$(28) \qquad \lambda_J := \sup\left\{\int_{-\infty}^t L_J(t,s)ds : t \in R\right\} \leqq \lambda \text{ for any } J > 0$$

holds, then for any (e,E) in $H(p,P)$, (3_H) has a unique almost periodic solution, and it is a unique R-bounded solution which satisfies (3_H) on R and its module is contained in $\mathrm{mod}(p,P)$.

(ii) If for some $J > 0$

$$\|p\| + \int_{-\infty}^{t} P_J(t,s)ds \leqq J \text{ for any } t \in R$$

and $\lambda_J < 1$ hold, then for any (e, E) in $H(p, P)$, (3_H) has a unique almost periodic solution in X_J, and it is a unique R-bounded solution in X_J which satisfies (3_H) on R and its module is contained in $\mathrm{mod}(p, P(X_J))$.

Proof. From (27) and (28) it is easy to see that for any (e, E) in $H(p, P)$, there is a K_J in $H(L_J)$ such that K_J satisfies (27) and (28) with $L_J = K_J$. Let $(A, \|\cdot\|)$ be the complete metric space of continuous almost periodic functions $\xi : R \to R^n$ with the supremum norm $\|\cdot\|$ such that $\mathrm{mod}(\xi)$ is contained in $\mathrm{mod}(p, P)$. From Lemma 3(ii), the map M defined by

$$(M\xi)(t) := e(t) - \int_{-\infty}^{t} E(t, s, \xi(s))ds, \qquad t \in R$$

maps A into A. Moreover, for any $\xi_i \in A$ with $\|\xi_i\| \leqq J$ $(i = 1, 2)$ for some $J > 0$ we have

$$|(M\xi_1)(t) - (M\xi_2)(t)| \leqq \int_{-\infty}^{t} K_J(t,s)|\xi_1(s) - \xi_2(s)|ds \leqq \lambda_J\|\xi_1 - \xi_2\|, \quad t \in R,$$

which together with (28) yields

$$\|M\xi_1 - M\xi_2\| \leqq \lambda\|\xi_1 - \xi_2\|.$$

Thus $M : A \to A$ is a contraction mapping. Hence M has a unique fixed point in A, which gives a unique almost periodic solution of (3_H), say $\pi(t)$. It is easy to see that from (27) and (28), $\pi(t)$ is a unique R-bounded solution which satisfies (3_H) on R, and that $\mathrm{mod}(\pi) \subset \mathrm{mod}(p, P(X_J)$.

(ii) This part can be proved similarly as in the proof of (i) by taking a subset S of A defined by

$$S := \{\ \xi \in A : \|\xi\| \leqq J \text{ and } \mathrm{mod}(\xi) \subset \mathrm{mod}(p, P(X_J))\}\,.$$

Although Theorems 1 and 4 give asymptotic behavior of R-bounded solutions of (3_H), they do not necessarily give asymptotic behavior of all solutions of (3_H). But, for linear equations, we can obtain asymptotic behavior of all solutions. Consider the linear equation

$$x(t) = p(t) - \int_{-\infty}^{t} P(t,s)x(s)ds, \qquad t \in R, \tag{29}$$

where $p : R \to R^n$ and $P : R \times R \to R^{n\times n}$ are continuous and almost periodic. Then we have the following theorem.

Theorem 5. If for some $\lambda < 1$

$$\int_{-\infty}^{t} |P(t,s)|ds \leqq \lambda \text{ for any } t \in R \tag{30}$$

holds, then (29) has a unique R-bounded solution which satisfies (29) on R, and it is an almost periodic solution with a module contained in $\mathrm{mod}(p, P)$, and globally attractive.

Proof. It is easy to see that for any $t_0 \in R$ and any bounded continuous function $\phi : (-\infty, t_0) \to R^n$, the solution $x(t) := x(t, t_0, \phi)$ of (29) satisfies

$$|x(t)| \leqq \max\bigl(\sup\{\Pi(s) : t_0 \leqq s \leqq t\},\ \sup\{|\phi(s)| : s < t_0\},\ |x(t_0+)|\bigr), \quad t \geqq t_0,$$

where $\Pi(s) := \sup\{|p(u)| : t_0 \leqq u \leqq s\}/(1-\lambda)$, $s \geqq t_0$. Thus, the conclusions are direct consequences of Theorems 1 and 4.

Now we show an example.

Example. Consider the scalar linear equation

$$x(t) = p(t) - m\int_{-\infty}^{t} e^{-t+s}b(t)x(s)ds, \qquad t \in R, \tag{31}$$

where $p : R \to R$ is continuous almost periodic, m is a constant with $|m| < 1$, and $b(t) := \sum_{k=1}^{\infty} 2^{-k}\cos kt$. Equation (31) is a special case of (29) with $n = 1$ and $P(t,s) = m\,e^{-t+s}b(t)$. Hence, (4) with $q(t) \equiv 0$, (5) with $Q(t,s,x) \equiv 0$, (27) with $L_J(t,s) = |m|e^{-t+s}b(t)$, and (30) with $\lambda = |m|$ hold. Thus, from Theorem 4, (31) has a unique R-bounded solution satisfying (31) on R, say $\pi(t)$, and it is an almost periodic solution with $\mathrm{mod}(\pi) \subset \mathrm{mod}(p, P)$, and globally attractive.

On the other hand, it is easy to see that $\pi(t)$ is a unique R^+-bounded solution of the equation

$$(32) \qquad x(t) = p(t) - m\int_{-\infty}^{0} e^{-t+s}b(t)\pi(s)ds - m\int_{0}^{t} e^{-t+s}b(t)x(s)ds, \quad t \in R^+,$$

which satisfies (32) on R^+. Moreover, from Corollary 1(ii) the almost periodic solution $\pi(t)$ of (32) is globally attractive.

Next we seek another condition for the uniqueness of R-bounded solutions of the equation

$$(33) \qquad x(t) = p(t) - \int_{-\infty}^{t} G(t,s)g(s,x(s))ds, \qquad t \in R,$$

where $p : R \to R^n$, $G : R \times R \to R^{n\times n}$ and $g : R \times R^n \to R^n$ are continuous, and

$$(34) \qquad G_s(t) \text{ is continuous, symmetric, and } G_{st}(t,s) \text{ is continuous,}$$

$$(35) \qquad \int_{-\infty}^{t} \left(|G(t,s)| + |G_s(t,s)|(t-s)^2 + |G_{st}(t,s)|(t-s)^2\right) ds \text{ is } R\text{-bounded,}$$

$$(36) \qquad \int_{-\infty}^{t-T} |G_s(t,s)|(t-s)^2 ds \to 0 \text{ uniformly for } t \in R \text{ as } T \to \infty,$$

$$(37) \qquad \lim_{s\to-\infty} sG(t,s) = 0 \text{ for each fixed } t \in R,$$

$$(38) \qquad G_{st}(t,s) \text{ is negative (positive) semi-definite,}$$

and

$$(39) \qquad g(t,x) - g(t,y) = C(t,x,y)(x-y) \text{ if } t \in R,\ x \in R^n \text{ and } y \in R^n,$$

where $C : R \times R^n \times R^n \to R^{n\times n}$ is continuous, symmetric, and positive (negative) definite, and for any $J > 0$ there are $A_J > 0$ and B_J with

$$A_J \leqq |C(t,x,y)| \leqq B_J \text{ if } t \in R,\ |x| \leqq J \text{ and } |y| \leqq J.$$

For (33), which is a special case of (3), we have the following theorem.

Theorem 6. If (34)–(39) hold, then (33) has at most one R-bounded solution which satisfies (33) on R.

This theorem can be proved using a function

$$V(t) := \int_{-\infty}^{t} \left(\int_{s}^{t} z^*(v) C(v) dv \right) G_s(t,s) \int_{s}^{t} C(v) z(v) dv \, ds, \quad t \in R,$$

where z^* denotes the transpose of z, $z(t) := x_1(t) - x_2(t)$, $t \in R$, $C(t) := C(t, x_1(t), x_2(t))$, $t \in R$, and where $x_1(t)$ and $x_2(t)$ are R-bounded solutions of (33) which satisfy (33) on R. For the details, see Theorem 3 in [3].

Combining Theorems 1 and 6, we have the following corollary.

Corollary 2. In addition to (34)–(39), if $p(t)$ and $G(t,s,x) := G(t,s)g(s,x)$ are almost periodic in t, and if (33) has an R-bounded solution with an initial time in R, then (33) has a unique almost periodic solution with a module contained in $\mathrm{mod}(p, G)$, and it is a unique R-bounded solution of (33) which satisfies (33) on R. Moreover, any R-bounded solution of (33) approaches the unique almost periodic solution as $t \to \infty$.

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