T. A. Burton
Department of Mathematics
Southern Illinois University
Carbondale, Illinois 62901

Admissible Controls in a PDE of Lurie Type

0.1 Introduction

We consider two forms of a Lurie system, the simplest of which is

$$\begin{aligned} u_t &= \Delta u + b(x)f(\sigma) + \int_{t-h}^{t} p(s, u(s,X), \nabla u(s,X))ds \\ \sigma' &= \int_\Omega c(X)u(t,X)dX - rf(\sigma), \end{aligned}$$

$u(t,X) = 0$ on $\partial\Omega$, Ω is a domain with smooth boundary, $\Delta u = u_{xx}+u_{yy}+u_{zz}$, $X = (x,y,z)$, and where

$$\sigma f(\sigma) > 0 \text{ if } \sigma \neq 0. \tag{*}$$

The exact forms are given in (15a), (16a), and (46).

The goal is to give conditions to ensure that all solutions satisfy

$$|\sigma(t)| + \int_\Omega u^2(t,X)dX \to 0 \text{ as } t \to \infty$$

without strengthening (*). That is, conditions are sought to ensure that every f satisfying (*) is admissible.

This is a control problem, the first stage of which is to show that all solutions are controllable to zero for every f satisfying (*) (this is absolute stability); the next stage (which we do not consider) is to minimize a cost functional with the assurance that solutions will tend to zero for the control function f which minimizes the cost functional. (c.f. Curtain and Pritchard [4; p. 3] and Nakagiri[10; p. 175].)

Appropriate conditions are given to ensure absolute stability and these use a result in [2] which gives a very surprising differential inequality for a Liapunov functional of the form

$$V'(t) \leq \alpha V(t)$$

derived from a standard inequality relating V' only to space variables.

0.2 Background

A classical control problem in ordinary differential equations, called the problem of Lurie, has several forms, one of which is

$$\left\{ \begin{aligned} x' &= Ax + bf(\sigma) \\ \sigma' &= c^T x - rf(\sigma) \end{aligned} \right. \tag{1}$$

where b and c are constant column vectors, r is a positive constant, A is an $n \times n$ constant matrix all of whose characteristic roots have negative real parts, f is continuous, and $\sigma f(\sigma) > 0$ if $\sigma \neq 0$. The problem is to give conditions to ensure that every solution tends to zero. Discussion of the problem may be found in [6], [7], [13], for example.

Lurie used the Liapunov function

$$V(x, \sigma) = x^T B x + \int_0^\sigma f(s) ds \tag{2}$$

with $B^T = B$ and $A^T B + BA = -D$ for B and D positive definite. That function has played a major role in the problem, as is seen in the extensive survey book of Lefschetz [7]. The derivative of V along a solution of (1) satisfies

$$V'(x(t), \sigma(t)) = -x^T D x + f(\sigma)[2b^T B + c^T]x - r f^2(\sigma) \tag{3}$$

and Lefschetz [7] showed that this is negative definite if and only if

$$r > (Bb + c/2)^T D^{-1} (Bb + c/2). \tag{4}$$

Still, unless $\int_0^\sigma f(s)ds \to \infty$ as $|\sigma| \to \infty$, one may not immediately conclude that solutions are bounded (and, therefore, tend to zero).

The simple remedy is to ask that admissible controls satisfy

$$f^2(\sigma) \geq \int_0^\sigma f(s) ds \ \ \text{(and so} \ \int_0^{\pm\infty} f(s) ds = +\infty) \tag{**}$$

so that $V' \leq -\gamma V$, from which we conclude that $V(t) \leq V(0)e^{-\gamma t}$ and that σ is bounded and tends to zero.

But LaSalle [6] showed that solutions of (1) are bounded if and only if

$$C^T A^{-1} b + r > 0, \tag{5}$$

and it is known that (4) implies (5), so (3) and (4) imply that all solutions of (1) tend to zero.

There is a complete parallel between Liapunov functions for (1) and the PDE of interest here, with the same difficulty of proving boundedness occurring. It was hoped that (5) would prove central for the PDE case as well, but LaSalle's idea does not seem to extend to that case. It may be noted that (5) has attracted considerable attention. For example, in [3] we showed that if $f(\sigma)/\sigma \to 0$ as $|\sigma| \to \infty$, then solutions of (1) are uniformly ultimately bounded if and only if (5) holds. Somolinos [12] strengthened that result to $f(\sigma)/\sigma$ bounded as $|\sigma| \to \infty$, among other results.

In [8] (see also [9] and [10]) Nakagiri introduced as a counterpart of (1) the equation

$$u_t = \Delta u + \int_{-h}^0 d\eta(s, X) u(t+s, X) ds + c(X) f(\sigma(t)) \tag{6}$$

where $\sigma(t) = \int_\Omega d(X) u(t, X) dX$ and $u = 0$ on $\partial\Omega$. This formulation is not parallel to (1), but rather to

$$\begin{cases} x' &= Ax + bf(\sigma) \\ \sigma &= d^T x, \end{cases}$$

as given by Lefschetz [7; pp. 39-40], so that

$$\sigma' = d^T x' = d^T[Ax + bf(\sigma)] = d^T Ax + d^T bf(\sigma)$$

which results in (1) when $d^T A = c^T$ and $d^T b = -r < 0$. We discuss a form of (6) in the last section.

Nakagiri solves his problem using semigroup theory and frequency domain techniques. It is very interesting to note that some of the same questions arise for (6) as for (1). In particular, Nakagiri asked conditions similar to (**) and required that

$$\sigma f(\sigma) \le k\sigma^2 \text{ and } \int_0^\sigma f(s)ds \to \infty \text{ as } |\sigma| \to \infty \tag{***}$$

in order for a control to be admissible. We will not need such conditions.

To bring into focus the work here and to introduce the main boundedness technique that we use, we summarize the analysis of a one-dimensional problem. Consider

$$\begin{cases} u_t &= g(u_x)_x + b(x)f(\sigma) + \int_{t-h}^t p(s, u(s,x), u_x(s,x))ds \\ \\ \sigma' &= \int_0^1 c(x)u(t,x)dx - rf(\sigma), \end{cases} \tag{7}$$

$$u(t,0) = u(t,1) = 0, \tag{8}$$

where $0 < \alpha \le dg(x)/dx$ for some $\alpha > 0$, $\int_0^1 p^2(t,u,u_x)dx \le \beta \int_0^1 u_x^2 ds$ for some $\beta > 0$, and b and c are at least L^2 functions. Under certain conditions the Liapunov functional

$$V_1(t) = \int_0^1 [(\frac{1}{2})u^2 + k\int_{-h}^0 \int_{t+s}^t p^2(v,u,u_x)dvds]dx + \int_0^\sigma f(s)ds \tag{9}$$

will satisfy

$$V_1'(t) \le -\lambda \int_0^1 [u^2 + f^2(\sigma) + \int_{t-h}^t p^2(s,u,u_x)ds]dx \tag{10}$$

If $\int_0^\sigma f(s)ds$ does not diverge with σ, then the question of boundedness raised concerning (1) occurs once more.

And a major point of this paper is to show that $\sigma(t)$ is bounded without any condition on f except (*).

In order to prove that solutions of the PDE are bounded we need a result in [2] which may be stated as follows. Let

$$x' = F(t,x) \tag{11}$$

and suppose that $V_1, P : [0,\infty) \times \mathbb{R}^{n+1} \to [0,\infty)$, $U : [0,\infty) \times \mathbb{R} \to [0,\infty)$, and $Q : [0,\infty) \times [0,\infty) \times \mathbb{R}^{n+1} \to [0,\infty)$ are continuous functions with

$$V_1(t,x) = P(t,x) + U(t,z), \tag{12}$$

where $x = (x_1, \dots, x_n, z)$, and we also suppose that

$$V_1'(t,x) \le -Q(P(t,x),t,x) \tag{13a}$$

with $Q(P,t,x) > 0$ if $P > 0$ and Q increasing in P. (In [2], this last condition was inadvertantly left out, but was used in the proof.)

Theorem B. Let (12) and (13) hold and suppose there is an $L > 0$ such that if $t_n \to \infty$ and $z_n \to \infty$ then

$$U(t_n, z_n) \to L, \quad U(t,z) < L \text{ for all } (t,z) \text{ with } z > 0. \tag{14}$$

If $V = V_1 - L$ and if $x(t)$ is a solution of (11) on $[t_o, \infty)$ with $\limsup_{t\to\infty} z(t) = \infty$, then

$$V'(t,x) \le -Q(V(t,x)/2, t, x) \text{ for } z(t) > 0. \tag{13b}$$

Our application will have (13b) as $V' \le -\phi V$, ϕ constant.

In this result the connection of $x(t)$ to (11) is immaterial. If $x(t)$ is any function on $[t_o, \infty)$ for which (13) holds, then the result is valid.

We have discussed two formulations of the Lurie problem in terms of (1) and (6). There is yet a third standard formulation (c.f., Lefschetz [7]) of the form

$$\begin{aligned} w_t &= \Delta w + \int_{t-h}^{t} p(s, u(s,X), \nabla u(s,X))ds + b(X)\mu(t), \\ \mu'(t) &= f(\sigma), \\ \sigma &= \int_\Omega d(X)w(t,X)dX - r\mu. \end{aligned}$$

This formulation can be transformed into a system closely related to (6). We do not consider it here.

0.3 Absolute stability

In this section we consider two systems, the first of which is

$$\begin{aligned} u_t &= \Delta u + b(X)f(\sigma) + \int_{t-h}^{t} p(s, u(s,X), \nabla u(s,X))ds, \\ \sigma' &= \int_\Omega c(X)u(t,X)dX - rf(\sigma), \end{aligned} \tag{15a}$$

$$\begin{aligned} u(t,X) &= 0 \text{ for } X \in \partial\Omega, \\ u(0,X) &= g^0(X), u(s,X) = g^1(s,X) \text{ a.e. } s \in [-h,0], X \in \Omega, \end{aligned} \tag{16a}$$

where f is continuous, (*) holds,

$$\int_\Omega |p(s,u,\nabla u)|^2 dX \le \beta \int_\Omega |\nabla u|^2 dX \text{ for some } \beta > 0,$$

$c(X)$ and $b(X)$ are at least $L^2(\Omega)$, r and h are positive constants, and p is at least continuous.

The Sobolev estimates which we will need also work with the quasi-linear term in

$$\begin{cases} u_t = g(u_x)_x + b(x)f(\sigma) + \int_{t-h}^{t} p(s, u(s,x), u_x(s,x))ds, \\ \sigma' = \int_0^1 c(x)u(t,x)dx - rf(\sigma), \end{cases} \tag{15b}$$

$$\begin{cases} u(t,0) &= u(t,1)=0 \\ u(0,x) &= g^0(x), u(s,x)=g^1(s,x) \text{ a.e. } s\in[-h,0), 0\le x\le 1, \end{cases} \tag{16b}$$

$\int_0^1 |p(s,u,u_x)|^2 dx \le \beta \int_0^1 u_x^2 dx$ for some $\beta > 0$, $c(x)$ and $b(x)$ are at least $L^2(0,1)$, r and h are positive constants, p is at least continuous, $dg(x)/dx \overset{def}{=} g'(x) \ge \alpha$ for some $\alpha > 0$.

Not only will the Sobolev estimates work for this system, but we will conclude absolute stability in the supremum norm for (15b), while we can only prove absolute stability in the L^2-norm for (15a). Nakagiri's stability is also in the L^2-norm. An existence result will be sketched in the next section, but our primary goal here is to establish strong a priori bounds on solution which do not require strenghthening of (*).

Theorem 1. Suppose that $b(X)$ and $c(X)$ are in $L^2(\Omega)$, $b(X)+c(X)$ is bounded, $g'(r) \ge \alpha > 0$, λ_1 is the first eigenvalue of $-\Delta$ on H_0^1 and $k - (\frac{1}{2}) = \gamma > 0$ with

$$\lambda_1(1-kh\beta) - (h/2) \overset{def}{=} \mu > 0, \tag{19}$$

and

$$\int_\Omega (b(X)+c(X))^2 dX < 4\mu r. \tag{20}$$

Then for each solution of (15a), (16a) on $[0,\infty)$ there is an $M>0$ with

$$\int_\Omega |\nabla u|^2 dX + \int_0^\sigma f(s)ds + \int_0^t \int_\Omega |\Delta u|^2 dX ds < M$$

and

$$|\sigma(t)| + \int_\Omega \left[u^2(t,X) + |\nabla u|^2 + \int_{t-h}^t p^2(s,u,\nabla u)ds\right] dX \to 0 \text{ as } t\to\infty.$$

Also, for each solution of (15b), (16b) on $[0,\infty)$ there is an $M>0$ with

$$\| u \| + \int_0^1 u_x^2(t,x)dx + \int_0^\sigma f(s)ds + \int_0^t \int_0^1 u_{xx}^2(t,x)dxdt < M$$

(where $\|\cdot\|$ is the supremum norm in x for fixed t) and $|u(t,x)| + |\sigma(t)| \to 0$ as $t\to\infty$.

Proof. Most of the proof of the nonlinear case (b) parallels that for the linear case (a). We focus mainly on (15a). For this solution, define

$$\begin{aligned} V_1(t) &= \textstyle\int_\Omega \left[(\frac{1}{2})u^2(t,X) + k\int_{-h}^0 \int_{t+s}^t p^2(v,u(v,X),\nabla u(v,X))dvds\right]dX \\ &\quad + \textstyle\int_0^\sigma f(s)ds \end{aligned} \tag{21}$$

so that

$$\begin{aligned} V_1'(t) &= \int_\Omega \{u[\Delta u + b(X)f(\sigma) + \int_{t-h}^t p(s,u(s,X),\nabla u(s,X))ds] \\ &\quad + hkp^2(t,u,\nabla u) - k\int_{t-h}^t p^2(s,u(s,X),\nabla u(s,X))ds \\ &\quad + f(\sigma)c(X)u(t,X)\}dX - rf^2(\sigma) \\ &= \int_\Omega \{-|\nabla u|^2 + (b+c)uf(\sigma) + hk\beta|\nabla u|^2 + (h/2)u^2 \\ &\quad -(k-(\frac{1}{2}))\int_{t-h}^t p^2(s,u(s,X),\nabla u(s,X))ds - (r/|\Omega|)f^2(\sigma)\}dx \end{aligned}$$

(where $|\Omega| = \int_\Omega dX$ and we have used the divergence theorem to say

$$\begin{aligned}
\int_\Omega u\Delta u dX &= -\int_\Omega |\nabla u|^2 dX) \\
&\leq \int_\Omega \{[-\lambda_1(1-kh\beta) + (h/2)]u^2 + (b+c)uf(\sigma) \\
&\quad -(r/|\Omega|)f^2(\sigma) \\
&\quad -(k-(\frac{1}{2}))\int_{t-h}^t p^2(s,u,\nabla u)ds\}dX \\
&= \int_\Omega \{-\mu u^2 + (b+c)uf(\sigma) - (r/|\Omega|)f^2(\sigma) \\
&\quad -\gamma\int_{t-h}^t p^2(s,u,\nabla u)ds\}dX \\
&= \int_\Omega \Big\{-\mu\{u^2 + [(b+c)/\mu]uf(\sigma) + [(b+c)/2\mu]^2 f^2(\sigma) \\
&\quad +\Big((r/|\Omega|\mu) - [(b+c)/2\mu]^2\Big) f^2(\sigma)\} \\
&\quad -\gamma\int_{t-h}^t p^2(s,u,\nabla u)ds\Big\} \\
&\leq -\overline{\lambda}\int_\Omega \{(u + [(b+c)/2\mu]f(\sigma))^2 + f^2(\sigma) \\
&\quad +\int_{t-h}^t p^2(s,u,\nabla u)ds\}dX \\
&\leq -\lambda\int_\Omega \{u^2 + f^2(\sigma) + \int_{t-h}^t p^2(s,u,\nabla u)ds\}dX
\end{aligned}$$

(by completing the square again starting with $f^2(\sigma)$ and using $b+c$ bounded) for λ, $\overline{\lambda} > 0$. Indeed, we can argue with a different $\lambda, \overline{\lambda} > 0$ that

$$V_1' \leq -\lambda\int_\Omega \{u^2 + |\nabla u|^2 + f^2(\sigma) + \int_{t-h}^t p^2(s,u,\nabla u)ds\}dX \tag{22}$$

by taking a sightly smaller μ when we replaced $|\nabla u|^2$ by u^2. We will also arrive at $|\nabla u|^2$ in another way.

Now define

$$W(t) = \int_\Omega |\nabla u|^2 dX = \int_\Omega (u_x^2 + u_y^2 + u_z^2)dX \tag{23}$$

so that

$$\begin{aligned}
W'(t) &= \int_\Omega 2(u_x u_{xt} + u_y u_{yt} + u_z u_{zt})dX \\
&= -\int_\Omega 2(u_{xx}u_t + u_{yy}u_t + u_{zz}u_t)dX
\end{aligned}$$

(by the divergence theorem and the boundary conditions)

$$\begin{aligned}
&= -2\int_\Omega \Delta u[\Delta u + fb(X)f(\sigma) + \int_{t-h}^t p(s,u,\nabla u)ds]dX \\
&\leq \int_\Omega \{[-2 + (h/K) + (1/K)](\Delta u)^2 + Kb^2(X)f^2(\sigma) \\
&\quad +K\int_{t-h}^t p^2(s,u,\nabla u)ds\}dX
\end{aligned}$$

for $K > 0$. Thus, since b is in L^2, if K is large and m is small then for

$$Z(t) = V_1(t) + mW(t) \tag{24}$$

we have (for a new $\lambda > 0$)

$$Z'(t) \le -\lambda \int_\Omega [(\Delta u)^2 + |\nabla u|^2 + f^2(\sigma) + \int_{t-h}^t p^2(s,u,\nabla u)ds]dX. \tag{25}$$

In fact, under these boundary conditions, Simpson and Spector [11: p. 26] show that there is a $\gamma > 0$ with

$$\gamma \int_\Omega |\nabla u|^2 dX \le \int_\Omega (\Delta u)^2 dX. \tag{26}$$

(This argument would allow us to delete the condition that $b + c$ is bounded, but would complicate subsequent arguments.) Thus, an integration of (25) will yield the inequality involving M of the theorem.

The case for (15b) uses the same V_1 where Ω is now (0,1) and $W(t) = \int_0^1 \int_0^{u_x} g(s)dsdx$.

We will now show that $V_1(t)$ actually tends to zero. Suppose that

$$\limsup_{t\to\infty} \sigma(t) = \infty.$$

Then it is clear that $\int_0^\sigma f(s)ds$ is bounded for $\sigma > 0$ since $V_1'(t) \le 0$; thus, we can define

$$L = \int_0^\infty f(s)ds$$

and

$$V(t) = V_1(t) - L.$$

Review (12) - (14) and Theorem B, letting

$$\begin{aligned} P(t,x) &= \int_\Omega \{(\frac{1}{2})u^2(t,X) + k\int_{-h}^0 \int_{t+s}^t p^2(v,u,\nabla u)dvds\}dX, \\ U(t,z) &= \int_0^\sigma f(s)ds, \end{aligned}$$

and

$$Q(P(t,x),t,x) = P(t,x)[\lambda/(hk+1)]$$

when we notice that

$$\int_{-h}^0 \int_{t+s}^t p^2(v,u,\nabla u)dvds \le h\int_{t-h}^t p^2(s,u,\nabla u)ds.$$

Note also that P is a function of t alone; this means that Theorem B will be valid in this setting.

Lemma. If $\limsup_{t\to\infty} \sigma(t) = \infty$, then $\sigma(t) \to \infty$.

Proof. Note that

$$|\int_\Omega c(X)u(t,X)dX|^2 \le (\int_\Omega c^2(X)dX)^2 + (\int_\Omega u^2(t,X)dX)^2$$

so the term on the left is bounded since $c \in L^2$. As the right-hand-side of σ' in (15a) is bounded for σ bounded, if there is a point $\sigma_0 \neq 0$ and a sequence $\{t_n\} \to \infty$ such that $|\sigma(t_n) - \sigma_0| \to 0$ as $n \to \infty$, then there is a $T > 0$ such that if $0 < \delta < (\frac{1}{2})|\sigma_0\ |$ then $|\sigma(t) - \sigma_0| \leq \delta$ for $t_n \leq t \leq t_n + T$ and n large. An integration of (22) will then show that $V_1(t) \to -\infty$ as $t \to \infty$, a contradiction. We therefore conclude that $\sigma(t) \to \infty$ as $t \to \infty$ and the lemma is proved. (This argument is part of a general theorem in [1].)

Hence, by Theorem B there is a $t_0 > 0$ and a constant $\phi > 0$ such that $V'(t) \leq -\phi V(t)$ for $t \geq t_0$. But

$$\begin{aligned} V'(t) &= -(|V'|)^{\frac{1}{2}}(|V'|)^{\frac{1}{2}} \leq -(\phi V)^{\frac{1}{2}}(|V_1'|)^{\frac{1}{2}} \\ &\leq -KV^{\frac{1}{2}}(|\sigma'|^2)^{\frac{1}{2}} = -KV^{\frac{1}{2}}|\sigma'| \end{aligned}$$

for some $K > 0$. That is

$$\begin{aligned} |\sigma'|^2 &= (\int_\Omega c(X)u(t,X)dX - rf(\sigma))^2 \\ &\leq 2(\int_\Omega c(X)u(t,X)dX)^2 + 2r^2 f^2(\sigma) \\ &\leq 2(\int_\Omega c^2(X)dX \int_\Omega u^2(t,X)dX) + 2r^2 f^2(\sigma) \\ &\leq c_1(\int_\Omega u^2(t,X)dX + f^2(\sigma)) \\ &\leq c_2|V_1'| \end{aligned}$$

for constants c_1 and c_2. We then have

$$V^{-\frac{1}{2}}V' \leq -K|\sigma'|$$

so

$$V^{\frac{1}{2}}(t) \leq V^{\frac{1}{2}}(t_0) - (K/2)|\sigma(t) - \sigma(t_0)|.$$

Thus $\sigma(t)$ is bounded. We can then integrate (22) and conclude that $\sigma(t) \to 0$ so $\int_0^\sigma f(s)ds \to 0$ as $t \to \infty$. (Here, we have used our previous argument with t_n and T.)

We now have

$$\begin{aligned} V_1'(t) &\leq -\lambda \int_\Omega \left[\int_{t-h}^t p^2(s,u,\nabla u)ds + u^2(t,X)\right] dX \\ &\leq -\lambda V_1(t) + H(t) \end{aligned}$$

where $\lambda > 0$ and $H(t) \to 0$ as $t \to \infty$. Hence,

$$V_1(t) \leq V_1(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}H(s)ds$$

and that integral tends to zero since it is the convolution of an L^1-function with a function tending to zero.

From (25) and (26), together with $V_1(t) \to 0$, it is the case that along any solution we have

$$\begin{aligned} Z'(t) &\leq -\lambda \int_\Omega [(\Delta u)^2 + |\nabla u|^2 + \int_{t-h}^t p^2(s,u,\nabla u)ds]dX \\ &\leq -\lambda Z(t) + D(t) \end{aligned}$$

where $D(t) \to 0$ as $t \to \infty$. Just as in the case of V_1 in the last paragraph, we argue that $Z(t) \to 0$ as $t \to \infty$ and so $\int_\Omega [u^2(t,X) + |\nabla u|^2] dX \to 0$ as $t \to \infty$. A parallel argument for (15b) yields $\| u(t,x) \| \leq \int_0^1 u_x^2(t,x)dx \to 0$ as $t \to \infty$. This completes the proof of Theorem 1.

0.4 Remarks on Existence

As mentioned earlier, our work was motivated by a paper of Nakagiri [8] concerning

$$u_t = \Delta u + \int_{-h}^{0} d\eta(s,x)u(t+s,x) + \phi(\sigma(t))c(x), t \geq 0, x \in \Omega, \tag{6}$$

$$\sigma(t) = \int_\Omega d(x)u(t,x)dx \ \text{ for } \ t \geq 0, \tag{27}$$

$$u(t,x) = 0 \ \text{ on } \ \partial\Omega, t \geq 0, \tag{28}$$

$$u(0,x) = g^0(x), u(s,x) = g^1(s,x) \ \text{ a.e., } \ s \in [-h,0), x \in \Omega. \tag{29}$$

The system is written as an abstract ODE

$$u'(t) = Au(t) + \int_{-h}^{0} d\eta(s)u(t+s) + \phi(\sigma(t))c, t \geq 0, \tag{30}$$

with (27) and (29) holding. Under certain additional assumptions, if $T(t)$ is the semigroup generated by A and if $W(t)$ is the unique fundamental solution of (c.f., [10; p. 175])

$$W(t) = \begin{cases} T(t) + \int_0^t T(t-s) \int_{-h}^0 d\eta(\xi)W(\xi+s)ds & ,t \geq 0 \\ 0 & ,t < 0 \end{cases} \tag{31}$$

and if

$$U_t(s) = \int_{-h}^{0} W(t-s+\xi)d\eta(\xi) \ \text{ a.e. } \ s \in [-h,0], \tag{32}$$

then there is at least one solution $u(t)$ of

$$\begin{aligned} u(t) &= W(t)g^0 + \int_{-h}^{0} U_t(s)g^1(s)ds \\ &\quad + \int_0^t W(t-s)\phi(\int_\Omega d(x)u(s,x)dx)cds, t \geq 0, \end{aligned} \tag{33}$$

and it is the strong solution of (6) when $c \in D(A)$ and

$$g^0 \in D(A), g^1 \in W^{1,2}([-h,0]; L^2(\Omega)), g^1(0) = g^0. \tag{34}$$

There are at least three distinct ways of linking a control and these are discussed in Lafschetz [7]. Nakagiri's formulation will not yield our system. However, existence theory for our system is now readily obtained from his work. Here is a sketch.

The linearity in the delay yields a global solution of (31) and the form of (33) is well suited to contraction mappings. Indeed, the term involving $U_t(s)$ in (33) drops out in the contraction mapping argument and modification of the Nakagiri scheme to fit a linear form of (15a) can be seen by considering the system

$$\begin{cases} u_t &= u_{xx} + b(x)f(\sigma) \\ \\ \sigma' &= \int_0^1 c(x)u(t,x)dx - rf(\sigma), \end{cases} \tag{35}$$

$$u(t,0) = u(t,1) = 0, \tag{36}$$

$$u(0,x) = g^0(x), \sigma(0) = \sigma_0. \tag{37}$$

Write

$$u_t = u_{xx}, u(t,0) = u(t,1) = 0 \tag{38}$$

as

$$u'(t) + A(u(t)) = 0 \tag{39}$$

and then write the system as

$$\begin{cases} u(t) &= e^{-At}g^0(x) + \int_0^t e^{-A(t-s)}b(x)f(\sigma(s))ds \\ \\ \sigma(t) &= \sigma_0 + \int_0^t [\int_0^1 c(x)u(s,x)dx - rf(\sigma(s))]ds. \end{cases} \tag{40}$$

When $g^0(x) \in L^2(0,1)$, then $e^{-At}g^0(x) \in D(A)$ for $t > 0$; and when $b(x)f(\sigma(t))$ is Hölder continuous and locally integrable, then the integral in the first equation of (40) is in the $D(A)$ (c.f., Henry [5; p. 50]). If f satisfies a local Lipschitz condition, then (40) will define a contraction mapping with a unique fixed point which will then solve (35).

0.5 Nakagiri's Equations

We turn now to

$$u_t = \Delta u + \int_{-h}^0 d\eta(s,X)u(t+s,X)ds + f(\sigma(t))c(X), \tag{41}$$

$$\sigma(t) = \int_\Omega d(X)u(t,X)dX, \tag{42}$$

$$u(t,X) = 0 \text{ on } \partial\Omega \text{ and } d(X) = 0 \text{ on } \partial\Omega. \tag{43}$$

Following the lead of Lefschetz [7; pp. 39-40] for systems of ODEs, we write

$$\begin{aligned} \sigma'(t) &= \int_\Omega d(X)u_t(t,X)dX \\ &= \int_\Omega d(X)[\Delta u + \int_{-h}^0 d\eta(s,X)u(t+s,X)ds + f(\sigma(t))c(X)]dX \\ &= \int_\Omega d(X)\Delta u dX + \int_\Omega d(X)\int_{-h}^0 d\eta(s,X)u(t+s,X)dsdX \\ &\quad + \int_\Omega d(X)c(X)dXf(\sigma). \end{aligned}$$

Existence results were stated with (33) and (34). Using the divergence theorem we suppose there is a function $b(X)$ with $\int_\Omega d(X)\Delta u dX = \int_\Omega b(X)\cdot \nabla u dX$ (that is, $b(X) = -\nabla d(X)$) and we ask that

$$\int_\Omega d(X)c(X)dX = -r < 0, b \in L^2, c \text{ and } b \text{ are bounded.} \tag{44}$$

Our work is based on construction of Liapunov functionals and different ones must be used if the delay is discrete. For brevity, then, we replace the delay term by

$$\int_{-h}^0 d\eta(s,X)u(t+s,X)ds \to \int_{t-h}^t p(s,X)u(s,X)ds \tag{45}$$

where p is continuous. Our main concern here is to show absolute stability without strengthening (*).

Our system will now be

$$\begin{cases} u_t &= \Delta u + \int_{t-h}^t p(s,X)u(s,X)ds + c(X)f(\sigma) \\ \sigma' &= \int_\Omega b(X)\cdot\nabla u dX + \int_\Omega d(X)\int_{t-h}^t p(s,X)u(s,X)dsdX - rf(\sigma), \end{cases} \tag{46}$$

$$u(t,X) = 0 \text{ on } \partial\Omega. \tag{47}$$

Theorem 2. Let λ_1 be the first eigenvalue of $-\Delta$ on H_0^1, α and β positive, $\alpha+\beta = 1$, $k > 1$,

$$\beta\lambda_1 - (h/2) - khp^2(t,X) \geq \gamma > 0, \tag{48}$$

$$-4\int_\Omega [d(X)c(X) + (h/2)d^2(X)]dX > \int_\Omega [(|b(X)|^2/\alpha) + (c^2(X)/\gamma)]dX + \mu \tag{49}$$

for some $\mu > 0$ and let (44) hold. If (u,σ) is any solution of (46) - (47) on $[0,\infty)$, then

$$|\sigma(t)| + \int_\Omega u^2(t,X)dX \to 0 \text{ as } t \to \infty.$$

Proof. The analog of (21) is

$$V_1(t) = \int_\Omega [(\frac{1}{2})u^2(t,X) + k\int_{-h}^0\int_{t+s}^t p^2(v,X)u^2(v,X)dvds]dX + \int_0^\sigma f(s)ds$$

and its derivative along a solution of (46) satisfies

$$\begin{aligned} V_1'(t) &= \int_\Omega [u\{\Delta u + \int_{t-h}^t p(s,X)u(s,X)ds + c(X)f(\sigma)\} \\ &\quad + f(\sigma)b(Xx)\cdot\nabla u + f(\sigma)d(X)\int_{t-h}^t p(s,X)u(s,X)ds]dX \end{aligned}$$

$$-rf^2(\sigma) + kh\int_\Omega p^2(t,X)u^2(t,X)dX$$
$$-k\int_\Omega\int_{t-h}^t p^2(s,X)u^2(s,X)dsdX$$
$$\le \int_\Omega\{-|\nabla u|^2 + (h/2)u^2 + (\frac{1}{2})\int_{t-h}^t p^2(s,X)u^2(s,X)ds$$
$$+c(X)uf(\sigma) + f(\sigma)b(X)\cdot\nabla u + (h/2)f^2(\sigma)d^2(X)$$
$$+(\frac{1}{2})\int_{t-h}^t p^2(s,X)u^2(s,X)ds + khp^2(t,X)u^2(t,X)$$
$$-k\int_{t-h}^t p^2(s,X)u^2(s,X)ds\}dX - rf^2(\sigma)$$

(using the divergence theorem)

$$\le \int_\Omega\{-\alpha|\nabla u|^2 + (h/2)u^2 - \beta\lambda_1 u^2 + c(X)uf(\sigma) + khp^2(t,X)u^2$$
$$+f(\sigma)b(X)\cdot\nabla u - (k-1)\int_{t-h}^t p^2(s,X)u^2(s,X)ds\}dX$$
$$-[r-(h/2)\int_\Omega d^2(X)dX]f^2(\sigma)$$

(using (48))

$$\le \int_\Omega\{-\alpha|\nabla u|^2 - \gamma u^2 + c(X)uf(\sigma) + |f(\sigma)||b(X)||\nabla u|$$
$$-(k-1)\int_{t-h}^t p^2(s,X)u^2(s,X)ds\}dX$$
$$-[r-(h/2)\int_\Omega d^2(X)dX]f^2(\sigma)$$
$$\le \int_\Omega\{-\alpha[|\nabla u|^2 - |f(\sigma)|(|b(X)|/\alpha)|\nabla u|$$
$$+[f^2(\sigma)/4\alpha^2]|b(X)|^2]$$
$$-\gamma[u^2 - (c(X)/\gamma)f(\sigma)u + (c^2(X)/4\gamma^2)f^2(\sigma)]$$
$$-(k-1)\int_{t-h}^t p^2(s,X)u^2(s,X)ds$$
$$r-(h/2)\int_\Omega d^2(X)dX - (|b(X)|^2/4\alpha)$$
$$-(c^2(X)/4\gamma)]f^2(\sigma)\}dX.$$

Using (44) and (49), we can find a $\overline{\mu} > 0$ with

$$V_1'(t) \le -\overline{\mu}\int_\Omega[u^2 + |\nabla u|^2 + f^2(\sigma) + \int_{t-h}^t p^2(s,X)u^2(s,X)ds]dX \tag{50}$$

The identical argument as was given in the proof of Theorem 1 will show that $V_1(t) \to 0$ as $t \to \infty$ and we complete the proof in the same way.

Remark. Theorem 2 does not seem to be comparable to that of Nakagiri. His η is more general than ours and his stability conditions and techniques are different. On the other hand, our class of admissible functions $f(\sigma)$ is much larger than his and includes all those for the ODE counterpart.

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