Uniform Boundedness by Averaging

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Abstract

We consider a system of functional differential equations with infinite delay and derive conditions on Liapunov functionals to ensure that solutions are uniformly bounded and uniformly ultimately bounded. The analysis is based on the method of finding a bound on the average values of unknown solutions and Jensen's inequality. Comparisons between our theorems and those existing in the literature are also given.

1 Introduction

Consider a system of functional differential equations

$$x'(t) = F(t, x_t), \quad x(t) \in R^n \tag{1.1}$$

in which $F(t,\phi)$ is a functional defined for $t \geq 0$ and $\phi \in C$, where C is the set of bounded continuous functions $\phi : (-\infty, 0] \to R^n$ with the supremum norm. For each $t \in R^+ = [0, +\infty)$, $C(t)$ denotes the set of continuous functions $\phi : [0,t] \to R^n$ with $\|\phi\| = \sup\{|\phi(s)\| : 0 \leq s \leq t\}$, where $|\cdot|$ is the Euclidean norm on R^n. We assume

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that for each $t_0 \geq 0$ and each $\phi \in C(t_0)$ there is at least one solution $x(t, t_0, \phi)$ of (1.1) defined on an interval $[t_0, \alpha)$ with $x_{t_0} = \phi$. Here, $x_t(s) = x(t+s)$ for $s \leq 0$. Moreover, if the solution remains bounded, then $\alpha = \infty$.

We are interested in conditions on Liapunov functionals which will ensure that solutions are uniformly bounded (UB) and uniformly ultimately bounded (UUB). Much discussion of these concepts and of the above mentioned existence properties may be found in Burton [3] or Yoshizawa [15], for example.

Definition 1.1. Solutions of (1.1) are UB if for each $B_1 > 0$ there exists $B_2 > 0$ such that $[t_0 \geq 0, \phi \in C, \|\phi\| < B_1, t \geq t_0]$ imply that $|x(t, t_0, \phi)| \leq B_2$.

Definition 1.2. Solutions of (1.1) are UUB if there is a $B > 0$ and for each $B_3 > 0$ there is a $T > 0$ such that $[t_0 \geq 0, \phi \in C, \|\phi\| < B_3, t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < B$.

These are generalizations of uniform stability and uniform asymptotic stability. An example will bring our work into focus.

Example 1.1. Let $D : [0, \infty) \to [0, 1)$ be continuous, $L^1[0, \infty)$, and let $\int_t^\infty D(u)du \in L^1[0, \infty)$. Suppose that $a, b : [0, \infty) \to [0, \infty)$ are continuous, that n is the quotient of odd positive integers, and that $p : [0, \infty) \to R$ is bounded and continuous. Finally, suppose that there is an $L > 0$ such that

$$-a(t) + \int_0^\infty D(s)ds + [b^2(t)/(2L)] \leq 0. \tag{1.2}$$

Note that $p(t)$ is bounded, but we will be most interested in the case in which $b(t)$ ranges from zero to infinity.

Consider the scalar equation

$$x' = -a(t)x^3 - x^n + \int_0^t D(t-s)x^3(s)ds + b(t)p(t) \tag{1.3}$$

and define a Liapunov functional V by

$$V(t, x_t) = (1/4)x^4 + (1/2)\int_0^t \int_{t-s}^\infty D(u)du x^6(s)ds \tag{1.4}$$

so that along a solution of (1.3) we obtain

$$V'(t,x_t) = -a(t)x^6 - x^{n+3} + x^3\int_0^t D(t-s)x^3(s)ds + x^3b(t)p(t)$$

$$+(1/2)\int_0^\infty D(u)dux^6 - (1/2)\int_0^t D(t-s)x^6(s)ds$$

$$\leq -a(t)x^6 - x^{n+3} + (1/2)\int_0^t D(t-s)(x^6(t)+x^6(s))ds + (1/2L)x^6b^2(t) + (L/2)p^2(t)$$

$$+(1/2)\int_0^\infty D(u)dux^6 - (1/2)\int_0^t D(t-s)x^6(s)ds$$

$$= [-a(t) + (1/2)\int_0^t D(s)ds + (1/2)\int_0^\infty D(u)du + (1/2L)b^2(t)]x^6$$

$$-x^{n+3} + (L/2)p^2(t)$$

so that

$$V'(t,x_t) \leq -x^{n+3} + (L/2)p^2(t).$$

This gives us a very familiar set of inequalities:

$$(1/4)x^4 \leq V(t,x_t) \leq (1/4)x^4 + \int_0^t \Phi(t-s)x^6(s)ds \tag{1.5}$$

and

$$V'(t,x_t) \leq -x^{n+3} + M \tag{1.6}$$

for some $M > 0$ and $\Phi(t) = \int_t^{+\infty} D(u)du/2$.

Investigators have struggled for more than fifty years to find combinations of terms in such inequalities which will yield UB and UUB. We will present several results here and the reader should find it very interesting to interpret them in terms of the three quantities x^4, x^6, x^{n+3}.

The ideas from which UB and UUB came were extensively studied by the Lefschetz school during the 1940's with much work on a Liénard equation being done by Cartwright and Littlewood with a view to proving the existence of a periodic solution. That type of work is given in detail in the book by Sansoni and Conti [14]. Levinson [11] noticed that these concepts were general and fundamental. That work stimulated research in asymptotic fixed point theory for proving the existence of periodic solutions in very general systems. One can trace that work from Browder [2] (followed by many other results), to Jones [9], and on to one of the most useful of all by Horn [8]. Applications of this type are found in Arino-Burton-Haddock [1], for example, in connection with Liapunov functionals sharing relations (1.5) and (1.6).

General relations of this type are frequently written as

$$W_1(|x|) \le V(t, x_t) \le W_2(|x|) + W_3(\int_0^t \Phi(t-s)W_4(|x(s)|)ds) \tag{1.7}$$

and

$$V'(t, x_t) \le -W_5(|x(t)|) + M \tag{1.8}$$

where the W_i are strictly increasing functions with $W_i(0) = 0$, called wedges. And that is the form we will consider here. There are three cases in which these very readily yield UB and UUB. If $W_4 = W_5$ and $\Phi'(t) \le 0$, then the analysis is simple, as may be seen in Burton and S. Zhang ([5], p.144). If (1.8) can actually be written as a differential inequality in V, rather than in x, say $V' \le g(t, V)$, then Lakshmikantham and Leela ([10], p. 214) gives a full discussion. If (1.8) involves the norm of the right-hand-side of the differential equation, then lengthy, but simple, analysis may be found in many places including Burton ([3], p. 275). More recent and general results of that type are also found in Makay ([12], [13]).

But when none of those three situations obtain, it becomes a very difficult problem. Yoshizawa ([15], p. 206) will bring into focus the type of analysis that is then needed. Hale ([6], p. 139) states that the conditions become so restrictive that he declines to discuss them.

Our work begins with the idea that (1.6) can generate an average value of a power of the unknown solution. This average can then be substituted into the integral in (1.4) to obtain a bound on V and, hence, on the unknown x^4.

2 The Main Result

Theorem 2.1. Suppose there exists a continuous function $\Phi : R^+ \to R^+$ with $\Phi \in L^1(R^+)$, wedges W_j with $W_1(r) \to +\infty$ as $r \to +\infty$, positive constants U, M with $W_5(U) > M$, and a continuous functional $V : R^+ \times C \to R^+$ such that for each $x \in C(t)$, the following conditions hold:

(i) $W_1(|x(t)|) \le V(t, x_t) \le W_2(|x(t)|) + W_3(\int_0^t \Phi(t-s) W_4(|x(s)|)ds)$,

(ii) $V'_{(1.1)}(t, x_t) \le -W_5(|x(t)|) + M$.

Then solutions of (1.1) are UB if and only if for each $K_1 > 0$, there exists $K_2 > 0$ such that if $x(t) = x(t, t_0, \phi)$ is a solution of (1.1) with $\|\phi\| \le K_1$, then

$$\int_{t_0}^{\bar{t}} \Phi(\bar{t} - s) W_4(|x(s)|)ds \le K_2 \tag{2.1}$$

whenever $v(s) < v(\bar{t})$ for $t_0 \le s < \bar{t}$, where $v(s) = V(s, x_s)$.

Proof. Suppose that solutions of (1.1) are UB. Then for each $B_1 > 0$, there exists $B_2 > 0$ such that $[t_0 \ge 0, \|\phi\| < B_1, t \ge t_0]$ imply $|x(t, t_0, \phi)| \le B_2$. We may assume that $B_2 > B_1$. Now let $x(t) = x(t, t_0, \phi)$ and $\int_0^{+\infty} \Phi(u)du = J$. Then for $t \ge t_0$, we have

$$\int_{t_0}^{t} \Phi(t-s) W_4(|x(s)|)ds \le \int_{t_0}^{t} \Phi(t-s) W_4(B_2)ds \le J W_4(B_2).$$

This implies that (2.1) holds for all $t \ge t_0$.

On the other hand, suppose that (2.1) holds. Let $x(t) = x(t, t_0, \phi)$ and $v(t) = V(t, x_t)$ with $\|\phi\| \le K_1$. Then we have either

(A) $\quad v(t) \le v(t_0)$ for all $t \ge t_0$ or

(B) $\quad v(s) < v(\bar{t})$ for some $\bar{t} > t_0$ and all $t_0 \le s < \bar{t}$.

If (A) holds, then

$$W_1(|x(t)|) \le v(t) \le v(t_0) \le W_2(K_1) + W_3(J W_4(K_1)).$$

Thus, $|x(t)| \le W_1^{-1}[W_2(K_1) + W_3(J W_4(K_1))]$. Now suppose that (B) holds. By the definition of $\bar{t}$, we have $W_5(|x(\bar{t})|) \le M$ and $|x(\bar{t})| \le W_5^{-1}(M)$. Note that $W_5^{-1}(M)$ is well defined since $W_5(U) > M$. It follows from (i) and (2.1) that

$$\begin{aligned} v(\bar{t}) &\le W_2(|x(\bar{t})|) + W_3[\int_0^{\bar{t}} \Phi(\bar{t} - s) W_4(|x(s)|)ds] \\ &= W_2(|x(\bar{t})|) + W_3[\int_0^{t_0} \Phi(\bar{t} - s) W_4(|x(s)|)ds + \int_{t_0}^{\bar{t}} \Phi(\bar{t} - s) W_4(|x(s)|)ds] \\ &\le W_2[W_5^{-1}(M)] + W_3[J W_4(K_1) + K_2]. \end{aligned}$$

Since $\bar{t}$ is arbitrary, we obtain for all $t \geq t_0$

$$\begin{aligned} v(t) &\leq W_2[W_5^{-1}(M)] + W_3[JW_4(K_1) + K_2] + v(t_0) \\ &\leq W_2[W_5^{-1}(M)] + W_3[JW_4(K_1) + K_2] + W_2(K_1) + W_3(JW_4(K_1)). \end{aligned}$$

This yields $|x(t)| \leq B_2$ for all $t \geq t_0$, where

$$B_2 = W_1^{-1}[W_2(W_5^{-1}(M)) + W_3(JW_4(K_1) + K_2) + W_2(K_1) + W_3(JW_4(K_1))].$$

Thus, solutions of (1.1) are UB.

Lemma 2.1. Suppose $\Phi : R^+ \to R^+$ is continuous with $\Phi(u),\ \Phi'(u)u \in L^1(R^+)$ and $q \in C([t_0, t], R^+)$. If there exist positive constants α and β such that

$$\frac{1}{t-s}\int_s^t q(u)du \leq \alpha + \frac{\beta}{t-s} \tag{2.2}$$

for all $t_0 \leq s < t$, then

$$\int_{t_0}^t \Phi(t-u)q(u)du \leq J^*\alpha + J'\beta \tag{2.3}$$

with $J \leq J^*$, where

$$J = \int_0^{+\infty} \Phi(u)du,\ J' = \int_0^{+\infty} |\Phi'(u)|du,\ J^* = \sup_{t\geq 0}[\Phi(t)t + \int_0^t |\Phi'(u)|udu]$$

Proof. First, observe that for each $b > 0$, we have

$$\int_0^b |\Phi'(u)|udu \geq |\int_0^b \Phi'(u)udu| = |\Phi(u)u\Big|_0^b - \int_0^b \Phi(u)du|$$

and

$$\int_b^{+\infty} |\Phi'(u)|du \geq |\int_b^{+\infty} \Phi'(u)du| = |\Phi(+\infty) - \Phi(b)|.$$

where $\Phi(+\infty) = 0$ since $\Phi(u),\ \Phi'(u)u \in L^1(R^+)$. It is clear from the first inequality that $J \leq J^*$. We also have

$$\Phi(b)b \leq \int_0^{+\infty} [\Phi(u) + |\Phi'(u)|udu] < +\infty$$

and $\int_b^{+\infty} |\Phi'(u)|du \geq \Phi(b)$ for all $b \geq 0$. Now integrating by parts on the left-hand side of (2.3) from t_0 to t and using (2.2), we obtain

$$\begin{aligned}
\int_{t_0}^t \Phi(t-s)q(s)ds &= \Phi(t-s)\Big(-\int_s^t q(u)du\Big)\Big|_{t_0}^t - \int_{t_0}^t \Phi'(t-s)\int_s^t q(u)duds \\
&\leq [\Phi(t-t_0)(t-t_0) + \int_{t_0}^t |\Phi'(t-s)|(t-s)ds]\alpha \\
&\quad +[\Phi(t-t_0) + \int_{t_0}^t |\Phi'(t-s)|ds]\beta \\
&= [\Phi(t-t_0)(t-t_0) + \int_0^{t-t_0} |\Phi'(u)|udu]\alpha \\
&\quad +[\Phi(t-t_0) + \int_0^{t-t_0} |\Phi'(u)|du]\beta \\
&\leq J^*\alpha + [\Phi(t-t_0) - \int_{t-t_0}^{+\infty} |\Phi'(u)|du + \int_0^{+\infty} |\Phi'(u)|du]\beta \\
&\leq J^*\alpha + J'\beta.
\end{aligned}$$

This completes the proof.

Remark 2.1. If $\Phi : R^+ \to R^+$ with $\Phi'(t) \leq 0$ and $\Phi \in L^1(R^+)$, then $\int_0^{+\infty} |\Phi'(u)|udu < +\infty$ and $J = J^*$. Indeed,

$$\Phi(t)t + \int_0^t |\Phi'(u)|udu = \Phi(t)t - \int_0^t \Phi'(u)udu = \int_0^t \Phi(u)du.$$

If $\Phi(t) = e^{-2t}\sin^2(t)$, then $\Phi(u), \Phi'(u)u \in L^1(R^+)$. However, $\Phi'(t) \leq 0$ for all $t \in R^+$ is not satisfied. Condition $\Phi'(t) \leq 0$ was used in the early work of Burton and S. Zhang [5], Burton and Hering [4], and B. Zhang [16].

Remark 2.2. The wedge W_2 in Theorem 2.1 can be replaced by $\overline{W}_2(r)+Q$, where $\overline{W}_2$ is a wedge and Q is a positive constant. If W_4 is bounded, then the right-hand side of (i) can be reduced to this case and solutions of (1.1) are UB by Theorem 2.1. Therefore, we assume that the constants J and J^* in Lemma 2.1 are positive throughout this paper.

Corollary 2.1. Suppose there exists a continuous function $\Phi : R^+ \to R^+$ with $\Phi(u), \Phi'(u)u \in L^1(R^+)$, wedges W_j with $W_1(r) \to +\infty$ as $r \to +\infty$, positive constants U, M with $W_5(U) > M$, and a continuous functional $V : R^+ \times C \to R^+$ such that (i) and (ii) hold for each $x \in C(t)$. Suppose also that for each $\alpha > 0$, there exists $\alpha^* > 0$

such that

$$\frac{1}{t-s}\int_s^t W_5(|x(u)|)du \le \alpha \text{ implies } \frac{1}{t-s}\int_s^t W_4(|x(u)|)du \le \alpha^* \tag{2.4}$$

for any $0 \le s < t,\ x \in C(t)$. Then solutions of (1.1) are UB.

Proof. Let $B_1 > 0$ and $x(t) = x(t, t_0, \phi)$ be a solution of (1.1) with $\|\phi\| \le B_1$. Define $v(t) = V(t, x_t)$. Suppose that there exists a $\bar{t} > t_0$ such that $v(s) < v(\bar{t})$ for all $t_0 \le s < \bar{t}$. Integrate (ii) from s to $\bar{t}$ to obtain

$$v(\bar{t}) - v(s) \le -\int_s^{\bar{t}} W_5(|x(u)|)ds + M(\bar{t} - s).$$

This implies

$$\frac{1}{\bar{t}-s}\int_s^{\bar{t}} W_5(|x(u)|)du \le M. \tag{2.5}$$

By (2.4), there exists $M^* > 0$ such that

$$\frac{1}{\bar{t}-s}\int_s^{\bar{t}} W_4(|x(u)|)du \le M^*. \tag{2.6}$$

Applying Lemma 2.1 with $q(u) = W_4(|x(u)|),\ \alpha = M^*$, and $\beta = 0$, we obtain (2.1). By Theorem 2.1, solutions of (1.1) are UB.

Remark 2.3. If $W_5(W_4^{-1}(r))$ is convex downward, then (2.4) holds. Indeed, by Jensen's Inequality, we have

$$\frac{1}{t-s}\int_s^t W_5(|x(u)|)du \ge W_5\Big[W_4^{-1}\Big(\int_s^t W_4(|x(u)|)du\Big/(t-s)\Big)\Big].$$

Moreover, since $W_5[W_4^{-1}(r)]$ is convex downward, there are positive constants a and b such that

$$W_5[W_4^{-1}(r)] \ge ar - b$$

for all $r \ge 0$. Thus, (ii) can be written as

$$V'(t) \le -\tilde{W}_4(|x(t)|) + \tilde{M}$$

where $\tilde{W}_4(r) = aW_4(r)$ and $\tilde{M} = M + b$.

Remark 2.4. In general, (2.4) is not true for arbitrary wedges. For example, let $W_4(r) = r$, $W_5(r) = \sqrt{r}$ and $x_m(u) = m^{3/2}e^{-2mu}$ for $m = 1, 2, \cdots$. Then

$$\int_0^1 W_5(|x_m(u)|)du = \frac{1}{m^{1/4}}(1 - e^{-m}) \leq 1.$$

However,

$$\int_0^1 W_4(|x_m(u)|)du = \frac{m^{1/2}}{2}(1 - e^{-2m}) \to +\infty$$

as $m \to +\infty$.

Theorem 2.2. Suppose that all conditions of Theorem 2.1 hold including (2.1). Then solutions of (1.1) are UUB if and only if there are constants B^*, B^{**} so that for each $B_1 > 0$, there exists a positive constant K such that for each solution $x(t) = x(t, t_0, \phi)$ of (1.1) with $\|\phi\| \leq B_1$, there exists a $\hat{t} \in [t_0 + h,\ t_0 + K + h]$ such that

$$\sup_{\hat{t}-h \leq s \leq \hat{t}} v(s) \leq B^* \tag{2.7}$$

and whenever $t > \hat{t}$ with $v(s) < v(t)$ for $\hat{t} - h \leq s < t$, then

$$\int_{\hat{t}-h}^{t} \Phi(t-s)W_4(|x(s)|)ds \leq B^{**} \tag{2.8}$$

where $v(s) = V(s, x_s)$ and $h > 0$ satisfies $W_4(B_2)\int_h^{+\infty} \Phi(u)du < 1$ with B_2 given in the definition of UB for B_1.

Proof. Since (2.1) holds, solutions of (1.1) are UB. For each $B_1 > 0$, there exists $B_2 > 0$ such that $[t_0 \geq 0, \|\phi\| \leq B_1, t \geq t_0]$ imply $|x(t, t_0, \phi)| < B_2$. First, suppose that solutions of (1.1) are UUB. We show that (2.7) and (2.8) hold. By the definition of UUB for bound B, for each $B_1 > 0$, there exists $T > 0$ such that $[t_0 \geq 0, \|\phi\| \leq B_1, t \geq T + t_0]$ imply $|x(t, t_0, \phi)| < B$. Let $K = T + h$ and $\hat{t} = t_0 + K + h$. Thus, for any $t \in [\hat{t} - h, \hat{t}]$, we have $t - h \geq \hat{t} - 2h = t_0 + T$ and $|x(t)| < B$ with

$$\begin{aligned}
v(t) &\leq W_2(|x(t)|) + W_3(\int_0^t \Phi(t-s)W_4(|x(s)|)ds) \\
&\leq W_2(B) + W_3\Big(\int_0^{t-h} \Phi(t-s)dsW_4(B_2) + \int_{t-h}^{t} \Phi(t-s)W_4(|x(s)|)ds\Big) \\
&\leq W_2(B) + W_3[1 + JW_4(B)] =: B^*.
\end{aligned}$$

This proves (2.7). It also follows from the definition of $\hat{t}$ that $|x(s)| < B$ for all $s \geq \hat{t} - h$. Thus,

$$\int_{\hat{t}-h}^{t} \Phi(t-s)W_4(|x(s)|)ds \leq JW_4(B) = B^{**}$$

for all $t \geq \hat{t}$ and (2.8) is satisfied.

Now suppose (2.7) and (2.8) hold. We will show that solutions of (1.1) are UUB. For $\hat{t} \in [t_0 + h,\ t_0 + K + h]$, we have either

(C) $\quad v(t) \leq \sup_{\hat{t}-h \leq s \leq \hat{t}} v(s)$ for all $t \geq \hat{t}$ or

(D) $\quad v(s) < v(\bar{t})$ for some $\bar{t} > \hat{t}$ and all $\hat{t} - h \leq s < \bar{t}$.

Let $T = K + h$. If (C) holds, then for $t \geq T + t_0 \geq \hat{t}$, $W_1(|x(t)|) \leq v(t) \leq \sup_{\hat{t}-h \leq s \leq \hat{t}} v(s) \leq B^*$ and $|x(t)| \leq W_1^{-1}(B^*)$. Next, suppose (D) holds. By the definition of $\bar{t}$, we have $|x(\bar{t})| \leq W_5^{-1}(M)$ and

$$\begin{aligned} v(\bar{t}) &\leq W_2(|x(\bar{t})|) + W_3(\int_0^{\bar{t}} \Phi(\bar{t}-s)W_4(|x(s)|)ds) \\ &\leq W_2[W_5^{-1}(M)] + W_3[\int_0^{\hat{t}-h} \Phi(\bar{t}-s)W_4(|x(s)|)ds + \int_{\hat{t}-h}^{\bar{t}} \Phi(\bar{t}-s)W_4(|x(s)|)ds] \\ &\leq W_2[W_5^{-1}(M)] + W_3[W_4(B_2)\int_{\bar{t}-\hat{t}+h}^{+\infty} \Phi(u)du + \int_{\hat{t}-h}^{\bar{t}} \Phi(\bar{t}-s)W_4(|x(s)|)ds] \\ &\leq W_2[W_5^{-1}(M)] + W_3[1 + B^{**}]. \end{aligned}$$

Since $\bar{t}$ is arbitrary, we have $|x(t)| \leq W_1^{-1}[W_2(W_5^{-1}(M)) + W_3(1 + B^{**})]$ for $t \geq \hat{t}$ if (D) holds. Let

$$B = W_1^{-1}(B^*) + W_1^{-1}[W_2(W_5^{-1}(M)) + W_3(1 + B^{**})].$$

Then $|x(t)| \leq B$ for all $t \geq T + t_0$. The proof is complete.

Lemma 2.2. Suppose there exists a continuous function $\Phi : R^+ \to R^+$ with $\Phi \in L^1(R^+)$, wedges W_j, positive constants U, M with $W_5(U) > M$, and a continuous functional $V : R^+ \times C \to R^+$ such that (i) and (ii) hold for each $x \in C(t)$. If solutions of (1.1) are UB, then for any positive constants B_1, γ, h, U^* with $U^* \leq U$ and $W_5(U^*) > M$, there exists a constant $T^* > 0$ with the following properties: for each solution $x(t) = x(t, t_0, \phi)$ of (1.1) with $\|\phi\| \leq B_1$, there exists a $\bar{t} \in [t_0 + h,\ t_0 + T^* + h]$ such that

$$|x(\bar{t})| \leq U^* \ \text{ and } \ v(s) \leq \gamma + v(\bar{t}) \tag{2.9}$$

for $s \in [\bar{t} - h, \bar{t}\,]$, where $v(s) = V(s, x_s)$.

Proof. Let $B_1 > 0$ be given and find $B_2 > 0$ satisfying the definition of UB. We may assume $B_1 < B_2$. Let $x(t) = x(t, t_0, \phi)$ be a solution of (1.1) with $\|\phi\| \leq B_1$. Then $|x(t, t_0, \phi)| < B_2$ for all $t \geq t_0$. Set $v(t) = V(t, x_t)$. By (i), we have $v(t) \leq W_2(B_2) + W_3(JW_4(B_2))$. It follows from (ii) that there exists a constant $L > 0$ such that $|x(t)| > U^*$ cannot hold for any interval of a length greater than or equal to L after t_0. Define

$$I_j = [t_0 + (j-1)(L+h),\ t_0 + j(L+h)], \quad j = 1, 2, \cdots.$$

On each I_j, there is the first $t_j^* \geq t_0 + (j-1)(L+h)$ such that $|x(t_j^*)| \leq U^*$ and $t_j^* \leq t_0 + j(L+h) - h$. Next, define

$$I_j^* = [t_j^*,\ t_0 + j(L+h)] \ \text{ and } \ v(t_j) = \max\{v(s) :\ s \in I_j^*\}$$

for some $t_j \in I_j^*$. Then $|x(t_j)| \leq U^*$. Now consider the intervals $L_j = [t_j - h,\ t_j]$ $j = 2, 3, \cdots$. For each j, there are two cases:

(I) $\quad v(t_j) + \gamma \geq v(s) \quad$ for all $s \in L_j \quad$ or

(II) $\quad v(t_j) + \gamma < v(s_j) \quad$ for some $s_j \in L_j$.

Notice that in case (II), $s_j \notin I_j^*$. Thus, $s_j \leq t_j^*$. We will show that if case (II) holds, then

$$v(t_j) + \gamma \leq v(t_{j-1}), \tag{2.10}$$

where $v(t_{j-1}) = \max\{v(s) : s \in I_{j-1}^*\}$. In fact, if $s_j \in [t_0 + (j-1)(L+h),\ t_j^*]$, then

$$v(t_j) + \gamma \leq v(s_j) \leq v(t_0 + (j-1)(L+h)) \leq v(t_{j-1}) \tag{2.11}$$

since $v(s) \leq -W_5(U^*) + M < 0$ for $s \in [t_0 + (j-1)(L+h), t_j^*]$ by the definition of t_j^*. If $s_j \leq t_0 + (j-1)(L+h)$, (2.10) is automatically satisfied by the definition of t_{j-1}. Thus, there is a positive integer N such that case (II) cannot hold on $N-1$ consecutive intervals $L_2, L_3, \cdots, L_N$. This implies that case (I) must occur on some L_{j^*} with $j^* \leq N$. Define $T^* = N(L+h)$ and $\bar{t} = t_{j^*}$. This proves the lemma.

The next result removes the restriction $\Phi'(t) \leq 0$ in ([5],p.144).

Theorem 2.3. Suppose there exists a continuous function $\Phi : R^+ \to R^+$ with $\Phi(u), \Phi'(u)u \in L^1(R^+)$, wedges W_j with $W_1(r) \to +\infty$ as $r \to +\infty$, positive constants

U, M with $W_5(U) > M$, and a continuous functional $V : R^+ \times C \to R^+$ such that (i) and (ii) hold for each $x \in C(t)$. If $W_5(r) = W_4(r)$ for all $r \geq 0$, then solutions of (1.1) are UB and UUB.

Proof. By Corollary 2.1, solutions of (1.1) are UB. We now show the UUB. For each $B_1 > 0$, there exists $B_2 > 0$ such that $[t_0 \geq 0, \|\phi\| \leq B_1, t \geq t_0]$ imply $|x(t, t_0, \phi)| \leq B_2$. Let $x(t) = x(t, t_0, \phi)$ and $v(s) = V(s, x_s)$. Choose $h > 0$ large enough so that $W_4(B_2) \int_h^{+\infty} \Phi(u)du < 1$. By Lemma 2.2, there exists a constant $T^* > 0$ depending on B_1 and a $\bar{t} \in [t_0 + h,\ t_0 + T^* + h]$ such that $|x(\bar{t})| \leq U$ and $v(s) \leq 1 + v(\bar{t})$. For $s \in [\bar{t} - h, \bar{t}\,]$, integrate (ii) from s to $\bar{t}$ to obtain

$$\begin{aligned} v(\bar{t}) &\leq v(s) - \int_s^{\bar{t}} W_4(|x(u)|)du + M(\bar{t} - s) \\ &\leq v(\bar{t}) + 1 - \int_s^{\bar{t}} W_4(|x(u)|)du + M(\bar{t} - s). \end{aligned}$$

Thus,

$$\frac{1}{\bar{t} - s} \int_s^{\bar{t}} W_4(|x(u)|)du \leq M + \frac{1}{(\bar{t} - s)}.$$

By Lemma 2.1 with $\alpha = M$ and $\beta = 1$, we obtain

$$\int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t} - u) W_4(|x(u)|)du \leq J^* M + J' =: M^{**}.$$

This yields

$$\begin{aligned} v(\bar{t}) &\leq W_2(|x(\bar{t})|) + W_3[\Big\{ \int_0^{\bar{t}-h} + \int_{\bar{t}-h}^{\bar{t}} \Big\} \Phi(\bar{t} - u) W_4(|x(u)|)du] \\ &\leq W_2(U) + W_3[W_4(B_2) \int_h^{+\infty} \Phi(u)du + M^{**}] \\ &\leq W_2(U) + W_3(1 + M^{**}) \end{aligned}$$

and

$$v(s) \leq 1 + v(\hat{t}) \leq 1 + W_2(U) + W_3(1 + M^{**}) \tag{2.12}$$

for all $s \in [\hat{t} - h, \hat{t}]$. Now let $t > \bar{t}$ such that $v(s) < v(t)$ for $\bar{t} - h \leq s < t$. Then $|x(t)| \leq U$. Integrate (ii) from s to t to obtain

$$0 \leq v(t) - v(s) \leq - \int_s^t W_4(|x(u)|)du + M(t - s).$$

Thus,

$$\frac{1}{t-s}\int_s^t W_4(|x(u)|)du \le M. \tag{2.13}$$

Applying Lemma 2.1 to (2.13) with $q(u) = W_4(|x(u)|)$ and having (2.12), we obtain (2.7) and (2.8) with $K = T^*$. Thus, solutions of (1.1) are UUB by Theorem 2.2. The proof is complete.

Theorem 2.4. Suppose there exists a continuous function $\Phi : R^+ \to R^+$ with $\Phi(u), \Phi'(u)u \in L^1(R^+)$, wedges W_j with $W_1(r) \to +\infty$ as $r \to +\infty$, positive constants U, M, and M^* with $W_5(U) > M$ and $M^* > M$, and a continuous functional $V : R^+ \times C \to R^+$ such that (i) and (ii) hold for each $x \in C(t)$. Suppose also that $W_4(r) \to +\infty$ as $r \to +\infty$ and there are positive constants σ, r_0 such that $r \ge r_0$ implies

$$\int_0^r W_5(W_4^{-1}(s))ds > (J^*/J)M^*W_4(W_1^{-1}[W_2(U) + W_3(\sigma + Jr)]) \tag{2.14}$$

where J and J^* are given in (2.3). Then solutions of (1.1) are UB and UUB.

Proof. We first show solutions of (1.1) are UB. Let $B_1 > 0$ and $\sigma > 0$ be given in (2.14). Choose $h > 0$ such that

$$W_4(B_1)\int_h^{+\infty} \Phi(u)du < \sigma \ \text{ and } \ \int_0^h \Phi(u)du \ge JM/M^*.$$

Let $x(t) = x(t, t_0, \phi)$ be a solution with $\|\phi\| \le B_1$. Then we have either

(A*) $v(t) \le \max\{v(s) : t_0 \le s \le t_0 + h\}$ for all $t \ge t_0 + h$ or

(B*) $v(s) < v(\bar{t})$ for some $\bar{t} > t_0 + h$ and all $t_0 \le s < \bar{t}$.

Notice that $\max\{v(s) : t_0 \le s \le t_0 + h\} \le v(t_0) + Mh$ by (ii) and $v(t_0) \le W_2(B_1) + W_3(JW_4(B_1))$. Thus, if (A*) holds, then

$$|x(t)| \le W_1^{-1}[W_2(B_1) + W_3(JW_4(B_1)) + Mh]$$

for all $t \ge t_0$. Now suppose (B*) holds. By the definition of $\bar{t}$ and (ii), we have $|x(\bar{t})| \le U$. Let $\hat{t} \in [t_0, \bar{t}\,]$ such that $|x(\hat{t})| = \max_{t_0 \le s \le \bar{t}} |x(s)|$. Then

$$W_1(|x(\hat{t})|) \ \le \ v(\hat{t}) \le v(\bar{t})$$

$$
\begin{aligned}
&\leq W_2(|x(\bar{t})|) + W_3[\int_0^{t_0} \Phi(\bar{t}-s)W_4(|x(s)|)ds + \int_{t_0}^{\bar{t}} \Phi(\bar{t}-s)W_4(|x(s)|)ds] \\
&\leq W_2(U) + W_3[W_4(B_1)\int_{\bar{t}-t_0}^{\infty} \Phi(u)du + \int_{t_0}^{\bar{t}} \Phi(\bar{t}-s)W_4(|x(s)|)ds] \\
&\leq W_2(U) + W_3(\sigma + X)
\end{aligned}
$$

where

$$X = \int_{t_0}^{\bar{t}} \Phi(\bar{t}-u)W_4(|x(u)|)du.$$

This yields

$$|x(\hat{t})| \leq W_1^{-1}[W_2(U) + W_3(\sigma + X)]$$

and

$$W_4(|x(\hat{t})|) \leq W_4(W_1^{-1}[W_2(U) + W_3(\sigma + X)]). \tag{2.15}$$

Next, define

$$W_6(r) = \int_0^r W_5[W_4^{-1}(u)]du.$$

Since the domain of W_4^{-1} is $[0, +\infty)$, W_6 is well defined, convex downward, and satisfies

$$W_6(r) \leq W_5[W_4^{-1}(r)]r^* \text{ on } 0 \leq r \leq r^*.$$

Particularly,

$$W_6(r) \leq W_5[W_4^{-1}(r)]W_4(|x(\hat{t})|) \text{ for } 0 \leq r \leq W_4(|x(\hat{t})|).$$

Thus,

$$W_6(W_4(|x(s)|)) \leq W_5(W_4^{-1}[W_4(|x(s)|)])W_4(|x(\hat{t})|) = W_5(|x(s)|)W_4(|x(\hat{t})|) \tag{2.16}$$

for $t_0 \leq s \leq \bar{t}$. For any $s \in [t_0, \bar{t}\,]$, we have

$$0 \leq v(\bar{t}) - v(s) \leq -\int_s^{\bar{t}} W_5(|x(u)|)du + M(\bar{t}-s).$$

Apply Lemma 2.1 to get

$$\int_{t_0}^{\bar{t}} \Phi(\bar{t}-u)W_5(|x(u)|)du \leq J^*M. \tag{2.17}$$

Using (2.16), (2.17), and Jensen's inequality, we obtain

$$\int_{t_0}^{\bar{t}} \Phi(\bar{t}-u)du W_6\Big[\frac{\int_{t_0}^{\bar{t}} \Phi(\bar{t}-u)W_4(|x(u)|)du}{\int_{t_0}^{\bar{t}} \Phi(\bar{t}-u)du}\Big] \leq J^* M W_4(|x(\hat{t})|).$$

Notice that

$$\int_{t_0}^{\bar{t}} \Phi(\bar{t}-u)du = \int_0^{\bar{t}-t_0} \Phi(u)du \geq \int_0^h \Phi(u)du \geq JM/M^*.$$

This implies

$$W_6(X/J) \leq \frac{J^* M}{\int_0^h \Phi(u)du} W_4(|x(\hat{t})|) \leq (J^*/J)M^* W_4(|x(\hat{t})|)$$

and

$$\int_0^{X/J} W_5[W_4^{-1}(u)]du \leq (J^*/J)M^* W_4(W_1^{-1}[W_2(U^*) + W_3(\sigma + X)]).$$

By (2.14), we must have $X \leq Jr_0$. Thus,

$$W_1(|x(\bar{t})|) \leq v(\bar{t}) \leq W_2(U) + W_3(\sigma + Jr_0). \tag{2.18}$$

Since $\bar{t}$ is arbitrary, we have for all $t \geq t_0$

$$\begin{aligned} W_1(|x(t)|) &\leq v(t) \leq W_2(U) + W_3(\sigma + Jr_0) + \max\{v(s) : t_0 \leq s \leq t_0 + h\} \\ &\leq W_2(U) + W_3(\sigma + Jr_0) + W_2(B_1) + W_3(JW_4(B_1)) + Mh \end{aligned}$$

and

$$|x(t)| \leq W_1^{-1}[W_2(U) + W_3(\sigma + Jr_0) + W_2(B_1) + W_3(JW_4(B_1)) + Mh] =: B_2.$$

This completes the proof of uniform boundedness.

Now we show that solutions of (1.1) are UUB. For the constants B_1, B_2 given above, $h > 0$ with $W_4(B_2)\int_h^{+\infty} \Phi(u)du < \sigma$, $U^* < U$ with $W_5(U^*) > M$, we define

$$\gamma = \min\Big\{\frac{M(M^* - M)J^*}{(M^* + M)J'},\ W_2(U) - W_2(U^*)\Big\}$$

where J^*, J' are given in (2.3). By Lemma 2.2 there exists $T^* > 0$ such that for each solution $x(t) = x(t, t_0, \phi)$ of (1.1) with $\|\phi\| \leq B_1$, there is a $\bar{t} \in [t_0+h, t_0+T^*+h]$ such that $|x(\bar{t})| \le U^*$ and $v(s) \le \gamma + v(\bar{t})$ for $s \in [\bar{t}-h, \bar{t}\,]$, where $v(s) = V(s, x_s)$. Choose $h > 0$ so that $\int_0^h \Phi(u)du \ge 2MJ/(M^*+M)$. Let $|x(t^*)| = \max\{|x(s)| : \bar{t}-h \le s \le \bar{t}\}$. Then

$$W_6(r) \le W_5[W_4^{-1}(r)]W_4(|x(t^*)|) \;\text{ for }\; 0 \le r \le W_4(|x(t^*)|)$$

and $W_6(W_4(|x(s)|)) \le W_5(|x(s)|)W_4(|x(t^*)|)$ for all $\bar{t}-h \le s \le \bar{t}$. For $s \in [\bar{t}-h,\, \bar{t}\,]$, we have

$$\begin{aligned} v(\bar{t}) &\le v(s) - \int_s^{\bar{t}} W_5(|x(u)|)du + M(\bar{t}-s) \\ &\le \gamma + v(\bar{t}) - \int_s^{\bar{t}} W_5(|x(u)|)du + M(\bar{t}-s). \end{aligned}$$

This implies

$$\frac{1}{\bar{t}-s}\int_s^{\bar{t}} W_5(|x(u)|)du \le M + \frac{\gamma}{\bar{t}-s}$$

for $\bar{t}-h \le s < \bar{t}$. By Lemma 2.1, we obtain

$$\int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t}-u)W_5(|x(u)|)du \le J^*M + J'\gamma.$$

By the definition of W_6 and Jensen's inequality, we have

$$\int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t}-u)du W_6\Big[\frac{\int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t}-u)W_4(|x(u)|)du}{\int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t}-u)du}\Big] \le (J^*M + J'\gamma)W_4(|x(t^*)|). \quad (2.19)$$

Define

$$X^* = \int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t}-u)W_4(|x(u)|)du.$$

Since $\int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t}-u)du = \int_0^h \Phi(u)du \ge 2MJ/(M^*+M)$, we get from (2.19) that

$$\int_0^{X^*/J} W_5[W_4^{-1}(u)]du \le \frac{M^*+M}{2MJ}[J^*M + J'\gamma]W_4(|x(t^*)|).$$

Notice also that

$$\begin{aligned} W_1(|x(t^*)|) &\le v(t^*) \le v(\bar{t}) + \gamma \\ &\le \gamma + W_2(U^*) + W_3[\sigma + \int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t}-u)W_4(|x(u)|)du] \\ &\le W_2(U) + W_3[\sigma + \int_{\bar{t}-h}^{\bar{t}} \Phi(\bar{t}-u)W_4(|x(u)|)du]. \end{aligned}$$

Thus,

$$W_4(|x(t^*)|) \leq W_4[W_1^{-1}(W_2(U) + W_3(\sigma + X^*))]$$

and

$$\begin{aligned}\int_0^{X^*/J} W_5[W_4^{-1}(u)]du &\leq [\frac{(M^*+M)J^*}{2J} + \frac{J'\gamma(M^*+M)}{2JM}]W_4(|x(t^*)|) \\ &\leq (J^*/J)M^*W_4[W_1^{-1}(W_2(U) + W_3(\sigma + X^*))].\end{aligned}$$

This implies that $X^* \leq Jr_0$ by (2.14). Thus,

$$v(\bar{t}) \leq W_2(U) + W_3[\sigma + Jr_0]$$

and

$$\sup_{\hat{t}-h \leq s \leq \hat{t}} v(s) \leq \gamma + W_2(U) + W_3[\sigma + Jr_0]. \tag{2.20}$$

Now let $t > \bar{t}$ such that $v(s) < v(t)$ for $\bar{t} - h \leq s < t$. Then $|x(t)| \leq U$. Integrate (ii) from s to t to obtain

$$0 \leq v(t) - v(s) \leq -\int_s^t W_5(|x(u)|du + M(t-s).$$

This yields

$$\frac{1}{t-s}\int_s^t W_5(|x(u)|)du \leq M$$

and

$$\int_{\bar{t}-h}^t \Phi(t-u)W_5(|x(u)|)du \leq J^*M$$

by Lemma 2.1. Let

$$|x(t^{**})| = \max\{|x(s)| : \bar{t} - h \leq s \leq t\}.$$

It follows from Jensen's inequality that

$$\int_{\bar{t}-h}^t \Phi(t-s)dsW_6\Big[\frac{\int_{\bar{t}-h}^t \Phi(t-s)W_4(|x(s)|)ds}{\int_{\bar{t}-h}^t \Phi(t-s)ds}\Big] \leq J^*MW_4(|x(t^{**})|). \tag{2.21}$$

Observing

$$\int_{\bar{t}-h}^t \Phi(t-s)ds = \int_0^{\bar{t}-t+h} \Phi(u)du \geq \frac{2MJ}{M^*+M}$$

and letting

$$X^{**} = \int_{\bar{t}-h}^{t} \Phi(t-u) W_4(|x(u)|) du,$$

we obtain from (2.21)

$$\int_0^{X^{**}/J} W_5[W_4^{-1}(u)] du \leq (J^*/J) M^* W_4(|x(t^{**})|).$$

By the definition of t, we have

$$\begin{aligned}
&W_1(|x(t^{**})|) \leq v(t^{**}) \leq v(t) \\
&\leq W_2(U) + W_3(\int_0^{\bar{t}-h} \Phi(t-u) W_4(|x(u)|) du + \int_{\bar{t}-h}^{t} \Phi(t-u) W_4(|x(u)|) du) \\
&\leq W_2(U) + W_3(\sigma + X^{**}).
\end{aligned}$$

This implies

$$\int_0^{X^{**}/J} W_5[W_4^{-1}(u)] du \leq (J^*/J) M^* W_4[W_1^{-1}(W_2(U) + W_3(\sigma + X^{**}))]$$

and $X^{**} \leq J r_0$ by (2.14). Thus, (2.7) and (2.8) hold. By Theorem 2.2, solutions of (1.1) are UUB. This completes the proof.

Remark 2.5. By Remark 2.1, if $\Phi'(t) \leq 0$, then $J^* = J$. In this case, (2.14) can be reduced to

$$\int_0^r W_5(W_4^{-1}(s)) ds > M^* W_4(W_1^{-1}[W_2(U) + W_3(\sigma + Jr)]) \tag{2.22}$$

for all $r \geq r_0$.

Example 2.1. Under condition (1.2), solutions of (1.3) are UB and UUB.

Indeed, let $\Phi(t) = \int_t^{+\infty} D(u) du/2$. Then $\Phi'(t) \leq 0$. Thus, $\Phi'(u)u \in L^1[0, +\infty)$ by Remark 2.1. Define $W_1(r) = W_2(r) = r^4/4, W_3(r) = r, W_4(r) = r^6$, and $W_5(r) = r^{n+3}$. Then (1.5) and (1.6) satisfy (i) and (ii) of Theorem 2.1. To show solutions of (1.3) are UB and UUB, we need to verify that (2.22) holds. Notice that $W_1^{-1}(r) = (4r)^{1/4}$ and $W_4^{-1}(r) = r^{1/6}$. Then

$$\int_0^r W_5[W_4^{-1}(s)] ds = \int_0^r s^{\frac{n+3}{6}} ds = \frac{6}{n+9} r^{\frac{n+9}{6}}.$$

For any $M^* > M$ and $\sigma > 0$, we have

$$\begin{aligned} & M^* W_4(W_1^{-1}[W_2(U) + W_3(\sigma + Jr)]) \\ = \ & M^*\Big([4(W_2(U) + \sigma + Jr)]^{1/4}\Big)^6 \\ = \ & M^*[4(W_2(U) + \sigma + Jr)]^{3/2}. \end{aligned}$$

Thus, there exists $r_0 > 0$ such that (2.22) holds since $n > 0$.

Corollary 2.2. Suppose there exists a continuous function $\Phi : R^+ \to R^+$ with $\Phi(u), \Phi'(u)u \in L^1(R^+)$, wedges W_j with $W_1(r) \to +\infty$ as $r \to +\infty$, positive constants U, M with $W_5(U) > J^*M/J$, where J^*, J are given in (2.3), and a continuous functional $V : R^+ \times C \to R^+$ such that (i) and (ii) hold for each $x \in C(t)$. Suppose also that $W_4(r) = r$ and there exists a positive constant r_0 such that $r \geq r_0$ implies

$$W_1(r) - W_3(Jr) > W_2(U) \tag{2.23}$$

where J is given in (2.3). Then solutions of (1.1) are UB and UUB.

Proof. Choose $W_5(U) > (J^*/J)M^* > (J^*/J)M$. By (2.23), we have

$$W_1(r + 1/J) > W_2(U) + W_3(1 + Jr)$$

for $r \geq r_0 - 1/J$. Thus,

$$r + 1/J > W_1^{-1}[W_2(U) + W_3(1 + Jr)]$$

for $r \geq r_0 - 1/J$. Since $W_5(U) > (J^*/J)M^*$, there exists a constant $\bar{r}_0 \geq r_0$ such that $r \geq \bar{r}$ implies

$$\int_0^r W_5(u)du > (J^*/J)M^*(r + 1/J).$$

Thus, for $r \geq \bar{r}$, we have

$$\int_0^r W_5(u)du > (J^*/J)M^* W_1^{-1}[W_2(U) + W_3(1 + Jr)]$$

which is equivalent to (2.14). By Theorem 2.4, solutions of (1.1) are UB and UUB.

Remark 2.6. Condition (2.23) is similar to those given by Hering [7], Yoshizawa ([15],p.202), and Zhang [17].

Corollary 2.3. Suppose there exists a continuous function $\Phi : R^+ \to R^+$ with $\Phi(u), \Phi'(u)u \in L^1(R^+)$, wedges W_j with $W_1(r) \to +\infty$ as $r \to +\infty$, positive constants U, M, and M^* with $W_5(U) > M$ and $M^* > M$, and a continuous functional $V : R^+ \times C \to R^+$ such that (i) and (ii) hold for each $x \in C(t)$. Suppose also that $W_4(r) \to +\infty$ as $r \to +\infty$, W_3 is uniformly continuous on R^+, and there exists a positive constant r_0 such that $r \geq r_0$ implies

$$\int_0^r W_5(W_4^{-1}(s))ds > (J^*/J)M^*W_4(W_1^{-1}[W_2(U) + W_3(Jr)])$$

where J is given in (2.3). Then solutions of (1.1) are UB and UUB.

Proof. Choose $U^* < U$ with $W_5(U^*) > M$ and let $\delta = W_5(U) - W_5(U^*)$. Since W_3 is uniformly continuous on R^+, there exists a constant $\sigma > 0$ such that $W_3(s + \sigma) - W_3(s) < \delta$ for all $s \in R^+$. Thus,

$$\begin{aligned}
&\int_0^r W_5(W_4^{-1}(s))ds > (J^*/J)M^*W_4(W_1^{-1}[W_2(U) + W_3(Jr)]) \\
&= (J^*/J)M^*W_4\{W_1^{-1}[W_2(U) + W_3(\sigma + Jr) - W_3(\sigma + Jr) + W_3(Jr)]\} \\
&\geq (J^*/J)M^*W_4\{W_1^{-1}[W_2(U) + W_3(\sigma + Jr) - \delta]\} \\
&\geq (J^*/J)M^*W_4\{W_1^{-1}[W_2(U^*) + W_3(\sigma + Jr)]\}.
\end{aligned}$$

This implies that (2.14) holds with U replaced by U^*. Thus, solutions of (1.1) are UB and UUB.

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