ASYMPTOTIC STABILITY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

T. A. Burton and G. Makay

Department of Mathematics, Southern Illinois University at Carbondale

Carbondale, Illinois 62901-4408

1. Introduction

We consider a system of functional differential equations with finite delay written as

$$x'(t) = f(t, x_t), \qquad ' = d/dt, \tag{1}$$

where $f : [0, \infty) \times \mathcal{C}_H \to \mathbf{R}^m$ is continuous and takes bounded sets into bounded sets and $f(t, 0) = 0$. Here, $(\mathcal{C}, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-h, 0] \to \mathbf{R}^m$ with the supremum norm, h is a non-negative constant, $\mathcal{C}_H$ is the open H-ball in $\mathcal{C}$, and $x_t(s) = x(t + s)$ for $-h \leq s \leq 0$. Standard existence theory shows that if $\phi \in \mathcal{C}_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ on $[t_0, t_0 + \alpha)$ satisfying (1) for $t > t_0$, $x_t(t_0, \phi) = \phi$ and α some positive constant; if there is a closed subset $B \subset \mathcal{C}_H$ such that the solution remains in B, then $\alpha = \infty$. Also, $|\cdot|$ will denote the norm in $\mathbf{R}^m$ with $|x| = \max_{1 \leq i \leq m} |x_i|$.

We are concerned here with asymptotic stability in the context of Liapunov's direct method. Thus, we are concerned with continuous, strictly increasing functions $W_i : [0, \infty) \to [0, \infty)$ with $W_i(0) = 0$, called wedges, and with Liapunov functionals V.

Definition: A continuous functional $V : [0, \infty) \times \mathcal{C}_H \to [0, \infty)$ which is locally Lipschitz in ϕ is called a Liapunov functional for (1) if there is a wedge W with

(i) $W(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$, and

(ii) $V'_{(1)}(t, x_t) = \limsup_{\delta \to 0} \frac{1}{\delta} \{V(t + \delta, x_{t+\delta}(t_0, \phi)) - V(t, x_t(t_0, \phi))\} \leq 0$.

Remark: A standard result states that if there is a Liapunov functional for (1), then $x = 0$ is stable. Definitions will be given in the next section.

The classical result on asymptotic stability may be traced back to Marachkov [9] through Krasovskii [7;pp. 151-154]. It may be stated as follows.

Theorem MK: Suppose there are a constant M, wedges W_i, and a Liapunov functional V (so $W_1(|\phi(0)|) \leq V(t, \phi)$ and $V(t, 0) = 0$) with

(i) $V'_{(1)}(t, x_t) \leq -W_2(|x(t)|)$ and

(ii) $|f(t, \phi)| \leq M$ if $t \geq 0$ and $\|\phi\| < H$.

Then $x = 0$ is asymptotically stable.

Condition (ii) is troublesome, since it excludes many examples of considerable interest. And there are several results which reduce or eliminate (ii). For example, we showed [2] that if

(iii) $V(t, \phi) \leq W_3(|x_t|_2)$,

where $|\cdot|_2$ is the L^2-norm, then uniform asymptotic stability would result. Other alternatives may be found in [3,4,5,6], for example.

We reduce (ii) in a variety of ways and obtain results on asymptotic stability, partial stability, and uniform asymptotic stability. We give an example in which we show that the zero solution of

$$x'' + tx' + x = 0 \tag{2}$$

is uniformly asymptotically stable.

The following is a simplified corollary to our results and is stated here to focus the paper.

Theorem A: Suppose there is a Liapunov functional V, wedges W_i, positive constants K and J, a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \leq K$ such that

(i) $V(t_n, \phi) \leq W_2(\|\phi\|)$,

(ii) $V'_{(1)}(t, x_t) \leq -W_3(|x(t)|)$ if $t_n - h \leq t \leq t_n$, and

(iii) $|f(t, \phi)| \leq J(t + 1)\ln(t + 2)$ for $t \geq 0$ and $\|\phi\| < H$.

Then $x = 0$ is AS.

2. Statement of results and examples

We now define the terminology to be used here.

Definition: The solution $x = 0$ of (1) is:

(a) *stable* if for each $\varepsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\|\phi\| < \delta, t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \varepsilon$;

(b) *uniformly stable (US)* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $[t_0 \geq 0, \|\phi\| < \delta, t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \varepsilon$;

(c) *asymptotically stable (AS)* if it is stable and if for each $t_0 \geq 0$ there is a $\gamma > 0$ such that $\|\phi\| < \gamma$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$;

(d) *uniformly asymptotically stable (UAS)* if it is US and if there is a $\gamma > 0$ and for each $\mu > 0$ there is a $T > 0$ such that $[t_0 \geq 0, \|\phi\| < \gamma, t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \mu$.

In preparation for our main result we remind the reader that if V is a Liapunov functional, then $W_1(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$, and $V'_{(1)}(t, x_t) \leq 0$. So that our result applies also to ODE's we introduce a positive number k which will replace h found in (1).

Theorem 1: Let $k > 0$, $k \geq h$, let V be a Liapunov functional for (1) (so that $W_1(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$, and $V'_{(1)}(t, x_t) \leq 0$) and $x = (x_1, ..., x_m)$. Consider the following conditions for a given i $(1 \leq i \leq m)$ and a given sequence $\{t_n\}$ with $t_n \uparrow \infty$:

(i) there are wedges W_i, U_i, Q_i,

(ii) there is a sequence $\left\{\lambda_n^{(i)}\right\}$ with $\lambda_n^{(i)} \geq \lambda > 0$, λ is a constant, and that

(iii) there are locally integrable functions $M_i, P_i : [0, \infty) \to [0, \infty)$ such that

(iv) either $M_i \equiv 0$ or for each $D > 0$ with $D/\lambda_n^{(i)} \leq k$ there is a sequence $\left\{c_n^{(i)}\right\}$, $c_n^{(i)} > 0$, such that if $a, b \in [t_n - k, t_n]$ with $a < b$, then $\int_a^b M_i(t)dt \leq \lambda_n^{(i)}(b - a)$ and $\int_{s_n}^{s_n + D/\lambda_n^{(i)}} P_i(s)ds \geq c_n^{(i)}$ for all $s_n \in [t_n - k, t_n - D/\lambda_n^{(i)}]$,

(v) $V'_{(1)}(t, x_t) \leq -P_i(t)U_i(|x_i|)$ for $\|x_t\| < H$ and $t \in [t_n - k, t_n]$, and

(vi) $V'_{(1)}(t, x_t) \leq -Q_i(|x'_i|) + M_i(t)$ for $\|x_t\| < H$ and $t \in [t_n - k, t_n]$ with Q_i convex downward.

We then have the following conclusions:

(I) If (i)-(vi) hold for all i satisfying $1 \leq i \leq m$ and for some $\{t_n\} \uparrow \infty$ with $c_n^{(i)} \geq c_0 > 0$ for all n and all i, if $t_n - t_{n-1}$ is bounded, and if $V(t, \phi) \leq W(\|\phi\|)$, then $x = 0$ is UAS.

(II) If (i)-(vi) hold for an arbitrary sequence $\{t_n\} \uparrow \infty$ and for some i satisfying $1 \leq i \leq m$, if $c_n^{(i)} \geq c_0 > 0$ for all n then any solution $x(t)$ which remains in $\mathcal{C}_H$ satisfies $x_i(t) \to 0$ as $t \to \infty$.

(III) If (i)-(vi) hold for all i satisfying $1 \leq i \leq m$ and for some sequence $\{t_n\} \uparrow \infty$, if $V(t_n, \phi) \leq W(\|\phi\|)$, if $c_n^{(i)} \geq c_n$ for $1 \leq i \leq m$ and some c_n with $\sum_{n=1}^{\infty} c_n = \infty$, then $x = 0$ is AS.

Remark: Theorem 1 is long because it is stated in terms of separate components of x; but an example will show that it is well worth the detail. However, to grasp the significance we will now state some useful corollaries.

Corollary 1: Suppose there is a Liapunov functional V, a locally integrable function $M : [0, \infty) \to [0, \infty)$ and a monotone increasing function $\lambda : [0, \infty) \to (1, \infty)$ such that if $0 < b - a < h$ then

(i) $\int_a^b M(t)dt \leq \lambda(b)(b - a)$ and $\int_1^{\infty} \frac{dt}{\lambda(t)} = \infty$.

Suppose also that there are wedges, a constant $K > 0$, and a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \leq K$ such that

(ii) $V(t_n, \phi) \leq W(\|\phi\|)$

and if $t_n - h \leq t \leq t_n$ then

(iii) $V'_{(1)}(t, x_t) \leq -W_2(|x(t)|)$ and

(iv) $V'_{(1)}(t, x_t) \leq -W_3(|x'(t)|) + M(t)$, W_3 is convex downward.

Then $x = 0$ is AS.

Corollary 2: Suppose there is a Liapunov functional V, wedges W_i, positive constants K and J, a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \leq K$ such that

(i) $V(t_n, \phi) \leq W_2(\|\phi\|)$,

(ii) $V'_{(1)}(t, x_t) \leq -W_3(|x(t)|)$ if $t_n - h \leq t \leq t_n$, and

(iii) $|f(t, \phi)| \leq J(t + 1)\ln(t + 2)$ for $t \geq 0$ and $\|\phi\| < H$.

Then $x = 0$ is AS.

Corollary 3: Suppose there are a Liapunov functional V and a wedge W_2 with

(i) $V(t, \phi) \leq W_2(\|\phi\|)$.

In addition, suppose there are locally integrable functions $M, P : [0, \infty) \to [0, \infty)$, a positive constant K, sequences $\{t_n\} \uparrow \infty$ and $\{\lambda_n\}$ with $t_n - t_{n-1} \leq K$, such that if $0 < b - a < h$ and if $t_n - h \leq t \leq t_n$ with $b \leq t_n$, then for each $D > 0$ there is a $c > 0$ with

(ii) $\int_a^b M(s)ds \le \lambda_n(b-a)$ and $\int_t^{t+D/\lambda_n} P(s)ds \ge c$,

(iii) $V'_{(1)}(t,x_t) \le -P(t)W_3(|x(t)|)$ for $t_n - h \le t \le t_n$, and

(iv) $V'_{(1)}(t,x_t) \le -W_4(|x'(t)|) + M(t)$, W_4 is convex downward.

Then $x=0$ if UAS.

Corollary 4: (Marachkov-Krasovskii) If there is a Liapunov functional V, wedges W_i, and a constant M such that

(i) $V(t,\phi) \le W_2(\|\phi\|)$,

(ii) $V'_{(1)}(t,x_t) \le -W_3(|x(t)|)$,

(iii) $|f(t,\phi)| \le M$ if $t \ge 0$ and $\|\phi\| < H$,

then $x=0$ is UAS.

We now give an example of Corollary 2.

Example 1: Let $a,b : [0,\infty) \to \mathbf{R}$ be continuous and suppose there are constants $c_1 \ge 1$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$ with

(a) $a(t) - c_1|b(t+1)| =: \alpha(t) \ge c_3$,

(b) there is a sequence $\{t_n\} \uparrow \infty$ and $K>0$ with $1 \le t_{n+1} - t_n \le K$ and $\int_{t_n-1}^{t_n} |b(s+1)|ds \le c_2$.

(c) $a(t) + |b(t)| \le c_4(t+1)\ln(t+2)$.

Then the zero solution of

$$x'(t) = -a(t)x + b(t)x(t-1) \tag{3}$$

is AS.

Proof: Define

$$V(t,x_t) = |x(t)| + c_1\int_{t-1}^{t} |b(s+1)||x(s)|ds$$

so that

$$\begin{aligned} V'_{(3)}(t,x_t) &\le -a(t)|x| + |b(t)||x(t-1)| + c_1|b(t+1)||x| - c_1|b(t)||x(t-1)| \\ &\le -[a(t) - c_1|b(t+1)|]|x| \le -\alpha(t)|x|. \end{aligned}$$

Take $H=1$ and $W(r)=r$. Then for $\|\phi\| < H$ we have

$$|\phi(0)| \le V(t,\phi), \qquad V(t,0) = 0$$

$$V(t_n,\phi) \le |\phi(0)| + c_1c_2\|\phi\|$$

and

$$V'(t,x_t) \leq -c_3|x(t)|.$$

The conditions of Corollary 2 are satisfied.

Examples of $a(t)$ and $b(t)$ are easily constructed so that this equation is not uniformly stable. Let $m(t) = -[t]\sin 2\pi t$, $r(t) = -[t](\cos 2\pi t - 1)/2\pi$ and $c(t) = |\sin\pi t| - \sin\pi t$, where $[\cdot]$ stands for the greatest integer function. Consider the scalar equation

$$x' = (m(t) - 1 - e^2\ln(t+1))x(t) + \frac{1}{2}c(t)(\ln t)x(t-1)$$

for $t \geq 1$. Note that

$$\int_n^{n+1} [t]\sin 2\pi t dt = n\int_n^{n+1} \sin 2\pi t dt = -\frac{n}{2\pi}(\cos 2\pi(n+1) - \cos 2\pi n) = 0$$

so that if $n \leq t < n+1$ then

$$r(t) = \int_0^t -[s]\sin 2\pi s ds = \frac{n}{2\pi}(\cos 2\pi t - 1) = \frac{[t]}{2\pi}(\cos 2\pi t - 1).$$

Let

$$V(t) = V(t,x_t) = e^{2r(t)}x^2 + \frac{1}{2}\int_{t-1}^t e^{2r(s+1)}\ln(s+1)c(s+1)x^2(s)ds$$

so that

$$\begin{aligned} V'(t) &\leq (-2m(t) + 2m(t) - 2 - 2e^2\ln(t+1))e^{2r(t)}x^2 + c(t)(\ln t)x(t)x(t-1)e^{2r(t)} \\ &+ \frac{1}{2}e^{2r(t+1)}\ln(t+1)c(t+1)x^2 - \frac{1}{2}e^{2r(t)}(\ln t)c(t)x^2(t-1) \\ &\leq -(2 + 2e^2\ln(t+1))e^{2r(t)}x^2 + (\ln t)e^{2r(t)}x^2 + \frac{c(t)}{2}(\ln t)e^{2r(t)}x^2(t-1) \\ &+ e^{2r(t+1)}\ln(t+1)x^2(t) - \frac{1}{2}e^{2r(t)}(\ln t)c(t)x^2(t-1). \end{aligned}$$

Now

$$e^{2r(t+1)} = e^{-2([t]+1)(\cos 2\pi t - 1)/2\pi} \leq e^2 e^{2r(t)}$$

so $V'(t) \leq -2x^2(t)$. Also, $V(t) \geq x^2(t)$. Finally, when n is even

$$V(n) = x^2 + \frac{1}{2}\int_{n-1}^n e^{2r(s+1)}\ln(s+1)(|\sin\pi(s+1)| - \sin\pi(s+1))x^2(s)ds = x^2.$$

Hence, the conditions of Corollary 2 are satisfied and $x = 0$ is AS.

Remark: This result will not follow from the work of Busenberg and Cooke [6] because they require that for each $\eta > 0$ there exists $\tau > 0$ such that $\int_t^{t+\eta} a(s)ds \leq \tau$. It will not follow from Burton [2] because that result requires that $V(t,\phi) \leq W_2(|\phi(0)|) + W_3(|\phi|_2)$, where $|\cdot|_2$ is the L^2-norm. It will not follow from Burton-Hatvani [5] for the same reason. It will not follow from Makay [8] because he requires $V(t,\phi) \leq W(\|\phi\|)$. It will not follow from Wang [12] because he requires uniform stability.

In the next example it is very easy to show AS by a variety of classical techniques [1,10,11]. But it requires all of the flexibility of Theorem 1 to show UAS.

Example 2: The zero solution of

$$x'' + tx' + x = 0, \qquad t \geq 1 \tag{4}$$

is UAS.

Proof: Write (4) as

$$\begin{aligned} x' &= -\frac{x+y}{t} \\ y' &= \frac{2x}{t} + \left(\frac{2}{t} - t\right) y. \end{aligned}$$

Let $(x_1, x_2) = (x, y)$ and define $V = (x^2 + y^2)/2$ so that

$$\begin{aligned} V' &= -\frac{1}{t}x^2 - \frac{1}{t}xy + \frac{2}{t}xy + \left(\frac{2}{t} - t\right) y^2 \\ &\leq -\frac{1}{t}x^2 + \frac{1}{2t}(x^2 + y^2) + \left(\frac{2}{t} - t\right) y^2 \\ &= -\frac{1}{2t}x^2 + \left(\frac{1}{2t} + \frac{2}{t} - t\right) y^2. \end{aligned}$$

Thus, for $t \geq 4$ we have (v) of Theorem 1 satisfied:

(v) $V' \leq -\frac{1}{2t}x^2 - \frac{t}{2}y^2 =: -P_1(t)U_1(|x_1|) - P_2(t)U_2(|x_2|)$.

(We remark that at this point we have $V' \leq -\frac{k}{t}V$ so the zero solution is AS.)

Again for $t \geq 4$ we have

$$V' \leq -\frac{1}{2t}(x^2 + y^2) \leq -\frac{1}{4t}|x + y|^2 = -\frac{t}{4}|x'(t)|^2$$

or

(vi) $V' \leq -\frac{1}{4}|x'(t)|^2 + 0 = -Q_1(|x_1'|) + M_1(t)$

so that for $i = 1$ we satisfy (vi) with $M_1(t) = 0$. Likewise, for $t \geq 4$ we have

$$\begin{aligned} V' &\leq -\frac{1}{2t}|x| - \frac{t}{2}|y| + \frac{1}{2t} + \frac{t}{2} \qquad \text{for } |(x,y)| \leq 1 \\ &\leq -\frac{1}{4}|y'| + t; \end{aligned}$$

thus, for $t \geq 4$ and $|(x,y)| \leq 1$ we have

(vi) $V' \leq -\frac{1}{4}|y'| + t =: -Q_2(|x_2'|) + M_2(t)$.

We see that (iv) is satisfied for $i = 1$, while for $i = 2$ we have

$$\begin{aligned} \int_a^b M_2(t)dt = \int_a^b t dt = \frac{t^2}{2}\bigg|_a^b &= \frac{b+a}{2}(b-a) \\ &\leq b(b-a); \end{aligned}$$

thus, if $k = 1$, $t_n = n$, $b \leq t_n$, then we have

$$\int_a^b M_2(t)dt \leq n(b-a) =: \lambda_n^{(2)}(b-a)$$

so that if $D > 0$, then for $n - 1 \leq s_n \leq n - \frac{D}{n}$ we have

$$\int_{s_n}^{s_n + D/n} P_2(t)dt \geq \int_{s_n}^{s_n + D/n} \frac{t}{2} dt \geq \frac{n-1}{2}\frac{D}{n} \geq \frac{D}{4}.$$

We have (ii) satisfied with $t_n = n$, $\lambda_n^{(2)} = n$, and (iv) satisfied for $i = 2$ with $c_n^{(2)} = D/4$. As V is autonomous, it is a Liapunov function and conditions of Theorem 1 (I) are satisfied. This completes the proof.

3. Proof of Theorem 1

We prove (I) first. Since V is a Liapunov functional we have $W_1(|\phi(0)|) \leq V(t,\phi)$ and $V'_{(1)}(t,x_t) \leq 0$. The additional assumptions that $V(t,\phi) \leq W(\|\phi\|)$ yields US. For $\varepsilon_1 = H$ find δ_1 of US and take $\gamma = \delta_1$ in the definition of UAS. Let $\mu > 0$ be given and find the δ_2 of US so that $[\|\phi\| < \delta_2, t_0 \geq 0, t \geq t_0]$ imply that $|x(t,t_0,\phi)| < \mu$.

We will find $T > 0$ such that if $\phi \in \mathcal{C}_\gamma$ and $t_0 \geq 0$, then $|x(t,t_0,\phi)| < \mu$ if $t \geq t_0 + T$. Let $x(t) = x(t,t_0,\phi)$ and $V(t) = V(t, x_t(t_0,\phi))$.

Consider the intervals $S_n = [t_n - k, t_n]$, where we may suppose, by renumbering, that $t_n - k \geq t_{n-1}$. For a given n, suppose that $\|x_t\| \geq \delta_2$. Then there is an $r_n \in S_n$ with $|x_i(r_n)| \geq \delta_2$ for some i. Let $-\alpha_n = V(t_n) - V(t_n - k)$.

(a) If $|x_i(t)| \geq \delta_2/2$ for $t \in S_n$, then by (v) we have $V'(t) \leq -P_i(t)U_i(\delta_2/2)$ on S_n. Let $D = k\lambda$, so that

$$-\alpha_n = V(t_n) - V(t_n - k) \leq -U_i(\delta_2/2) \int_{t_n-k}^{t_n} P_i(s)ds \leq -c_n^{(i)} U_i(\delta_2/2).$$

(b) If (a) fails, the there are $p_n < q_n$ with $[p_n, q_n] \subset S_n$ and with $|x_i(t)|$ between $\delta_2/2$ and δ_2 on $[p_n, q_n]$; to be definite, say $|x_i(p_n)| = \delta_2/2$ and $|x_i(q_n)| = \delta_2$. To simplify arithmetic in Jensen's inequality, let $k \leq 1$. Then we integrate (vi), use Jensen's inequality, and have

$$-\alpha_n \leq V(q_n) - V(p_n) \leq -Q_i \left(\int_{p_n}^{q_n} |x_i'(s)| ds \right) + \int_{p_n}^{q_n} M_i(s)ds \leq -Q_i(\delta_2/2) + (q_n - p_n)\lambda_n^{(i)}.$$

If $M_i = 0$, then $\alpha_n \geq Q_i(\delta_2/2)$.

(bi) If $\alpha_n \geq Q_i(\delta_2/2)/2$, this will suffice for our proof.

(bii) If $\alpha_n < Q_i(\delta_2/2)/2$, then $D := Q_i(\delta_2/2)/2 \leq (q_n - p_n)\lambda_n^{(i)}$. We then integrate (v) and have

$$-\alpha_n \leq V(q_n) - V(p_n) \leq -U_i(\delta_2/2) \int_{p_n}^{q_n} P_i(s)ds \leq -U_i(\delta_2/2) \int_{p_n}^{p_n + D/\lambda_n^{(i)}} P_i(s)ds \leq -c_n^{(i)} U_i(\delta_2/2).$$

From (a), (b), (bi) and (bii) we find

$$\alpha_n \geq \min_i [c_n^{(i)} U_i(\delta_2/2), Q_i(\delta_2/2)/2] \geq \min_i [c_0 U_i(\delta_2/2), Q_i(\delta_2/2)/2] =: \alpha.$$

If $t > t_n$, then

$$0 \leq V(t) \leq V(t_0) - n\alpha \leq W(\delta_1) - n\alpha,$$

a contradiction if $n > W(\delta_1)/\alpha$. Now there is a $k > 0$ with $t_n - t_{n-1} \leq k$ so we may select $N > W(\delta_1)/\alpha$ and then $T = Nk$. This completes the proof of (I).

The other proofs are parallel. We must only change t_n for (II), while in (III) we need to change t_n and $c_n^{(i)}$.

To prove (II) we first note that it is not vacuous. The zero solution is stable so there are solutions remaining in $\mathcal{C}_H$. Suppose that $x(t)$ remains in $\mathcal{C}_H$ and $x_i(t) \not\to 0$ as $t \to \infty$.

Then there is an $\varepsilon > 0$ and a sequence $\{t_n\} \uparrow \infty$ with $t_{n+1} \geq t_n + k$ and $|x_i(t_n)| \geq \varepsilon$. Let $S_n = [t_n - k, t_n]$ and $-\alpha_n = V(t_n) - V(t_n - k)$ where $V(t) = V(t, x_t)$. Using the same proof as in (I) we have

$$\alpha_n \geq \min_i [c_0 U_i(\varepsilon/2), Q_i(\varepsilon/2)/2] =: \alpha.$$

If $t > t_n$, then $0 \leq V(t) \leq V(t_0) - n\alpha$, a contradiction for large n. This proves (II).

To prove (III), we note again that it is not vacuous, as in (II), and we consider a solution $x(t)$ remaining in $\mathcal{C}_H$ on an interval $[t_0, \infty)$. Suppose that $x(t) \not\to 0$ and note that $V'(t, x_t) \leq 0$ so that if $t \geq t_n$ then $W_1(|x(t)|) \leq V(t, x_t) \leq V(t_n, x_{t_n}) \leq W(\|x_{t_n}\|)$; thus there is an $\varepsilon > 0$ with $\|x_{t_n}\| \geq \varepsilon$ and so there is an i for each n with $|x_i(r_n)| \geq \varepsilon$, where $r_n \in [t_n - h, t_n]$. Let $S_n = [t_n - k, t_n]$. Once again the same proof gives

$$\alpha_n \geq \min_i [c_n^{(i)} U_i(\varepsilon/2), Q_i(\varepsilon/2)/2] \geq \min_i [c_n U_i(\varepsilon/2), Q_i(\varepsilon/2)/2]. \tag{$*$}$$

Since $t > t_n$ yields

$$\begin{aligned} 0 \leq V(t, x_t) &\leq V(t_1, x_{t_1}) - \sum_{i=2}^{n} \alpha_i \\ &\leq W(\|x_{t_1}\|) - \sum_{i=2}^{n} \alpha_i, \end{aligned} \tag{$**$}$$

the second choice in (*) can hold only for finitely many n. Since $\sum_{n=0}^{\infty} c_n = \infty$, a contradiction results in (**) for large n. This completes the proof.

4. Proofs of the corollaries

First, note that Corollary 1 is just a statement of Theorem 1 (III) without a separate statement for each component. Also, $\lambda_n = \lambda(t_n)$ will suffice, since $P(t) = 1$ and so

$$\int_{s_n}^{s_n + D/\lambda_n} 1 dt = \int_{s_n}^{s_n + D/\lambda(t_n)} dt = \frac{D}{\lambda(t_n)} =: c_n$$

and $\sum c_n$ diverges since $\int_1^\infty \frac{dt}{\lambda(t)}$ diverges and λ is increasing.

Corollary 2 follows from Corollary 1 when we note that (iv) of Corollary 1 is satisfied, because for $\|\phi\| < 1$ we have

$$V'(t, x_t) \leq -W_2(|x(t)|) \leq -|f(t, x_t)| + J(t+1)\ln(t+2)$$

and $M(t) = J(t+1)\ln(t+2)$ satisfies condition (iv) of Corollary 1.

Corollary 3 plays the role for Theorem 1 (I) that Corollary 1 plays for Theorem 1 (III). It merely avoids the component conditions.

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