# A FIXED POINT THEOREM OF KRASNOSELSKII-SCHAEFER TYPE

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Abstract. In this paper we focus on three fixed point theorems and an integral equation. Schaefer's fixed point theorem will yield a $T$-periodic solution of

$$x(t) = a(t) + \int_{t-h}^{t} D(t,s)g(s,x(s))\,ds \tag{1}$$

if $D$ and $g$ satisfy certain sign conditions independent of their magnitude. A combination of the contraction mapping theorem and Schauder's theorem (known as Krasnoselskii's theorem) will yield a $T$-periodic solution of

$$x(t) = f(t,x(t)) + \int_{t-h}^{t} D(t,s)g(s,x(s))\,ds \tag{2}$$

if $f$ defines a contraction and if $D$ and $g$ are small enough.

We prove a fixed point theorem which is a combination of the contraction mapping theorem and Schaefer's theorem which yields a $T$-periodic solution of (2) when $f$ defines a contraction mapping, while $D$ and $g$ satisfy the aforementioned sign conditions.

**1. Introduction.** We are interested in proving that equations of the type

$$x(t) = f(t,x(t)) - \int_{t-h}^{t} D(t,s)g(s,x(s))\,ds \tag{1}$$

possess a $T$-periodic solution when $D$ is essentially a positive kernel and $f$ is a contraction. In particular, $D$ may be large.

Equations of this form are interesting in their own right. It is an equation with memory: the present value of $x$ depends on its past history.

But (1) can also arise from a much more familiar problem such as

$$x'(t) = -a(t)x(t) - g(t,x(t)) \tag{D}$$

where $a(t+T) = a(t)$ and $g(t+T,x) = g(t,x)$ for some $T > 0$. Krasnoselskii (cf. Schauder [14] and Smart [15; p. 31]) observed that in a variety of problems the inversion of a perturbed differential operator yields a contraction and a compact map. For

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example, in (D) we write

$$\left(x \exp \int_0^t a(s)\,ds\right)' = -g(t,x) \exp \int_0^t a(s)\,ds$$

and integrate from $t-T$ to $t$ obtaining

$$(ID) \qquad x(t) = x(t-T) \exp - \int_{t-T}^t a(s)\,ds - \int_{t-T}^t g(u,x(u)) \left[\exp - \int_u^t a(s)\,ds\right] du.$$

If $\exp - \int_{t-T}^t a(s)\,ds = Q < 1$ and if $(\mathcal{B}, \|\cdot\|)$ is the Banach space of continuous $T$-periodic functions $\varphi : R \to R$, then (ID) can be expressed as

$$(ID) \qquad \varphi(t) = (B\varphi)(t) + (A\varphi)(t)$$

where $B$ is a contraction and $A$ maps bounded subsets of $\mathcal{B}$ into compact subsets of $\mathcal{B}$. In fact, $B$ can take a portion of the integral which might not be well-behaved in some sense. The above integral maps bounded sets of T-periodic functions into equicontinuous sets, as may be seen in the last section in the proof of Lemma 2.

Contraction mappings shrink sets. Krasnoselskii showed that if $B\varphi + A\psi$ shrinks a certain set, then there will be a fixed point, a solution of (ID), which is in $\mathcal{B}$ and, hence, is periodic. His result can be stated as follows (Krasnoselskii [8; p. 370] or Smart [15; p. 31]). Concerning the terminology of compact mapping used in this theorem, Krasnoselskii is using the convention of Smart [15;p. 25] to mean the following: Let $A$ map a set $M$ into a topological space $X$. If $AM$ is contained in a compact subset of $X$, we say that $A$ is compact. In particular, $M$ need not be bounded.

**Theorem K.** *Let $M$ be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Suppose that $A$ and $B$ map $M$ into $\mathcal{B}$ such that*

*(i) $x, y \in M \Rightarrow Ax + By \in M$,*

*(ii) $A$ is compact and continuous,*

*(iii) $B$ is a contraction mapping.*

*Then $\exists\ y \in M$ with $y = Ay + By$.*

As we will emphasize again soon, the assumptions of the theorem are verified *directly from the functions appearing* in the operator equation (ID) and the proof rests on Schauder's second fixed point theorem.

There is a theorem of Schaefer ([13] or Smart [15; [ p. 29]) which competes with Schauder's and which usually yields much more, but it also requires much more. Schaefer's theorem requires that we have an a priori bound on *utterly unknown solutions* of an operator equation $\varphi = \lambda A\varphi$ for $0 < \lambda < 1$, in contrast with Schauder's which requires conditions on the *clearly visible* mapping $A$. Schaefer's theorem may be stated as follows. This is Smart's formulation; Schaefer proved the result for a locally convex space.

**Theorem S.** *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, $H$ a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset $X$ of $\mathcal{B}$. Then either*

*(i) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or*

*(ii) the set of all such solutions $x$, for $0 < \lambda < 1$, is unbounded.*

The problem which we focus on here is this: Can we substitute Schaefer-type conditions on $A$ for Krasnoselskii's Schauder-type conditions? We show that we can and that there are interesting applications. In particular, for (1) Krasnoselskii would require $f$ to be a contraction and $Dg$ to be small, while we allow $Dg$ to be large, provided $D$ and $g$ satisfy certain conditions.

**2. A fixed point theorem.** We start with an equation

$$x = Bx + Ax$$

where $B$ is a contraction. Now contractions shrink functions, but we are led to the homotopy equation

$$x = \lambda B(x/\lambda) + \lambda Ax$$

and we still need $\lambda B(x/\lambda)$ to shrink functions. Our first result shows that it does; applications depend heavily on this.

**Proposition.** *If $(\mathcal{B}, \|\cdot\|)$ is a normed space, if $0 < \lambda < 1$, and if $B : \mathcal{B} \to \mathcal{B}$ is a contraction mapping with contraction constant $\alpha$, then $\lambda B \frac{1}{\lambda} : \mathcal{B} \to \mathcal{B}$ is also a contraction mapping with contraction constant $\alpha$, independent of $\lambda$; in particular*

$$\|\lambda B(x/\lambda)\| \leq \alpha\|x\| + \|B0\|.$$

Proof.

To see that $\lambda B \frac{1}{\lambda}$ is a contraction, $x \in \mathcal{B} \Rightarrow x/\lambda \in \mathcal{B} \Rightarrow B(x/\lambda) \in \mathcal{B} \Rightarrow \lambda B(x/\lambda) \in \mathcal{B}$; moreover, $x,\ y \in \mathcal{B} \Rightarrow$

$$\|\lambda B(x/\lambda) - \lambda B(y/\lambda)\| = \lambda\|B(x/\lambda) - B(y/\lambda)\|$$

$$\leq \lambda\alpha\|(x/\lambda) - (y/\lambda)\| = \alpha\|x - y\|.$$

To obtain the bound, for any $x \in \mathcal{B}$ we have

$$\|\lambda B(x/\lambda)\| = \lambda\|B(x/\lambda)\| =$$

$$\lambda(\|B(x/\lambda) - B0 + B0\|)$$

$$\leq \lambda(\|B(x/\lambda) - B0\| + \|B0\|)$$

$$\leq \lambda(\alpha\|(x/\lambda) - 0\| + \|B0\|)$$

$$= (\lambda\alpha/\lambda)\|x\| + \|B0\|,$$

as required.

**Theorem 1.** *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $A,\ B : \mathcal{B} \to \mathcal{B}$, $B$ a contraction with contraction constant $\alpha < 1$, and $A$ continuous with $A$ mapping bounded sets into compact sets. Either*

*(i) $x = \lambda B(x/\lambda) + \lambda Ax$ has a solution in $\mathcal{B}$ for $\lambda = 1$, or*

*(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.*

Proof.

By the proposition, $\lambda B \frac{1}{\lambda}$ is a contraction mapping from $\mathcal{B}$ into $\mathcal{B}$. Consequently, for each $y \in \mathcal{B}$, the mapping $x \to \lambda B(x/\lambda) + \lambda Ay$ is also a contraction with unique solution $x = \lambda B(x/\lambda) + \lambda Ay$ in $\mathcal{B}$. This yields

$$\frac{x}{\lambda} = B\frac{x}{\lambda} + Ay$$

or

$$(I-B)\frac{x}{\lambda} = Ay$$

so

$$\frac{x}{\lambda} = (I-B)^{-1}Ay$$

and

$$(*) \qquad x = \lambda(I-B)^{-1}Ay.$$

Now $(I-B)^{-1}$ exists and is continuous (cf. Smart [15; p. 32]). Since $A$ is continuous and maps bounded sets into compact sets, so does $(I-B)^{-1}A$ (The proof given by Kreyszig [9; p. 412, 656] is valid for general metric spaces.).

By Schaefer's theorem, either (*) has a solution with $x = y$ for $\lambda = 1$ (hence, (i) has a solution for $\lambda = 1$), or the set of all such solutions, $0 < \lambda < 1$, is unbounded. This completes the proof.

**3. An example.** Equation (1) is related to a large class of important problems going back at least to Volterra [16] who suggested that the growth of solutions of

$$x' = -\int_0^t k(t-s)g(x(s))\,ds$$

could be controlled if $xg(x) > 0$ for $x \neq 0$, $k(t) > 0$, $k'(t) < 0$, $k''(t) > 0$. Appropriate details were provided by Levin [10] through construction of a very clever Liapunov function. Subsequently, both that equation and its integral equation counterpart were widely studied in the literature ([6], [7], [11], [12]) both by means of Liapunov functions and by transform theory.

We have studied integrodifferential equations in ([3], [4]) and variants of (1) in ([1], [2], [5]) when $f(t,x)$ is independent of $x$ using Schaefer's theorem or an analog; but Schaefer's theorem does not apply here since $f(t,x)$ can not define a compact mapping. Thus, we are interested in Krasnoselskii's theorem. But we want the kernel to be free to grow large; this means that we will not be able to satisfy Krasnoselskii's conditions. That was the motivation for our theorem.

Let $(\mathcal{B}, \|\cdot\|)$ be the Banach space of continuous $T$-periodic functions $\varphi : R \to R$ with the supremum norm. Consider (1) and suppose there is a $T > 0$ and $\alpha \in (0,1)$ with:

$$f(t+T,x) = f(t,x), \quad D(t+T,s+T) = D(t,s), \quad g(t+T,x) = g(t,x), \tag{2}$$

$$t-h \le s \le t \text{ implies that } D_s(t,t-h) \ge 0, \tag{3}$$

$$D_{st}(t,s) \le 0, \quad D(t,t-h) = 0,$$

$$|f(t,x) - f(t,y)| \le \alpha|x-y|, \quad xg(t,x) \ge 0, \tag{4}$$

$$\forall\, k > 0\, \exists\, P > 0\, \exists\, \beta > 0 \text{ with} \tag{5}$$

$$2\lambda[-(1-\alpha)xg(t,x) + k|g(t,x)|\,] \le \lambda[P - \beta|g(t,x)|\,],$$

$$f, g, \text{ and } D_{st} \text{ are continuous.} \tag{6}$$

**Theorem 2.** *If (2)–(6) hold, then (1) has a $T$-periodic solution.*

Proof.

Define a mapping $B : \mathcal{B} \to \mathcal{B}$ by $\varphi \in \mathcal{B}$ implies

$$(B\varphi)(t) = f(t, \varphi(t)). \tag{7}$$

**Lemma 1.** *If $B$ is defined by (7), then $B$ is a contraction mapping from $\mathcal{B}$ into $\mathcal{B}$ with contraction constant $\alpha$ of (4).*

Proof.

If $\varphi$, $\psi \in \mathcal{B}$, then

$$\begin{aligned} \|B\varphi - B\psi\| &= \sup_{t\in[0,T]} |(B\varphi)(t) - (B\psi)(t)| \\ &= \sup_{t\in[0,T]} |f(t,\varphi(t)) - f(t,\psi(t))| \\ &\le \sup_{t\in[0,T]} \alpha|\varphi(t) - \psi(t)| = \alpha\|\varphi - \psi\|, \end{aligned}$$

as required.

Define a mapping $A : \mathcal{B} \to \mathcal{B}$ by $\varphi \in \mathcal{B}$ implies

$$(A\varphi)(t) = -\int_{t-h}^{t} D(t,s)g(s,\varphi(s))\,ds. \tag{8}$$

**Lemma 2.** *If $A$ is defined by (8) then $A : \mathcal{B} \to \mathcal{B}$, $A$ is continuous, $A$ maps bounded sets into compact sets.*

Proof.

This is a standard exercise. For completeness, here are the details. Let $k > 0$ be given, $\varphi \in \mathcal{B}$ be an arbitrary element with $\|\varphi\| \le k$. By the continuity of $D$, if $0 \le t \le T$ and $-h \le s \le T$, there is an $M > 0$ with

$$(I) \qquad |D(t,s)| \le M.$$

By the uniform continuity of $D$ on $[0,T] \times [-h,T]$, for each $\varepsilon > 0$ there is a $\delta_1 > 0$ such that $u, v \in [0,T]$, $s$, $t \in [-h,T]$, $|u - v| + |s - t| \le \delta_1$ implies

$$(II) \qquad |D(u,s) - D(v,t)| \le \varepsilon.$$

Next, since $g$ is continuous on $[0,T] \times [-k,k]$ and periodic in $t$ there is an $L > 0$ with

$$(III) \qquad |g(s,x)| \le L \text{ for } s \in R \text{ and } x \in [-k,k].$$

In fact, by the uniform continuity of $g$ on that set, for any $\varepsilon > 0$ there is a positive $\delta_2(\varepsilon) < T$ such that if $s$, $t \in [0,T]$ and $x$, $y \in [-k,k]$ with $|s-t| + |x-y| \le \delta_2$, then $|g(s,x) - g(t,y)| \le \varepsilon$ and, by the periodicity,

$$(IV) \qquad |g(s,x) - g(t,y)| \le \varepsilon \text{ for } s,t \in R \text{ and } x,y \in [-k,k]$$

with $|s-t| + |x-y| < \delta_2$.

The assertions about $A$ will now follow. If $\varphi \in \mathcal{B}$, a change of variable shows that $A\varphi$ is $T$-periodic. Clearly, if $\varphi \in \mathcal{B}$, then $A\varphi$ is continuous. Thus, $A\varphi \in \mathcal{B}$.

We now show that $A$ maps bounded sets into compact sets. First,

$$(V) \qquad \{A\varphi : \varphi \in \mathcal{B} \text{ and } \|\varphi\| \le k\} \text{ is equicontinuous.}$$

To see this, note that if $u < v$, then

$$(A\varphi)(u) - (A\varphi)(v) = -\int_{u-h}^{v-h} D(u,s)g(s,\varphi(s))\,ds$$
$$-\int_{v-h}^{u} [D(u,s) - D(v,s)]g(s,\varphi(s))\,ds$$
$$+\int_{u}^{v} D(v,s)g(s,\varphi(s))\,ds.$$

By (I)-(III), for all $u,\, v \in [0,T]$ with $|u-v| < \delta_1$, if $\varphi \in \mathcal{B}$ and $\|\varphi\| \le k$, then

$$\begin{aligned} |(A\varphi)(u) - (A\varphi)(v)| &\le ML|u-v| + L|u-v+h|\varepsilon \\ &\quad + ML|u-v| \qquad (VI) \\ &\le 2ML\delta_1 + L(\delta_1 + h)\varepsilon. \end{aligned}$$

Next, for $\varphi \in \mathcal{B}$ and $\|\varphi\| \le k$, it follows from (I) and (III) that $|(A\varphi)(u)| \le LMh$ so that

$$(VII) \qquad \|A\varphi\| \le LMh \text{ for } \varphi \in \mathcal{B} \text{ and } \|\varphi\| \le k.$$

By Ascoli's theorem $A$ maps bounded sets into compact sets.

To see that $A$ is continuous, fix $\varphi$ and $\psi \in \mathcal{B}$ with $\|\varphi - \psi\| < \delta_2$, $\|\varphi\| \le k$, $\|\psi\| \le k$. Then for $0 \le u \le T$ we have by (I) and (IV) that

$$\begin{aligned} |(A\varphi)(u) - (A\psi)(u)| &\le \int_{u-h}^{u} |D(u,s)|\, |g(s,\varphi(s)) - g(s,\psi(s))|\, ds \\ &\le Mh\varepsilon. \end{aligned}$$

This completes the proof of Lemma 2.

Next, notice that if $B : \mathcal{B} \to \mathcal{B}$ is defined by

$$(Bx)(t) = f(t, x(t)), \text{ then } \left(\lambda B \frac{x}{\lambda}\right)(t) = \lambda f\left(t, \frac{x(t)}{\lambda}\right).$$

**Lemma 3.** *There is a $K \ge 0$ such that if $0 < \lambda < 1$ and if $x \in \mathcal{B}$ solves*

$$(1_\lambda) \qquad x(t) = \lambda f\left(t, \frac{x}{\lambda}\right) - \lambda \int_{t-h}^{t} D(t,s) g(s, x(s))\, ds$$

*then $\|x\| \le K$.*

Proof.

Let $x \in \mathcal{B}$ solve $(1_\lambda)$ and define

$$V(t) = \lambda^2 \int_{t-h}^{t} D_s(t,s) \left( \int_s^t g(v, x(v))\, dv \right)^2 ds.$$

This is a type of Liapunov function obtained from $(1_\lambda)$ by squaring $x - \lambda f$, integrating by parts, and using the Schwarz inequality.

Now $D_{st}(t,s) \le 0$ so

$$V'(t) \le -\lambda^2 D_s(t,t-h)\Big(\int_{t-h}^t g(v,x(v))\,dv\Big)^2$$
$$+2\lambda^2 g(t,x)\int_{t-h}^t D_s(t,s)\int_s^t g(v,x(v))\,dv\,ds.$$

The first term on the right-hand-side is not positive by (3); if we integrate the last term by parts and use (3) again we have

$$V'(t) \le 2\lambda g(t,x)\int_{t-h}^t \lambda D(t,s)g(s,x(s))\,ds$$
$$= 2\lambda g(t,x)\Big[\lambda f(t,\frac{x}{\lambda}) - x(t)\Big]$$

from $(1_\lambda)$. But by the reasoning in the proposition,

$$|\lambda f(t,\frac{x}{\lambda})| \le \alpha|x(t)| + |f(t,0)| \le \alpha|x(t)| + k$$

for some $k > 0$. Thus,

$$V'(t) \le 2\lambda\{|g(t,x)|[\alpha|x| + k] - xg(t,x)\}$$
$$= 2\lambda[|\alpha x g(t,x)| + k|g(t,x)| - xg(t,x)]$$
$$\le 2\lambda[-(1-\alpha)xg(t,x) + k|g(t,x)|\,].$$

As $\alpha < 1$, from (5) we have

$$V'(t) \le \lambda[-\beta|g(t,x)| + P].$$

Thus, $x \in \mathcal{B}$ implies $V \in \mathcal{B}$ so that

$$0 = V(T) - V(0) \le \lambda\Big[-\beta\int_0^T |g(t,x(t))|\,dt + PT\Big]$$

or

(9) $$\int_0^T |g(t,x(t))|\,dt \le PT/\beta$$

since $\lambda > 0$. As $g(t,x(t)) \in \mathcal{B}$, there is an $n > 0$ with

(10) $$\int_{t-h}^t |g(t,x(t))|\,dt \le n.$$

Taking $M = \max_{-h \le s \le t \le T} |D(t,s)|$, then from the proposition, $(1_\lambda)$, and (10) we have

$$\begin{aligned} |x(t)| &\le \left|\lambda f\left(t, \frac{x}{\lambda}\right)\right| + \lambda \left| \int_{t-h}^{t} D(t,s) g(s, x(s))\, ds \right| \\ &\le \alpha |x(t)| + k + Mn \end{aligned}$$

or

$$\|x\| \le (Mn + k)/(1-\alpha),$$

as required. Application of Theorem 1 completes the proof.

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