Asymptotic Behavior of Nonlinear Functional Differential Equations by Schauder's Theorem

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Abstract. In this paper we consider several equations of the general type

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - r(t))$$

and use Schauder's fixed point theorem to obtain stability, uniform stability, asymptotic stability, and uniform asymptotic stability. Distributed delays are among the types considered.

1. Introduction

In a series of papers [2-6] we have studied stability problems by means of a variety of fixed point theorems. Much of this study has been motivated by difficulties encountered in Liapunov's direct method. In many cases those difficulties have been circumvented by use of fixed point methods.

The first problem encountered in Liapunov theory is the construction of a Liapunov function. The parallel problem in fixed point theory is the construction of a suitable set and a fixed point mapping of that set into itself. It is also necessary to decide on the proper topology. If the differential equation is of the form

$$x'(t) = -a(t)x(t) + f(t, x_t)$$

where x dominates f in some sense near zero, then the variation of parameters formula yields

$$x(t) = x(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} f(s, x_s)ds$$

and a suitable fixed point mapping can be as simple as

$$(P\phi)(t) = x_0 e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} f(s, \phi_s)ds,$$

a suitable fixed point of which solves the equation. We discussed this method in a large number of examples in [4].

When the leading term is sublinear or superlinear, then methods must be developed for arriving at suitable mappings. Here, we deal with several forms related to

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - r(t))$$

and develop a fixed point mapping which may be useful in a variety of superlinear problems. Furthermore, the literature suggests that there may be much to learn about such problems.

As a starting point, Hale [7; p. 117] shows that for $a(t), b(t)$ bounded continuous functions with $a(t) \geq \delta > 0, 0 < q < 1, |b(t)| < q\delta$, and r a positive constant, then along solutions of

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - r)$$

the Liapunov functional

$$V(t, x_t) = \frac{1}{4}x^4(t) + \frac{\delta}{2}\int_{t-r}^t x^6(s)ds$$

satisfies

$$\begin{aligned} V'(t, x_t) &= -a(t)x^6(t) + b(t)x^3(t)x^3(t - r) + \frac{\delta}{2}x^6(t - r) \\ &\leq -\eta(x^6(t) + x^6(t - r)) \end{aligned}$$

for some $\eta > 0$. This implies uniform asymptotic stability by a standard theorem [7; p. 105]. It is the boundedness of $a(t), b(t)$ which is crucial in his result.

By contrast we showed in [2] using fixed point theory that the condition $b(t)/a(t) \to 0$ as $t \to \infty$ can play a crucial role in compactness arguments yielding asymptotic stability.

In this paper we more carefully construct the mapping set so that we achieve asymptotic stability with fewer conditions on $a(t), b(t)$ than in either of the above investigations. The work depends on the following lemma.

Let r_0 be a fixed nonnegative constant and let $h : [-r_0, \infty) \to [1, \infty)$ be any strictly increasing and continuous function with $h(-r_0) = 1, h(t) \to \infty$ as $t \to \infty$. For any $t_0 \in R^+ := [0, \infty)$, let C_{t_0} be the space of continuous functions $\phi : [t_0 - r_0, \infty) \to R := (-\infty, \infty)$ with

$$\|\phi\|_h := \sup\left\{ \frac{|\phi(t)|}{h(t-t_0)} : t \geq t_0 - r_0 \right\} < \infty.$$

Then, clearly $\| \cdot \|_h$ is a norm on C_{t_0}, and $(C_{t_0}, \| \cdot \|_h)$ is a Banach space.

First we state a lemma without proof (See Burton [1; p. 169]).

Lemma. If the set $\{\phi_k(t)\}$ of R-valued functions on $[t_0 - r_0, \infty)$ is uniformly bounded and equi-continuous, then there is a bounded and continuous function ϕ and a subsequence $\{\phi_{k_j}(t)\}$ such that $\|\phi_{k_j} - \phi\|_h \to 0$ as $j \to \infty$.

2. Asymptotic behavior of solutions

Consider the scalar nonlinear equation

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - r(t)), \qquad t \in R^+ \tag{2.1}$$

where $a, r : R^+ \to R^+$ and $b : R^+ \to R$ are continuous. Let α be any fixed number with $0 < \alpha \leq 1/\sqrt{3}$. We assume that there are constants $r_0 \geq 0$ and $\gamma > 0$ so that

$$t - r(t) \geq -r_0, \tag{2.2}$$

$$\sigma = \sigma(t_0) := \sup_{t \geq t_0} \int_{t_0}^{t} (\gamma^3 |b(s)| - a(s)) ds < \infty \text{ for any } t_0 \in R^+, \tag{2.3}$$

and

$$\sup_{t \geq t_0 \geq 0} \left(\frac{\frac{1}{2\delta^2} + \int_{t_0}^{t} (a(s) - \gamma^3 |b(s)|) ds}{\frac{1}{2\delta^2} + \int_{t_0}^{\tau(t)} (a(s) - \gamma^3 |b(s)|) ds} \right)^{1/2} \leq \gamma \text{ for any } \delta \in (0, \eta], \tag{2.4}$$

where $\tau = \tau(t) := \max(t_0, t - r(t))$, and η is a number defined by

$$\eta = \eta(t_0) := \left(\frac{1}{\alpha^2} + 2\sigma(t_0) \right)^{-1/2}. \tag{2.5}$$

Corresponding to Equation (2.1), consider the scalar nonlinear equation

$$q' = (\gamma^3 |b(t)| - a(t)) q^3, \qquad t \in R^+. \tag{2.6}$$

Let $q:[t_0-r_0,\infty)\to R$ be a continuous function such that $q(t)=\eta$ on $[t_0-r_0,t_0]$, and $q(t)$ is the unique solution of the initial value problem

$$q' = (\gamma^3|b(t)| - a(t))q^3, \qquad q(t_0)=\eta, \qquad t\ge t_0.$$

Then $q(t)$ can be expressed as

$$\begin{aligned} q(t) &= \eta e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)du} a(s)(q(s)-q^3(s))ds + \gamma^3 \int_{t_0}^t e^{-\int_s^t a(u)du}|b(s)|q^3(s)ds \\ &= \left(\frac{1}{\eta^2} + 2\int_{t_0}^t (a(s)-\gamma^3|b(s)|)ds\right)^{-1/2}, \qquad t\ge t_0, \end{aligned} \tag{2.7}$$

which together with (2.3) and (2.5), implies

$$0 < q(t) \le \alpha, \qquad t \ge t_0. \tag{2.8}$$

Concerning the stabilities of the zero solution of Equation (2.1), we have the following theorem.

Theorem 2.1. Suppose that (2.2)-(2.4) hold. Then we have:

(i) The zero solution of Equation (2.1) is stable.

(ii) If we have $\sigma^* := \sup\{\sigma(t) : t\in R^+\} < \infty$, then the zero solution of Equation (2.1) is uniformly stable.

(iii) If we have

$$\int_{t_0}^t (a(s)-\gamma^3|b(s)|)ds \to \infty \qquad \text{as } t\to\infty, \tag{2.9}$$

then the zero solution of Equation (2.1) is asymptotically stable.

(iv) In addition to $\sigma^*<\infty$, if we have

$$\int_{t_0}^t (a(s)-\gamma^3|b(s)|)ds \to \infty \qquad \text{uniformly for } t_0\in R^+ \text{as } t\to\infty, \tag{2.10}$$

then the zero solution of Equation (2.1) is uniformly asymptotically stable.

Proof. (i) It is easy to see that the zero solution of Equation (2.6) is stable. Thus, for any $\epsilon\in(0,\alpha]$ and $t_0\in R^+$, there is a $\delta=\delta(\epsilon,t_0)$ such that $0<\delta\le\eta$, and that for any $t_0 \in R^+$ and q_0 with $|q_0| \le \delta$, we have $|q(t, t_0, q_0)| < \epsilon$ for all $t \ge t_0$. For the t_0, let $(C_{t_0}, \|\cdot\|_h)$ be the Banach space of continuous functions $\phi : [t_0 - r_0, \infty) \to R$ with the norm $\|\cdot\|_h$. For a continuous function $\psi : [-r_0, 0] \to R$ with $\sup_{-r_0 \le \theta \le 0} |\psi(\theta)| \le \delta$, let S be the set of continuous functions $\phi : [t_0 - r_0, \infty) \to R$ such that $\phi(t) = \psi(t - t_0)$ for $t_0 - r_0 \le t \le t_0$, $|\phi(t)| \le q(t)$ for $t \ge t_0$, and $|\phi(t_1) - \phi(t_2)| \le L|t_1 - t_2|$ for $t_1, t_2 \in R^+$ with $t_0 \le \tau_1 \le t_1, t_2 \le \tau_2$, where $q(t)$ is defined by (2.7) with $\eta = \delta$, and where $L : R^+ \times R^+ \to R^+$ is a function defined by

$$L(\tau_1, \tau_2) := \max\{2a(t)\alpha + (a(t) + \gamma^3|b(t)|)\alpha^3 : \tau_1 \le t \le \tau_2\}.$$

Since we have (2.8), we obtain

$$|q'(t)| \le (a(t) + \gamma^3|b(t)|)\alpha^3, \qquad t \ge t_0.$$

Thus the function $\xi(t)$ defined by

$$\xi(t) := \begin{cases} \psi(t - t_0), & t_0 - r_0 \le t \le t_0 \\ \frac{\psi(0)q(t)}{\delta}, & t \ge t_0 \end{cases}$$

is an element of S, and from Lemma, S is a compact convex nonempty subset of C_{t_0}. Define a mapping P for $\phi \in S$ by

$$(P\phi)(t) := \begin{cases} \psi(t - t_0), & t_0 - r_0 \le t \le t_0 \\ \psi(0)e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)du} a(s)(\phi(s) - \phi^3(s))ds \\ + \int_{t_0}^t e^{-\int_s^t a(u)du} b(s)\phi^3(s - r(s))ds, & t \ge t_0. \end{cases}$$

Then we have $(P\phi)(t) = \psi(t - t_0)$ for $t_0 - r_0 \le t \le t_0$, and from (2.4) and (2.7) with $\eta = \delta$ we obtain

$$\begin{aligned} |(P\phi)(t)| &\le \delta e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)du} a(s)(q(s) - q^3(s))ds \\ &\quad + \int_{t_0}^t e^{-\int_s^t a(u)du} |b(s)| q^3(s - r(s))ds \\ &\le \delta e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)du} a(s)(q(s) - q^3(s))ds \\ &\quad + \gamma^3 \int_{t_0}^t e^{-\int_s^t a(u)du} |b(s)| q^3(s)ds \\ &= q(t), \qquad t \ge t_0. \end{aligned}$$

Moreover, it is easy to see that

$$(P\phi)'(t) = -a(t)(P\phi)(t) + a(t)(\phi(t) - \phi^3(t)) + b(t)\phi^3(t - r(t)), \qquad t > t_0,$$

which implies

$$\begin{aligned}|(P\phi)'(t)| &\leq a(t)q(t) + a(t)(q(t) - q^3(t)) + |b(t)|q^3(t - r(t)) \\ &\leq 2a(t)\alpha + (a(t) + \gamma^3|b(t)|)\alpha^3, \qquad t > t_0,\end{aligned}$$

and hence, P maps S into S. Clearly P is continuous. Thus, by Schauder's first theorem, P has a fixed point ϕ in S and that is the solution $x(t, t_0, \phi)$ of Equation (2.1) which satisfies

$$|x(t, t_0, \psi)| \leq q(t) = q(t, t_0, \delta) < \epsilon, \qquad t \geq t_0,$$

and hence, the zero solution of Equation (2.1) is stable.

(ii) If $\sigma^* < \infty$, then the zero solution of Equation (2.6) is uniformly stable. Since the uniform stability of the zero solution of Equation (2.1) can be similarly proved as in the proof of (i), we omit the details.

(iii) Assumption (2.9) implies that $q(t) \to 0$ as $t \to \infty$, and hence, the zero solution of Equation (2.6) is asymptotically stable. Since the asymptotic stability of the zero solution of Equation (2.1) can be similarly proved as in the proof of (i), we omit the details.

(iv) Assumptions $\sigma^* < \infty$ and (2.10) imply that the zero solution of Equation (2.6) is uniformly asymptotically stable. Since the uniform asymptotic stability of the zero solution of Equation (2.1) can be similarly proved as in the proof of (i), we omit the details.

Now we show two examples.

Example 2.1. Define functions $a, b, r : R^+ \to R^+$ by $a(t) := 2 + |t \sin t|, b(t) := \max(1, 1 + 2t \sin t)/27$ and $r(t) := 1/(t + 1)$, and let $\alpha = 1/\sqrt{3}$. Then it is easily seen that (2.2) with $r_0 = 1$, (2.3) and (2.4) with $\gamma = 3$ hold, and $\sigma^* = \infty$. Thus, concerning the stabilities of the zero solution of the equation

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - r(t)), \qquad t \in R^+,$$

Theorem 2.1 does not assure unifrom stability, but assures stability.

Example 2.2. Define a function $c : R^+ \to R^+$ by

$$c(t) := \begin{cases} n - n^2|t - 2n\pi|, & |t - 2n\pi| \le \frac{1}{n} \\ 0, & \text{otherwise,} \end{cases}$$

where n denotes positive integers, and define a function $b : R^+ \to R^+$ by $b(t) := \max(c(t), 1 + \cos t)$. Let $a(t) = 9b(t)$, $r(t) = 1$ and $\alpha = 1/\sqrt{3}$. Then it is easily seen that (2.2) with $r_0 = 1$, (2.3) and (2.4) with $\gamma = 2$ hold, and $\eta = 1/\sqrt{3}$. Moreover, $\sigma^* < \infty$ and (2.10) hold. Thus, by Theorem 2.1, the zero solution of the equation

$$x'(t) = -9b(t)x^3(t) + b(t)x^3(t-1), \qquad t \in R^+ \tag{2.11}$$

is uniformly asymptotically stable.

In Burton [2], under the assumption

$$a(t) \ge |b(t+r)| + k \qquad \text{for some } k > 0, \tag{2.12}$$

the uniform asymptotic stability of the zero solution of Equation (2.1) with $r(t) \equiv r$ is discussed by using the Liapunov functional

$$V(t,\phi) = |\phi(0)| + \int_{t-r}^{t} |b(s+r)|\phi^3(s-t)ds.$$

But we cannot apply this method since (2.12) does not hold for Equation (2.11). Moreover, in Burton [2], under the assumption

$$-a(t) + \frac{1}{4}b^2(t) + 1 \le -k \qquad \text{for some } k > 0, \qquad t \in R^+, \tag{2.13}$$

the uniform asymptotic stability of the zero solution of Equation (2.1) with $r(t) \equiv r$ is discussed by using the Liapunov functional

$$V(t,\phi) = \frac{1}{4}\phi^4(0) + \int_{t-r}^{t} \phi^6(s-t)ds.$$

But we cannot apply this method since (2.13) does not hold for Equation (2.11).

Next we show another example.

Example 2.3. Let $a(t) = 9/(t+2)$, $b(t) = 1/(t+2)$, $r(t) = 1$ and $\alpha = 1/\sqrt{3}$. Then it is easily seen that (2.2) with $r_0 = 1$, (2.3) and (2.4) with $\gamma = 2$ hold, and $\eta = 1/\sqrt{3}$. Moreover, (2.10) does not hold, but $\sigma^* < \infty$ and (2.9) hold. Thus, concerning the stabilities of the zero solution of the equation

$$x'(t) = -\frac{9}{t+2}x^3(t) + \frac{1}{t+2}x^3(t-1), \qquad t \in R^+,$$

Theorem 2.1 does not assure uniform asymptotic stability, but assures uniform stability and asymptotic stability.

Next consider the scalar nonlinear integrodifferential equation

$$x'(t) = -a(t)x^3(t) + \int_{t-r(t))}^{t} b(t,s)x^3(s)ds, \qquad t \in R^+, \tag{2.14}$$

where $a, r : R^+ \to R^+$ and $b : R^+ \times R \to R$ are continuous. Let α be any fixed number with $0 < \alpha \le 1/\sqrt{3}$. We assume that (2.2) holds, and that there is a constant $\gamma > 0$ so that

$$\sigma = \sigma(t_0) := \sup_{t \ge t_0} \int_{t_0}^{t} (\gamma^3 \int_{s-r(s)}^{s} |b(s,u)|du - a(s))ds < \infty \text{ for any } t_0 \in R^+ \tag{2.15}$$

and

$$\sup_{t \ge t_0 \ge 0} \left\{ \sup_{\tau \le v \le t} \left(\frac{\frac{1}{2\delta^2} + \int_{t_0}^{t} (a(s) - \gamma^3 \int_{s-r(s)}^{s} |b(s,u)|du)ds}{\frac{1}{2\delta^2} + \int_{t_0}^{v} (a(s) - \gamma^3 \int_{s-r(s)}^{s} |b(s,u)|du)ds} \right)^{1/2} \right\} \le \gamma \text{ for any } \delta \in (0, \eta], \tag{2.16}$$

where $\tau = \tau(t) := \max(t_0, t - r(t))$, and η is defined by (2.5).

Corresponding to Equation (2.14), consider the scalar nonlinear equation

$$q' = (\gamma^3 \int_{t-r(t)}^{t} |b(t,s)|ds - a(t))q^3, \qquad t \in R^+.$$

Let $q : [t_0 - r_0, \infty) \to R^+$ be a continuous function such that $q(t) = \eta$ on $[t_0 - r_0, t_0]$, and that $q(t)$ is the unique solution of the initial value problem

$$q' = (\gamma^3 \int_{t-r(t)}^{t} |b(t,s)|ds - a(t))q^3, \qquad q(t_0) = \eta, \qquad t \ge t_0.$$

Then $q(t)$ can be expressed as

$$q(t) = \left(\frac{1}{\eta^2} + 2\int_{t_0}^{t}(a(s) - \gamma^3\int_{s-r(s)}^{s}|b(s,u)|du)ds\right)^{-1/2}, \qquad t \geq t_0,$$

which together with (2.5) and (2.15), implies (2.8).

Concerning the stabilities of the zero solution of Equation (2.14), we have the following theorem.

Theorem 2.2. Suppose that (2.2), (2.15) and (2.16) hold. Then we have:

(i) The zero solution of Equation (2.14) is stable.

(ii) If we have $\sigma^* := \sup\{\sigma(t) : t \in R^+\} < \infty$, then the zero solution of Equation (2.14) is uniformly stable.

(iii) If we have

$$\int_{t_0}^{t}(a(s) - \gamma^3\int_{s-r(s)}^{s}|b(s,u)|du)ds \to \infty \qquad \text{as } t \to \infty, \tag{2.17}$$

then the zero solution of Equation (2.14) is asymptotically stable.

(iv) In addition to $\sigma^* < \infty$, if we have

$$\int_{t_0}^{t}(a(s) - \gamma^3\int_{s-r(s)}^{s}|b(s,u)|du)ds \to \infty \qquad \text{uniformly for } t_0 \in R^+ \text{ as } t \to \infty, \tag{2.18}$$

then the zero solution of Equation (2.14) is uniformly asymptotically stable.

This theorem can be easily proved by taking the set S in the proof of Theorem 2.1 for the above function $q(t)$ and a function $L = L(\tau_1, \tau_2)$ with

$$2a(t)\alpha + (a(t) + \gamma^3\int_{t-r(t)}^{t}|b(t,s)|ds)\alpha^3 \leq L \text{ for } \tau_1 \leq t \leq \tau_2,$$

and by defining a mapping P for $\phi \in S$ by

$$(P\phi)(t) := \begin{cases} \psi(t-t_0), & t_0 - r_0 \leq t \leq t_0 \\ \psi(0)e^{-\int_{t_0}^{t}a(u)du} + \int_{t_0}^{t}e^{-\int_s^t a(u)du}a(s)(\phi(s) - \phi^3(s))ds & \\ +\int_{t_0}^{t}e^{-\int_s^t a(u)du}\int_{s-r(s)}^{s}b(s,u)\phi^3(u)duds, & t > t_0. \end{cases}$$

So we omit the details of the proof.

Now we show two examples.

Example 2.4. Let $a(t) = 9(1+\sin t), b(t,s) = 1+\sin t, r(t) = 1$ and $\alpha = 1/\sqrt{3}$. Then it is easliy seen that (2.2) with $r_0 = 1$, (2.15) and (2.16) with $\gamma = 2$ hold, and $\eta = 1/\sqrt{3}$. Moreover, $\sigma^* < \infty$ and (2.18) hold. Thus, by Theorem 2.2, the zero solution of the equation

$$x'(t) = -9(1+\sin t)x^3(t) + (1+\sin t)\int_{t-1}^{t} x^3(s)ds, \qquad t \in R^+$$

is uniformly asymptotically stable.

Example 2.5. Let $a(t) = 9/(t+2), b(t,s) = 1/(t+2), r(t) = 1$ and $\alpha = 1/\sqrt{3}$. Then it is easily seen that (2.2) with $r_0 = 1$, (2.15) and (2.16) with $\gamma = 2$ hold, and $\eta = 1/\sqrt{3}$. Moreover, (2.18) does not hold, but $\sigma^* < \infty$ and (2.17) hold. Thus, concerning the stabilities of the zero solution of the equation

$$x'(t) = -\frac{9}{t+2}x^3(t) + \frac{1}{t+2}\int_{t-1}^{t} x^3(s)ds, \qquad t \in R^+,$$

Theorem 2.2 does not assure uniform asymptotic stability, but assures uniform stability and asymptotic stability.

Next consider the scalar nonlinear equation

$$x'(t) = -a(t)x^3(t) + b(t)x(t-r(t))x^2(t), \qquad t \in R^+, \tag{2.19}$$

where $a, r : R^+ \to R^+$ and $b : R^+ \to R$ are continuous. Let α be any fixed number with $0 < \alpha \le 1/\sqrt{3}$. We assume that (2.2) holds, and that there is a constant $\gamma > 0$ so that

$$\sigma = \sigma(t_0) := \sup_{t \ge t_0} \int_{t_0}^{t} (\gamma|b(s)| - a(s))ds < \infty \text{ for any } t_0 \in R^+ \tag{2.20}$$

and

$$\sup_{t \ge t_0 \ge 0} \left(\frac{\frac{1}{2\delta^2} + \int_{t_0}^{t}(a(s) - \gamma|b(s)|)ds}{\frac{1}{2\delta^2} + \int_{t_0}^{\tau}(a(s) - \gamma|(b(s)|)ds} \right)^{1/2} \le \gamma \text{ for any } \delta \in (0, \eta], \tag{2.21}$$

where $\tau = \tau(t) := \max(t_0, t - r(t))$, and η is define by (2.5).

Corresponding to Equation (2.19), consider the scalar nonlinear equation

$$q' = (\gamma|b(t)| - a(t))q^3, \qquad t \in R^+.$$

Let $q : [t_0 - r_0, \infty) \to R$ be a continuous function such that $q(t) = \eta$ on $[t_0 - r_0, t_0]$, and that $q(t)$ is the unique solution of the initial value problem

$$q' = (\gamma|b(t)| - a(t))q^3, \qquad q(t_0) = \eta, \qquad t \geq t_0.$$

Then $q(t)$ can be expressed as

$$q(t) = \left(\frac{1}{\eta^2} + 2\int_{t_0}^{t}(a(s) - \gamma|b(s)|)ds\right)^{-1/2}, \qquad t \geq t_0,$$

which together with (2.5) and (2.20), implies (2.8).

Concerning the stabilities of the zero solution of Equation (2.19), we have the following theorem.

Theorem 2.3. Suppose that (2.2), (2.20) and (2.21) hold. Then we have:

(i) The zero solution of Equation (2.19) is stable.

(ii) If we have $\sigma^* := \sup\{\sigma(t) : t \in R^+\} < \infty$, then the zero solution of Equation (2.19) is uniformly stable.

(iii) If we have

$$\int_{t_0}^{t}(a(s) - \gamma|b(s)|)ds \to \infty \qquad \text{as } t \to \infty, \tag{2.22}$$

then the zero solution of Equation (2.19) is asymptotically stable.

(iv) In addition to $\sigma^* < \infty$, if we have

$$\int_{t_0}^{t}(a(s) - \gamma|b(s)|)ds \to \infty \qquad \text{uniformly for } t_0 \in R^+ \text{ as } t \to \infty, \tag{2.23}$$

then the zero solution of Equation (2.19) is uniformly asymptotically stable.

This theorem can be easily proved by taking the set S in the proof of Theorem 2.1 for the above function $q(t)$ and a function $L = L(\tau_1, \tau_2)$ with

$$2a(t)\alpha + (a(t) + \gamma|b(t)|)\alpha^3 \leq L, \qquad \tau_1 \leq t \leq \tau_2,$$

and by defining a mapping P for $\phi \in S$ by

$$(P\phi)(t) := \begin{cases} \psi(t - t_0), & t_0 - r_0 \le t \le t_0, \\ \psi(0)e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)du} a(s)(\phi(s) - \phi^3(s))ds \\ + \int_{t_0}^t e^{-\int_s^t a(u)du} b(s)\phi(s - r(s))\phi^2(s)ds, & t > t_0. \end{cases}$$

So we omit the details of the proof.

Remark 2.1. Theorem 2.3 is deeply related to Theorem 11.1 in Burton-Furumochi [4; p. 111].

Now we show two examples.

Example 2.6. Let $a(t) = 3(1 + \sin t)$, $b(t) = 1 + \sin t$, $r(t) = 1$ and $\alpha = 1/\sqrt{3}$. Then it is easily seen that (2.2) with $r_0 = 1$, (2.20) and (2.21) with $\gamma = 2$ hold, and $\eta = 1/\sqrt{3}$. Moreover, $\sigma^* < \infty$ and (2.23) hold. Thus, by Theorem 2.3, the zero solution of the equation

$$x'(t) = -3(1 + \sin t)x^3(t) + (1 + \sin t)x(t - 1)x^2(t), \qquad t \in R^+$$

is uniformly asymptotically stable.

Example 2.7. Let $a(t) = 3/(t + 2)$, $b(t) = 1/(t + 2)$, $r(t) = 1$ and $\alpha = 1/\sqrt{3}$. Then it is easily seen that (2.2) with $r_0 = 1$, (2.20) and (2.21) with $\gamma = 2$ hold, and $\eta = 1/\sqrt{3}$. Moreover, (2.23) does not hold, but $\sigma^* < \infty$ and (2.22) hold. Thus, concerning the stabilities of the zero solution of the equation

$$x'(t) = -\frac{3}{t + 2}x^3(t) + \frac{1}{t + 2}x(t - 1)x^2(t), \qquad t \in R^+,$$

Theorem 2.3 does not assure uniform asymptotic stability, but assures uniform stability and asymptotic stability.

Finally, for completeness of our discussion of Equation (2.1), consider the scalar nonlinear equation

$$x'(t) = -a(t)x^3(t) + b(t)x^2(t - r(t))x(t), \qquad t \in R^+, \tag{2.24}$$

where $a, r : R^+ \to R^+$ and $b : R^+ \to R$ are continuous. Let α be any fixed number with $0 < \alpha \leq 1/\sqrt{3}$. We assume that (2.2) holds, and that there is a constant $\gamma > 0$ so that

$$\sigma = \sigma(t_0) := \sup_{t \geq t_0} \int_{t_0}^{t} (\gamma^2 |b(s)| - a(s)) ds < \infty \text{ for any } t_0 \in R^+ \tag{2.25}$$

and

$$\sup_{t \geq t_0 \geq 0} \left(\frac{\frac{1}{2\delta^2} + \int_{t_0}^{t} (a(s) - \gamma^2 |b(s)|) ds}{\frac{1}{2\delta^2} + \int_{t_0}^{\tau} (a(s) - \gamma^2 |b(s)|) ds} \right)^{1/2} \leq \gamma \text{ for any } \delta \in (0, \eta], \tag{2.26}$$

where $\tau = \tau(t) := \max(t_0, t - r(t))$, and η is defined by (2.5).

Corresponding to Equation (2.24), consider the scalar nonlinear equation

$$q'(t) = (\gamma^2 |b(t)| - a(t)) q^3, \qquad t \in R^+.$$

Let $q : [t_0 - r_0, \infty) \to R$ be a continuous function such that $q(t) = \eta$ on $[t_0 - r_0, t_0]$, and that $q(t)$ is the unique solution of the initial value problem

$$q' = (\gamma^2 |b(t)| - a(t)) q^3, \qquad q(t_0) = \eta, \qquad t \geq t_0.$$

Then $q(t)$ can be expressed as

$$q(t) = \left(\frac{1}{\eta^2} + 2 \int_{t_0}^{t} (a(s) - \gamma^2 |b(s)|) ds \right)^{-1/2}, \qquad t \geq t_0,$$

which together with (2.5) and (2.25), implies (2.8).

Concerning the stabilities of the zero solution of Equation (2.24), we have the following theorem.

Theorem 2.4. Suppose that (2.2), (2.25) and (2.26) hold. Then we have:

(i) The zero solution of Equation (2.24) is stable.

(ii) If we have $\sigma^* := \sup\{\sigma(t) : t \in R^+\} < \infty$, then the zero solution of Equation (2.24) is uniformly stable.

(iii) If we have

$$\int_{t_0}^{t} (a(s) - \gamma^2 |b(s)|) ds \to \infty \qquad \text{as } t \to \infty, \tag{2.27}$$

then the zero solution of Equation (2.24) is asymptotically stable.

(iv) In addition to $\sigma^* < \infty$, if we have

$$\int_{t_0}^{t} (a(s) - \gamma^2 |b(s)|) ds \to \infty \qquad \text{uniformly for } t_0 \in R^+ \text{ as } t \to \infty, \tag{2.28}$$

then the zero solution of Equation (2.24) is uniformly asymptotically stable.

This theorem can be easily proved by taking the set S in the proof of Theorem 2.1 for the above function $q(t)$ and a function $L = L(\tau_1, \tau_2)$ with

$$2a(t)\alpha + (a(t) + \gamma^2 |b(t)|)\alpha^3 \leq L, \qquad \tau_1 \leq t \leq \tau_2,$$

and by defining a mapping P for $\phi \in S$ by

$$(P\phi)(t) := \begin{cases} \psi(t - t_0), & t_0 - r_0 \leq t \leq t_0, \\ \psi(0)e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_s^t a(u)du} a(s)(\phi(s) - \phi^3(s))ds & \\ + \int_{t_0}^{t} e^{-\int_s^t a(u)du} b(s)\phi^2(s - r(s))\phi(s)ds, & t > t_0. \end{cases}$$

So we omit the details of the proof.

Remark 2.2. Theorem 2.4 is deeply related to Theorem 11.2 in Burton-Furumochi [4; p. 112].

Now we show two examples.

Example 2.8. Let $a(t) = 5(1 + \sin t)$, $b(t) = 1 + \sin t$, $r(t) = 1$ and $\alpha = 1/\sqrt{3}$. Then it is easily seen that (2.2) with $r_0 = 1$, (2.25) and (2.26) with $\gamma = 2$ hold, and $\eta = 1/\sqrt{3}$. Moreover, $\sigma^* < \infty$ and (2.28) hold. Thus, by Theorem 2.4, the zero solution of the equation

$$x'(t) = -5(1 + \sin t)x^3(t) + (1 + \sin t)x^2(t - 1)x(t), \qquad t \in R^+$$

is uniformly asymptotically stable.

Example 2.9. Let $a(t) = 5/(t + 2)$, $b(t) = 1/(t + 2)$, $r(t) = 1$ and $\alpha = 1/\sqrt{3}$. Then it is easily seen that (2.2) with $r_0 = 1$, (2.25) and (2.26) with $\gamma = 2$ hold, and

$\eta = 1/\sqrt{3}$. Moreover, (2.28) does not hold, but $\sigma^* < \infty$ and (2.27) hold. Thus, concerning the stabilities of the zero solution of the equation

$$x'(t) = -\frac{5}{t+2}x^3(t) + \frac{1}{t+2}x^2(t-1)x(t), \qquad t \in R^+,$$

Theorem 2.4 does not assure uniform asymptotic stability, but assures uniform stability and asymptotic stability.

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