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THE CASE FOR STABILITY BY FIXED POINT THEORY

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Abstract. In this paper we introduce stability by fixed point theory. We focus on an elementary example which is suitable for undergraduate students, a more sophisticated example with many real-world applications, and an example of a type not previously seen in stability theory. This is a brief expository paper. The examples are taken from papers which have previously appeared in the literature.
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1 Introduction

In 1892 Liapunov published a major work on stability of ordinary differential equations based on positive definite functions and the chain rule. It was the foundation of stability theory as we know it today for ordinary, functional, and partial differential equations, as well as overlap into control theory and integral equations. We have studied and contributed to it for more than forty years. Indeed, no day is complete without at least a brief investigation into some compelling aspect of the theory.

Yet, various difficulties with the theory persist and it does seem that other avenues need to be investigated. About ten years ago several investigators began a study into the idea that some of those difficulties might be overcome by means of fixed point theory. We will not survey those results here, but the interested reader can find out what has been learned by reading the papers listed in references [1-3, 5-20, 22-23, 27-28, 30-31]. Moreover, a monograph [16] on the subject should soon be available.

In this brief expository paper we will discuss three examples which illustrate some of the advantages of applying contraction mapping theory to stability problems. The first example has several applications, but we like to think of it as a sophisticated continuation of a problem frequently found in elementary texts in differential equations involving saving and investing money for retirement. The second example concerns many fundamental real-world problems including biological problems, nuclear reactors, viscoelasticity, and neural networks which are simply solved by means of contraction mappings.

Not only does fixed point theory solve old problems, but it does two other fascinating things. First, through its application there appear new results which we had neither sought nor expected; they are a surprise and a delight. Second, new kinds of results are proved of a type never before considered in the literature. We feel that it should be a most productive area of study for many years to come.

The most striking feature of stability by fixed point theory is that one can walk in off the street with just the variation of parameters formula, the definition of a complete metric space, and the basic contraction mapping theorem in order to solve sophisticated problems which have frustrated investigators for decades using other methods. As those three aforementioned concepts are standard and readily found in texts, they will not be stated here.

A significant advantage of fixed point methods over Liapunov's direct method is that conditions of the former are often averages, while those of the latter are usually pointwise. Moreover, John Appleby [1,2] has shown in papers yet to appear that the problems we consider can be perturbed by stochastic terms with the fixed point technique still yielding stability. These are two important features for application to real-world problems.

2 A problem for undergraduates

So much of stability theory for delay equations starts with an asymptotically stable ordinary differential equation onto which we add a delay term which turns out to be a harmless perturbation. We strive in all of our work to avoid such a form and our second example does so. However, these equations can be useful in giving us that first look at the subject, thereby avoiding two steps of complications. Hence, we start with just such a problem. It is a linear scalar differential equation with a delay. As we would like to interest undergraduate students and instructors in this method, we will begin with motivation which continues from a known elementary problem. As with the elementary problem, we will approximate a discrete scheme with an assumption of continuity. To offer some intuition about the example of this section we give a heuristic derivation of equations such as are covered here, together with a very familiar context. The seasoned investigator of functional differential equations will find this superficial and even distracting. For this we apologize and recommend that such readers drop down to Equation (1) below.

The following problem is copied from an elementary text on differential equations by Boyce and DiPrima [4; p. 55]: "A young person with no initial capital invests k dollars per year at an annual interest rate r. Assume that investments are made continuously and that interest is compounded continuously. If $r = 7.5\%$, determine k so that one million dollars will be available at the end of forty years."

It is solved by writing

$$S' = .075S + k, \quad S(0) = 0,$$

and solving for $S(40)$. Several things are idealized in the problem, but still it is a fair model. It is noted there that in certain contexts continuous investment yields roughly the same as daily investment and it allows the student the opportunity to see the power of differential equations in giving a simple solution to an otherwise tedious problem.

Now the forty years is up and for computational convenience instead of the one million dollars let us say that the person has \$900,000 to invest and to live off the proceeds. During times of low interest rates a financial advisor may recommend bank certificates of deposit of 90-day maturity, automatically renewed at the existing interest rate, but laddered so that \$10,000 of the total matures every day and both principal and interest are reinvested. This enables the investor to quickly take advantage of rising rates and to lock in high interest long-term instruments if they become available. We imagine that this is changed to continuous reinvestment, just as the elementary problem imagined continuous investment of k dollars per year. If the total value is again $S(t)$, then from just the investment we would have

$$S'(t) = b(t)S(t - (1/4)).$$

The $b(t)$ represents a product. One factor is the fraction of the total amount of $S(t - (1/4))$ which was invested three months earlier and matured today. The other factor is the interest being offered at that time. In addition, the person withdraws a percentage of the total $S(t)$ continuously for living expenses, resulting in an equation

$$S'(t) = -a(t)S(t) + b(t)S(t - (1/4)), \quad S(t) = \psi(t) \text{ for } -(1/4) \le t \le 0.$$

Here, the initial condition is an initial function $\psi : [-(1/4), 0] \to R$ with $\psi(t)$ being exactly that amount $S(t)$ which was invested at time t.

We can draw several conclusions of the following type. First, if the solutions are bounded, then times are likely to become difficult since inflation will eat away at the value and medical bills will increase with time; at this time, some studies have shown that those retiring with income sufficient to meet three times their current need approach desperate conditions within fifteen years. Next, we can ask if solutions will tend to zero. If they do, the person will be destined for the poor farm. At a minimum, the retiree must adjust the withdrawals so that the conditions of our theorem are not met.

Clearly, in this example it will make sense for both $a(t)$ and $b(t)$ to vary; $a(t)$ can be negative the day the income tax refund check arrives, and $b(t)$ can be negative when the bank fails and the FDIC assumes control. This problem is certainly as interesting as the text book problem and our point is that its solution is well within the grasp of undergraduate mathematics students.

The monograph [16] presents a vast body of differential equations solved in a completely elementary way. This enables an instructor to offer a mathematically honest course to undergraduates covering exciting problems in engineering, physics, chemistry, and mathematical biology, for example. It also offers a rich field for research.

We now leave the heuristic explanation and turn to the formal problem. Let a, b, r be continuous, $r(t) \geq 0$,

$$x' = -a(t)x(t) + b(t)x(t - r(t)), \tag{1}$$

where $-\int_0^t a(s)ds$ is bounded above, while

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \leq \alpha < 1. \tag{2}$$

The following result is just one of about twenty of this general nature which can be found in [17].

Theorem 1. If $\psi : (-\infty, 0] \to R$ is bounded and continuous then $x(t, 0, \psi)$ exists, is unique, and is bounded for $t \geq 0$.

Proof. Notice in (1) that $x'(0) = -a(0)x(0) + b(0)x(-r(0))$. Thus, (1) requires an initial condition consisting of an initial function defined at least on $[-r(0), 0]$. If we do not stipulate that $t - r(t)$ is bounded below, then we must ask that $\psi : (-\infty, 0] \to R$ so that $x(t) = \psi(t)$ for $t \leq 0$. Then we solve (1) for the solution for $t \geq 0$. It will turn out that the solution, denoted by $x(t, 0, \psi)$, can have a corner at $t = 0$, but it will become smooth as $t - r(t)$ increases past zero. This knowledge is not needed to solve the problem. Treating the term with the delay as a forcing function, we employ the elementary variation of parameters formula to (1) and obtain

$$x(t) = e^{-\int_0^t a(s)ds}\psi(0) + \int_0^t e^{-\int_s^t a(u)du}b(s)x(s - r(s))ds.$$

This is an integral equation and it will be used to define a mapping. Here is the idea.

We will construct a set M of functions which are acceptable candidates for the solution. Thus, the set M will consist of bounded and continuous functions $\phi : (-\infty, \infty) \to R$ which satisfy $\phi(t) = \psi(t)$ for $t \leq 0$. The set M will look like a broom with the handle being the initial function ψ. The functions in M are points and if ϕ, η are points in M then the distance between them is denoted by $\|\phi - \eta\| =: \sup_{-\infty \leq t \leq \infty} |\phi(t) - \eta(t)|$; this is the supremum metric. Under that metric, M is a complete metric space. Banach's contraction mapping theorem states that if $P : M \to M$ with the property that $\phi, \eta \in M$ implies that $\|P\phi - P\eta\| \leq \alpha\|\phi - \eta\|$ for some $\alpha < 1$, then there is one and only one point $\phi \in M$ with $P\phi = \phi$. Since the integral equation is used to define the mapping, the fixed point will satisfy the integral equation, it will satisfy the initial condition, it will reside in M, and hence it will be a bounded function.

Here are the details. For $\phi \in M$ define

$$(P\phi)(t) = \psi(t) \quad \text{if} \quad t \le 0 \quad \text{and for} \quad t \ge 0,$$
$$(P\phi)(t) = e^{-\int_0^t a(s)ds}\psi(0) + \int_0^t e^{-\int_s^t a(u)du} b(s)\phi(s - r(s))ds.$$

Then $P\phi$ is bounded and continuous. If $\phi, \eta \in M$ then

$$\begin{aligned}|(P\phi)(t) - (P\eta)(t)| \le& \\ &\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds|\phi(s - r(s)) - \eta(s - r(s))|ds \\ &\le \alpha\|\phi - \eta\|.\end{aligned}$$

The contraction implies that there is a unique fixed point satisfying (1), residing in M and being bounded.

In one step we have proved that there exists a solution, it is unique, it satisfies the differential equation and the initial condition, and it is bounded.

Our problem here is particularly simple because it is scalar and half-linear. But there are a few small tricks which make the same technique applicable to more sophisticated problems. In [16] we offer roughly one hundred such problems worked in fine detail. Our next example is typical of that class.

3 A more sophisticated problem

In 1928 Volterra [29] modelled a biological problem with the following equation. Let $L > 0$, $xg(x) > 0$ when $x \ne 0$, $a : [0, L] \to R$ and consider the initial value problem

$$x'(t) = -\int_{t-L}^{t} a(t - s)g(x(s))ds, \;\; \psi : [-L, 0] \to R. \tag{3}$$

We need ψ because we must define $x'(0)$:

$$x'(0) = -\int_{-L}^{0} a(-s)g(\psi(s))ds.$$

The solution is denoted by $x(t, 0, \psi)$, where $x(t, 0, \psi) = \psi(t)$, $-L \le t \le 0$.

The problem of Volterra was to give conditions showing that $x(t, 0, \psi)$ is bounded and tends to zero.

Volterra's equation turned out to be very important. In 1954 Ergen [21] modeled a circulating fuel nuclear reactor with it; there, x is the neutron density. The equation is also a model for one-dimensional viscoelasticity;

there, x is the strain and a is the relaxation function. It has also been used to model neural networks, to name just a few applications. Moreover, it is one of the central models for a variety of theoems in the theory of Liapunov's direct method; Krasovskii [24] features it prominently.

Concerning the solution of Volterra's problem, in 1928 he suggested a way to construct a Liapunov functional. Levin almost found it in 1963 [25].

Theorem 2 (Levin-Nohel 1964 [26]) Let

$$(i) \quad a(L) = 0, a'(t) \leq 0, a''(t) \geq 0, a''(t) \text{ not } \equiv 0,$$

$$(ii) \quad \int_0^x g(s)ds \to \infty \text{ as } x \to \pm\infty.$$

For a given ψ and for $x(t) = x(t, 0, \psi)$ the functional

$$(iii) \quad V(t) = \int_0^x g(u)du - (1/2)\int_{t-L}^t a'(t-v)[\int_v^t g(x(s))ds]^2 dv \geq 0$$

satisfies

$$(iv) \quad 2V'(t) = a'(L)[\int_{t-L}^t g(x(s))ds]^2 - \int_{t-L}^t a''(t-s)[\int_s^t g(x(v)dv]^2 ds$$

and

$$(v) \quad x^{(j)}(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for} \quad j = 0, 1, 2.$$

Critique

1. It is a marvelous result.
2. The big question is this: Is

$$(i) \quad a(L) = 0,\ a'(t) \leq 0,\ a''(t) \geq 0,\ a''(t) \quad \text{not} \quad \equiv 0,$$

a verifiable condition in real-world problems?

We will give a fixed point proof of asymptotic stability in which we replace (i) by averages, but generally ask more of g. This is the motive and intent of the fixed point technique. But in virtually every one of our fixed point solutions we find that we have discovered something new which was neither sought nor expected. In this case, we find that (ii), which is so essential in the Liapunov argument, is totally absent in the fixed point argument. It is a free and unexpected contribution of the fixed point method. This gives us serious motivation to examine important classical problems to see what more the fixed point method will give the problem.

We proceed now to a fixed point solution to Volterra's problem. Suppose that there is a function $g : R \to R$ satisfying:

$$|g(x) - g(y)| \leq K|x - y| \tag{4}$$

for some $K > 0$ and all $x, y \in R$. This helps to ensure that there is a contraction, but it also avoids a far more sinister problem against which we must be continually vigilant when using fixed point theory. We allow so much freedom in sign conditions that finite escape time is an ever present possbility. The Lipschitz growth will prevent that. In addition we need

$$\frac{g(x)}{x} \geq 0 \quad \text{and} \quad \lim_{x \to 0} \frac{g(x)}{x} \quad \text{exits,} \tag{5}$$

and sometimes

$$\frac{g(x)}{x} \geq \beta \tag{6}$$

for some $\beta > 0$.

Consider the scalar equation

$$x' = -\int_{t-L}^{t} p(s-t) g(x(s)) ds \tag{7}$$

with $L > 0$, p continuous,

$$\int_{-L}^{0} p(s) ds = 1, \tag{8}$$

(This is notation; we only need $\int_{-L}^{0} p(s)ds > 0$ since the difference can be absorbed by g.) and for the K of (4) let

$$2K \int_{-L}^{0} |p(v)v| dv =: \alpha < 1. \tag{9}$$

We can offer 4 ways to solve this problem. In [8] we treat this problem with a different technique which allows a solution with no more information than was displayed in the proof of Theorem 1. Here, we ask for an existence theorem, and it introduces us to another very useful technique in fixed point theory. This theorem is one of several of this general nature which were first published in [14].

Theorem 3. If (4), (5), (8), and (9) hold, then every solution of (7) is bounded. If (6) also holds, then every solution tends to zero.

Proof. Let $\psi : [-L, 0] \to R$ be a given continuous initial function and let $x_1(t) := x(t, 0, \psi)$ be the unique resulting solution. By the growth condition on g, $x_1(t)$ exists on $[0, \infty)$. If we add and subtract $g(x)$ we can write the equation as

$$x' = -g(x) + \frac{d}{dt} \int_{-L}^{0} p(s) \int_{t+s}^{t} g(x(u)) du ds.$$

Define a continuous non-negative function $a : [0, \infty) \to [0, \infty)$ by

$$a(t) := \frac{g(x_1(t))}{x_1(t)}.$$

Since a is the quotient of continuous functions it is continuous when assigned the limit at $x_1(t) = 0$, if such a point exists.

Thus, for the fixed solution, our equation is

$$x' = -a(t)x + \frac{d}{dt}\int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))duds$$

which, by the variation of parameters formula, followed by integration by parts, can then be written as

$$\begin{aligned}
x(t) &= \psi(0)e^{-\int_0^t a(s)ds} \\
&\quad + \int_0^t e^{-\int_v^t a(s)ds}\frac{d}{dv}\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))dudsdv \\
&= \psi(0)e^{-\int_0^t a(s)ds} + e^{-\int_v^t a(s)ds}\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))duds\Big|_0^t \\
&\quad - \int_0^t a(v)e^{-\int_v^t a(s)ds}\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))dudsdv \\
&= \psi(0)e^{-\int_0^t a(s)ds} + \int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))duds \\
&\quad - e^{-\int_0^t a(s)ds}\int_{-L}^{0} p(s)\int_{s}^{0} g(\psi(u))duds \\
&\quad - \int_0^t e^{-\int_v^t a(s)ds}a(v)\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))dudsdv.
\end{aligned}$$

Let

$$M = \{\phi : [-L,\infty) \to R | \phi_0 = \psi, \phi \in C, \phi \quad \text{bounded}\}$$

and define $P : M \to M$ using the above equation in $x(t)$. The notation $\phi_0 = \psi$ means that $\phi(t) = \psi(t)$ on $[-L, 0]$. For $\phi \in M$ define $(P\phi)(t) = \psi(t)$ if $-L \le t \le 0$. If $t \ge 0$, then define

$$\begin{aligned}
(P\phi)(t) &= \psi(0)e^{-\int_0^t a(s)ds} \\
&\quad + \int_{-L}^{0} p(s)\int_{t+s}^{t} g(\phi(u))duds \\
&\quad - e^{-\int_0^t a(s)ds}\int_{-L}^{0} p(s)\int_{s}^{0} g(\psi(u))duds \\
&\quad - \int_0^t e^{-\int_v^t a(s)ds}a(v)\int_{-L}^{0} p(s)\int_{v+s}^{v} g(\phi(u))dudsdv.
\end{aligned}$$

To see that P is a contraction, if $\phi, \eta \in M$ then

$$\begin{aligned}
&|(P\phi)(t) - (P\eta)(t)| \\
&\leq \int_{-L}^{0} |p(s)| \int_{t+s}^{t} |g(\phi(u)) - g(\eta(u))| du ds \\
&+ \int_{0}^{t} e^{-\int_v^t a(s)ds} a(v) \int_{-L}^{0} |p(s)| \int_{v+s}^{v} |g(\phi(u)) - g(\eta(u))| du ds dv \\
&\leq 2K\|\phi - \eta\| \int_{-L}^{0} |p(s)s| ds \\
&\leq \alpha \|\phi - \eta\|.
\end{aligned}$$

There is a unique fixed point, a bounded solution.

If $\frac{g(x)}{x} \geq \beta > 0$, then add to M the condition that $\phi(t) \to 0$ as $t \to \infty$. We can show $(P\phi)(t) \to 0$ when $\phi(t) \to 0$ so the fixed point tends to 0.

4 Unusual results

In Theorem 3 we were surprised to find that condition (ii) of Theorem 2 was not needed. Now we come to a different kind of surprise. Consider the equation

$$x'(t) = -a(t)x^{2n+1}(t-r) - b(t)x^{2n+3}(t-r) \tag{10}$$

where n is a positive integer. Can we borrow $b(t)$ and add it to $a(t)$, replacing (10) by

$$x' = -[a(t) + b(t)]x^{2n+1}(t-r)?$$

Theorem 4. Let $a(t) + b(t) = c(t) \geq 0$, and for $L^2 = \frac{2n+1}{2n+3}$ let

$$(1 - L^2) \sup_{t \geq 0} \int_{0}^{t} e^{-\int_s^t c(u+r)du} |b(s)| ds + 2 \sup_{t \geq 0} \int_{t-r}^{t} c(s+r) ds < 1. \tag{11}$$

Then there is a $\delta > 0$ such that if the function ψ satisfies $\|\psi\| < \delta$ then $|x(t, 0, \psi)| < L$.

A proof of this result and many of a related nature can be found in [6] and [10].

As an example, let

$$a(t) = 1 - 2\sin t, \quad b(t) = 2\sin t, \quad c(t) = 1.$$

Then

$$\int_{0}^{t} e^{-\int_s^t 1du} |b(s)| ds \leq 2$$

so to satisfy (11) we need

$$2\frac{2}{2n+3} + 2r < 1$$

or

$$r < \frac{2n-1}{4n+6}.$$

Notice that we could not have borrowed if $n = 1$.

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