Stability by Fixed Point Theory or Liapunov Theory: A Comparison

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ABSTRACT. Liapunov's direct method has been very effective in establishing stability results for a wide variety of differential equations. Yet, there is a large set of problems for which it has been ineffective. In a series of papers we have examined particular problems which have offered great difficulities for that theory and have presented solutions by means of various fixed point theorems. In this note we look at the linear scalar delay equation $x'(t) = -a(t)x(t-r)$ in which r is a positive constant, $a(t)$ is a continuous function which may change sign. While Liapunov's direct method usually asks pointwise conditions, the stability result we offer asks conditions of an averaging nature. A parallel discussion is also given for $x' = A(t)x - \int_0^t C(t,s)x(s)ds$ where $A(t)$ may change sign or be identically zero.

1. Introduction

This paper is one in a series of investigations (cf. [3,4-6]) in which we have looked at problems which were especially challenging for stability analysis using Liapunov's direct method. We show that many of these problems can be solved using fixed point theory.

Each of these papers has focused on a different type of problem, with a view to developing a set of fixed point techniques that would attack most of the difficulties encountered in Liapunov's direct method. For example, the first paper [4] dealt with problems which contained an asymptotically stable linear term so that they could be written as a fixed point mapping equation by means of the variation of parameters formula; the proofs of stability were all by contraction mappings. That paper was in the spirit of so much Liapunov theory in which the delay term is proved to be a harmless perturbation of the stable linear term. The interesting feature was that the conditions for stability were averages, as opposed to the pointwise conditions usually present in Liapunov theory.

By contrast, [3,5] focused on the idea of Krasnoselskii [12] (cf. Smart [19; p. 31] and Schauder [16]) that the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. In [3] we treated an equation in which the functions were unbounded in t, the delay was not necessarily bounded, the delay was not necessarily differentiable, and $t - r(t)$ did not necessarily tend to infinity. All of these properties are the ones which have presented such large difficulties for Liapunov theory (cf. Seifert [17]).

In all of these investigations we have focused on concrete examples. Here, we first examine the equation

$$x'(t) = -a(t)x(t-r) \tag{1}$$

in which $a : [0, \infty) \to R$ is continuous and r is a positive constant. The reader can carry out similar computations for

$$x'(t) = -a(t)x(t-r) - b(t)x(t)$$

where $b(t)$ is not necessarily positive.

To focus on the difficulty of these problems we look at an example

$$x'(t) = -(1 + 2\sin t)x(t-r)$$

and refer the reader to Hale [10; p. 55]. As t varies the equation takes on many known forms, some of which are stable and some are unstable. At $t = (3/2)\pi$ we have

$$x' = x(t-r)$$

and that equation is unstable for any $r > 0$. At $t = \pi/2$ we have

$$x' = -3x(t-r)$$

and that equation is stable if $0 \le r < \pi/6$. It is certainly reasonable to believe that a very restrictive condition on r is needed for stability and that a proof of stability will likely be very difficult.

Next, we examine

$$x' = A(t)x - \int_0^t C(t,s)x(s)ds$$

in which A can change sign, be zero, or even be positive for all t. Using Liapunov functionals, a "sharp" stability result [8] was obtained in 1983 which we now compare with a fixed point result.

Basic Liapunov theory from several points of view is found in [1,2,10,13-15,20]. Construction of many types of Liapunov functionals is found in [1,2,7-10,13-15,17,20,21]. Serban [18] has recently used fixed point theory to prove stability in difference equations.

2. Stability

Let $a : [0,\infty) \to R$ be bounded and continuous, let r be a positive constant, and let

$$x' = -a(t)x(t-r). \tag{1}$$

Although we can treat solutions with any initial time, we will always look at a solution $x(t) := x(t,0,\psi)$ where $\psi : [-r,0] \to R$ is a given continuous initial function and $x(t,0,\psi) = \psi(t)$ on $[-r,0]$. It is then known that there is a unique continuous solution $x(t)$ satisfying (1) for $t > 0$ and with $x(t) = \psi(t)$ on $[-r,0]$.

With such ψ in mind, we can write (1) as

$$x' = -a(t+r)x(t) + (d/dt)\int_{t-r}^t a(s+r)x(s)ds \tag{2}$$

so that by the variation of parameters formula, followed by integration by parts, we have

$$x(t) = x(0)e^{-\int_0^t a(s+r)ds} + \int_{t-r}^t a(u+r)x(u)du - e^{-\int_0^t a(u+r)du}\int_{-r}^0 a(u+r)x(u)du$$

$$-\int_0^t a(s+r)e^{-\int_s^t a(u+r)du}\int_{s-r}^s a(u+r)x(u)duds. \tag{3}$$

In a space to be defined and with a mapping defined from (3) we will find that we have a contraction mapping just in case there is a constant $\alpha < 1$ with

$$\int_{t-r}^t |a(u+r)|du + \int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du}\int_{s-r}^s |a(u+r)|duds \le \alpha. \tag{4}$$

As we are interested in asymptotic stability we will need

$$\int_0^t a(s+r)ds \to \infty \text{ as } t \to \infty. \tag{5}$$

This paper compares results from a certain application of fixed point theory with a certain common Liapunov functional. In theory, there is no comparison at all. It is known that if we have a strong type of stability, then there exists a Liapunov functional of a certain type. The fact that we can not find that Liapunov functional gives validity to this type of comparison. With that in mind, from (4) it is easy to see one of the advantages of fixed point theory over Liapunov theory. The latter requires $a(t+r) > 0$. If $a(t+r) \geq 0$, then a very good bound is obtained in (4) with little effort (see Example 2). If $a(t+r)$ changes sign then (4) can still hold, although a good bound on the second integral is more difficult (see Example 3).

We attempted to prove a form of the following result in [4; p. 97] but were unable to do so and we left the principle difficulty as a hypothesis. This was very unsatisfactory and there is a simple proof.

THEOREM 1. Let (4) and (5) hold. Then for every continuous initial function $\psi : [-r, 0] \to R$ the solution $x(t, 0, \psi)$ is bounded and tends to zero as $t \to \infty$.

Proof. Let $(B, \|\cdot\|)$ be the Banach space of bounded and continuous functions $\phi : [-r, \infty) \to R$ with the supremum norm. Let $(S, \|\cdot\|)$ be the complete metric space with supremum norm consisting of functions $\phi \in B$ such that $\phi(t) = \psi(t)$ on $[-r, 0]$ and $\phi(t) \to 0$ as $t \to \infty$.

Define $P : S \to S$ by

$$(P\phi)(t) = \psi(t) \text{ on } [-r, 0]$$

$$(P\phi)(t) = \psi(0)e^{-\int_0^t a(s+r)ds} + \int_{t-r}^t a(u+r)\phi(u)du$$

$$-e^{-\int_0^t a(u+r)du} \int_{-r}^0 a(u+r)\psi(u)du - \int_0^t a(s+r)e^{-\int_s^t a(u+r)du} \int_{s-r}^s a(u+r)\phi(u)duds.$$

Clearly, $P\phi$ is continuous, $(P\phi)(0) = \psi(0)$, and from (4) it follows that $P\phi$ is bounded. Also, P is a contraction by (4).

We can show that the last term tends to zero by using the classical proof that the convolution of an L^1-function with a function tending to zero, does also tend to zero. Here are the details. Let $\phi \in S$ be fixed and let $0 < T < t$. Denote the supremum of $|\phi|$ by $\|\phi\|$ and the supremum of $|\phi|$ on $[T, \infty)$ by $\|\phi\|_{[T,\infty)}$. Consider (4) and (5). We have

$$\int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du} \int_{s-r}^s |a(u+r)\phi(u)|duds$$

$$\leq \int_0^T |a(s+r)|e^{-\int_s^T a(u+r)du} \int_{s-r}^s |a(u+r)|duds\|\phi\|e^{-\int_T^t a(u+r)du}$$

$$+ \int_T^t |a(s+r)|e^{-\int_s^t a(u+r)du} \int_{s-r}^s |a(u+r)|duds\|\phi\|_{[T-r,\infty)}$$

$$\leq \alpha\|\phi\|e^{-\int_T^t a(u+r)du} + \alpha\|\phi\|_{[T-r,\infty)}.$$

For a given $\epsilon > 0$ take T so large that $\alpha\|\phi\|_{[T-r,\infty)} < \epsilon/2$. For that fixed T, take t^* so large that $\alpha\|\phi\|e^{-\int_T^t a(u+r)du} < \epsilon/2$ for all $t > t^*$. We then have that last term smaller than ϵ for all $t > t^*$. Thus, $P : S \to S$ is a contraction with unique fixed point in S.

REMARK. It is known that for (1) if solutions are bounded then the zero solution is Liapunov stable.

This example is designed to be compared with Example 4 using Liapunov's direct method.

EXAMPLE 1. In (1), let

$$a(t) = 1.1 + \sin t.$$

The conditions of Theorem 1 are satisfied if

$$2(1.1r + 2\sin(r/2)) < 1.$$

This is approximated by $0 < r < .2$.

Proof. We first estimate

$$\int_{t-r}^t |a(u+r)|du = \int_t^{t+r} (1.1 + \sin u)du.$$

It is easy to see that for $r < 1$ this integral is dominated by

$$\int_{(\pi-r)/2}^{(\pi+r)/2}(1.1+\sin u)du = 1.1r + 2\sin(r/2).$$

The second integral in (4) is easily seen to be bounded by

$$1.1r + 2\sin(r/2)$$

as well. Hence, to satisfy (4) we need

$$2(1.1r + 2\sin(r/2)) < 1.$$

A very rough estimate (taking $\sin(r/2) = r/2$) yields the requirement

$$0 < r < \frac{1}{4.2}.$$

We will see that this can be compared to a result using a Liapunov functional and, in this case, the Liapunov functional yields a significantly better result. But the next two examples reveal something equally interesting. In a later example the fixed point result is better than that of the Liapunov functional.

1. If $a(t) \geq 0$, then the Liapunov functional fails to address the problem, while the fixed point theorem yields a result fully consistent with that of Example 1. Again, it is a good result, obtained with little effort.

2. If $a(t)$ becomes negative, then the Liapunov functional fails, while the fixed point theorem yields a stability result which is significantly poorer than in the first two cases because of inherent diffiiculties in estimating the integrals in (4).

EXAMPLE 2. In (12), let

$$a(t) = 1 + \sin t.$$

The conditions of Theorem 1 are satisfied if

$$2(r + 2\sin(r/2)) < 1.$$

Proof. The first integral in (4) is again estimated in the same way as in Example 1 by

$$r + 2\sin(r/2).$$

The second integral in (4) is estimated by the same amount. We need

$$2(r + 2\sin(r/2)) < 1.$$

A crude estimate with $\sin(r/2) = r/2$ asks that $r < 1/4$.

REMARK Intuitively, Example 1 should be more strongly stable than Example 2. Yet, a look at (4) readily reveals why our results state the opposite. We conjecture that a different fixed point mapping might reverse the relation.

EXAMPLE 3. Consider the scalar equation

$$x'(t) = -(1 + 2\sin t)x(t - r) \tag{6}$$

where $0 < r < 1$ and to be restricted further later. The stability must not only be derived from $x(t - r)$, but the coefficient function

$$a(t) = 1 + 2\sin t \tag{7}$$

changes sign. We need to establish some bounds toward fulfilling the conditions of Theorem 1. These are given in (10) below. No effort is made to establish best possible bounds. When there is a need for better bounds, they can certainly be derived. The point being made here is that the coefficient can change sign.

We will conclude asymptotic stability when

$$(r + 4\sin(r/2))(2 + 2e^2) < 1.$$

This is approximately $0 \le r < .02$.

Proof.

LEMMA 1. For $a(t) = 1 + 2\sin t$ and $0 < r < 1$ we have

$$\int_{t-r}^{t} |a(u + r)|du \le 2((r/2) + 2\sin(r/2)). \tag{8}$$

Proof. We have

$$\int_{t-r}^{t} |a(u + r)|du = \int_{t}^{t+r} |1 + 2\sin u|du$$

$$\leq \int_{(\pi-r)/2}^{(\pi+r)/2} (1+2\sin u)du$$
$$= 2((r/2)+2\sin(r/2)).$$

With work, the next result can be greatly improved. But our point here is to obtain a quick result allowing $a(t)$ to be negative some of the time.

LEMMA 2. Under the same conditions

$$J := \int_0^t |1+2\sin(s+r)|e^{-\int_s^t (1+2\sin(u+r))du} \int_{s-r}^s |(1+2\sin(u+r))|duds$$

(9) $$\leq (r+4\sin(r/2))(1+2e^2).$$

Proof. Note that

$$\int_{s-r}^s |(1+2\sin(u+r))|du \leq r+4\sin(r/2)$$

by the mean value theorem. Thus,

$$J \leq (r+4\sin(r/2)) \int_0^t (1+2\sin(s+r)+2)e^{-\int_s^t (1+2\sin(u+r))du} ds$$
$$\leq (r+4\sin(r/2))\left(1+2\int_0^t e^{-\int_s^t (1+2\sin(u+r))du} ds\right)$$
$$\leq (r+4\sin(r/2))\left(1+2e^2\int_0^t e^{-\int_s^t du} ds\right)$$
$$\leq (r+4\sin(r/2))(1+2e^2).$$

Putting the results of the two lemmas together and referring to (4) we require that

(10) $$(r+4\sin(r/2))(2+2e^2) < 1.$$

3. A Liapunov Functional

We return now to

(1) $$x' = -a(t)x(t-r)$$

and

$$x' = -a(t+r)x + (d/dt)\int_{t-r}^{t} a(s+r)x(s)ds. \tag{2}$$

THEOREM 2. If there is a $\delta > 0$ with

$$a(t+r) \geq \delta, \text{ for all } t \geq 0, \tag{11}$$

an $\epsilon > 0$ with

$$a(t+r)\int_{t-r}^{t} a(s+r)ds - 2 + r \leq -\epsilon \text{ for all } t \geq 0, \tag{12}$$

and if there is a $\gamma > 0$ with

$$\gamma[a(t) + a(t+r)] \leq (\epsilon/2)a(t+r) \text{ for all } t \geq 0 \tag{13}$$

then the zero solution of (1) is uniformly asymptotically stable.

Proof. From (2) we have

$$(x - \int_{t-r}^{t} a(s+r)x(s)ds)' = -a(t+r)x$$

and we select a first Liapunov functional as

$$V_1(t, x_t) = \left(x - \int_{t-r}^{t} a(s+r)x(s)ds\right)^2 + \int_{-r}^{0}\int_{t+s}^{t} a(u+r)x^2(u)duds$$

so that the derivative along a solution of (2) is

$$V_1'(t, x_t) = 2\left(x - \int_{t-r}^{t} a(s+r)x(s)ds\right)(-a(t+r)x)$$

$$+ \int_{-r}^{0} a(t+r)x^2(t)du - \int_{-r}^{0} a(t+s+r)x^2(t+s)ds$$

$$\leq -2a(t+r)x^2 + a^2(t+r)\int_{t-r}^{t} a(s+r)dsx^2 + \int_{t-r}^{t} a(s+r)x^2(s)ds$$

$$+a(t+r)x^2(t)r - \int_{t-r}^{t} a(s+r)x^2(s)ds$$

$$= [-2a(t+r) + a^2(t+r)\int_{t-r}^{t} a(s+r)ds + ra(t+r)]x^2$$

$$= a(t+r)[-2 + a(t+r)\int_{t-r}^{t} a(s+r)ds + r]x^2$$

$$\leq -\epsilon a(t+r)x^2$$

(In Remark 1 below, use this and (13).)

$$\leq -\epsilon\delta x^2.$$

Now, we need to define a second Liapunov functional and add them together to make a positive definite Liapunov functional. Define

$$V_2(t, x_t) = \gamma[x^2 + \int_{t-r}^{t} a(s+r)x^2(s)ds]$$

so that the derivative along a solution of (1) is

$$V_2'(t, x_t) \leq \gamma[-2a(t)xx(t-r) + a(t+r)x^2 - a(t)x^2(t-r)]$$

$$\leq \gamma[a(t)x^2 + a(t)x^2(t-r) + a(t+r)x^2 - a(t)x^2(t-r)]$$

$$= \gamma[a(t) + a(t+r)]x^2$$

$$\leq (\epsilon/2)a(t+r)x^2.$$

If we define

$$V(t, x_t) = V_1(t, x_t) + V_2(t, x_t)$$

then we have

$$V'(t, x_t) \leq -(\epsilon/2)\delta x^2.$$

We can now find wedges with

$$W_1(|x(t)|) \leq V(t, x_t) \leq W_2(\|x_t\|)$$

and, since (11) and (12) imply that $x(t)$ is bounded, conclude that the zero solution is uniformly asymptotically stable.

Theorem 2 requires (11) so it can not be compared to Example 2 or 3. But it does compare very favorably with Example 1.

EXAMPLE 4. Let

$$a(t) = 1.1 + \sin t.$$

Theorem 2 holds if there is an $\epsilon > 0$ with

$$2.1(1.1r + 2\sin(r/2)) - 2 + r < -\epsilon.$$

Proof. Note that (11) is satisfied with $\delta = .1$. To satisfy (12) we have

$$(1.1 + \sin(t+r))\int_{t-r}^{t} (1.1 + \sin(s+r))ds - 2 + r$$

$$\leq (1.1 + \sin(t+r))(1.1r + 2\sin(r/2)) - 2 + r$$

$$\leq 2.1(1.1r + 2\sin(r/2)) - 2 + r < -\epsilon.$$

We make another very rough estimate by taking $\sin(r/2) = r/2$ and say that we need

$$5.41r - 2 < -\epsilon$$

or

$$r < 2/5.41.$$

REMARK 1. If the conditions of Theorem 2 hold, except that $a(t+r) \geq \delta > 0$ is replaced by $a(t+r) \geq 0$ and $\int_0^\infty a(t)dt = \infty$, then asymptotic stability can be concluded. In fact, by $V_1'(t, x_t) \leq -\epsilon a(t+r)x^2$, we have $\int_0^\infty a(t+r)x^2(t)dt < \infty$. This implies that $\liminf_{t\to\infty} x^2(t) = 0$. Notice that

$$\left|\frac{d}{dt}x^2(t)\right| = 2a(t)|x(t)||x(t-r| \leq a(t)|x(t)|^2 + a(t)|x(t-r)|^2.$$

Since $a(t)|x(t-r)|^2$ and $a(t)|x(t)|^2$ are in $L^1[0,\infty)$, we have $\frac{d}{dt}x^2(t) \in L^1[0,\infty)$. Thus, $\lim_{t\to\infty} x(t) = 0$.

REMARK 2. We do not see a clear way to interpret a relation between the fixed point condition (4) and the Liapunov condition (12). Sometimes one is better than the other. Sometimes the fixed point condition gives results when the Lapunov condition can not. But the next problem will conclude with a much clearer relation.

4. A Volterra equation

The Liapunov functional defined in Section 3 was motivated by the one derived in [8] for

$$x'(t) = A(t)x + \int_0^t C(t,s)x(s)ds \tag{14}$$

where we suppose there is a function $G(t,s)$ with $(\partial G(t,s)/\partial t) = C(t,s)$; for example, $G(t,s) = -\int_t^\infty C(u,s)du$. Then we can write (14) as

$$x' = A(t)x - G(t,t)x + (d/dt)\int_0^t G(t,s)x(s)ds$$

or

$$x' = Q(t)x + (d/dt)\int_0^t G(t,s)x(s)ds. \tag{15}$$

In [8] there is the following basic result for such equations.

THEOREM 3. Suppose there are constants Q_1, Q_2, J, R with $R < 2$ such that

$$0 < Q_1 \le |Q(t)| \le Q_2, \tag{16}$$

$$\int_0^t |G(t,s)|ds \le J < 1, \tag{17}$$

and

$$\int_0^t |G(t,s)|ds + \int_t^\infty |G(u,t)|du \le RQ_1/Q_2. \tag{18}$$

Suppose also that there is a continuous function $h : [0,\infty) \to [0,\infty)$ with $|G(t,s)| \le h(t-s)$ and $h(u) \to 0$ as $t \to \infty$. Then the zero solution of (14) is stable if and only if $Q(t) < 0$.

The proof is based on a Liapunov functional

$$V(t,x_t)=(x-\int_0^t G(t,s)x(s)ds)^2+Q_2\int_0^t\int_t^\infty |G(u,s)|du x^2(s)ds.$$

Slight additional assumptions (not stated here) yield asymptotic stability.

We may obtain a corollary stating that if Q is constant and G is of convolution type then (16)-(18) can be replaced by $\int_0^t |G(t-s)|ds \le J < 1$. And this demands exactly half as much as the fixed point theorem requires.

The next example shows asymptotic stability using this theorem when $n > 24$. After Theorem 6 we will consider the same equation using fixed point techniques and obtain asymptotic stability for $n > 4.2$. Thus, we will see that sometimes the fixed point theory yields better results and sometimes the Liapunov function does.

EXAMPLE 5. Let

$$x' = (\sin t)x - (1.1)(n-1)\int_0^t (1+t-s)^{-n}x(s)ds$$

where $n > 25.1$. Then the conditions of Theorem 3 hold and the zero solution is stable (actually it is asymptotically stable).

Proof. The equation can be written as

$$x' = (\sin t - 1.1)x + (d/dt)(1.1)\int_0^t (1+t-s)^{1-n}x(s)ds$$

with

$$G(t-s) = (1.1)(1+t-s)^{1-n}.$$

We have

$$Q_1 = .1, Q_2 = 2.1.$$

To satisfy (18) we calculate

$$\int_0^t (1.1)(1+t-s)^{1-n}ds + \int_t^\infty (1.1)(1+u-t)^{1-n}du \le 2.2/(n-2).$$

Then

$$RQ_1/Q_2 \le 2(.1)/2.1 = 2/21.$$

A sufficient condition for (18) is $2.2/(n-2) < 2/21$ or $n > 25.1$.

We are interested in comparing this result to those obtained by fixed point theorems. Two cases will be considered. First, it is interesting to see what happens in the dividing case when $Q(t) = 0$. It turns out to be stable and periodic perturbations generate periodic response in the limiting equations. Next, we replace the conditions by averaging conditions and obtain asymptotic stability results.

4a. The case $Q(t) = 0$.

The case

$$Q(t) = 0 \tag{19}$$

is interesting in its own right and the method of proof in the above theorem completely fails in that case. We examine it using fixed point theory.

THEOREM 4. Let Q be defined in (15) and let (17) and (19) hold. Then for each $x(0) \neq 0$ there is a unique bounded and continuous solution $x(t, 0, x(0))$ of (14) on $[0, \infty)$. Moreover, that solution does not converge to zero.

Proof. Let $(B, \|\cdot\|)$ be the Banach space of bounded continuous $\phi : [0, \infty) \to R$ with the supremum norm and define $P : B \to B$ by $\phi \in B$ implies

$$(P\phi)(t) = x(0) + \int_0^t G(t,s)\phi(s)ds. \tag{20}$$

Clearly, if $P\phi = \phi$ then $\phi(0) = x(0)$, ϕ is continuous, and $\phi'(t) = (d/dt)\int_0^t G(t,s)\phi(s)ds$ and so ϕ satisfies (15) and (14). Also, P does map B into itself and we have

$$|(P\phi)(t) - (P\psi)(t)| \leq \int_0^t |G(t,s)||\phi(s) - \psi(s)|ds \leq J\|\phi - \psi\|$$

so the unique fixed point is established. If $\phi(t) \to 0$ then we readily prove that

$$\int_0^t G(t,s)\phi(s)ds \to 0$$

leaving $\phi(t) \to x(0) \neq 0$, a contradiction.

To see that the zero solution is stable, every solution $x(t, 0, x(0))$ of (14) can be expressed as $x(t) = x(0)x(t, 0, 1)$. If $|x(t, 0, 1)| \leq M$, then for a given $\epsilon > 0$ we take $\delta = \epsilon/M$. Then $|x(0)| \leq \delta$ implies $|x(t)| \leq \delta M = \epsilon$. This completes the proof.

It is interesting to determine just how strongly stable the zero solution of (14) is when (17) and (19) hold. One measure of strength is to perturb (14) with a continuous function having a bounded integral. Then the same proof will yield a bounded continuous solution. Specialize that to ask that the perturbation be periodic. Then develop the limiting equation. We find that the limiting equation will then have a periodic solution. Here are the details.

Consider

$$x' = A(t)x + \int_0^t C(t, s)x(s)ds + p(t)$$

and suppose there is a positive constant T with

$$(21)\quad p(t + T) = p(t), \int_0^T p(s)ds = 0, C(t + T, s + T) = C(t, s), G(t + T, s + T) = G(t, s).$$

The limiting equation is obtained by defining $y(t) = x(t + nT)$ for $n = 1, 2, 3, ...$ and then letting $n \to \infty$. We write the result as

$$z' = A(t)z + \int_{-\infty}^t C(t, s)z(s)ds + p(t) \tag{22}$$

under the assumption that

$$\int_{-\infty}^t |C(t, s)|ds \tag{23}$$

is bounded and continuous.

Write (22) as in (15) in the form

$$z' = Q(t)z + (d/dt)\int_{-\infty}^t G(t, s)z(s)ds + p(t). \tag{24}$$

THEOREM 5. In (24) we suppose that (21) holds and that

$$Q(t) = 0 \text{ and } \int_{-\infty}^t |G(t, s)|ds \leq J < 1. \tag{25}$$

Then (22) has a T-periodic solution.

Proof. Let $(B_T, \|\cdot\|)$ be the Banach space of continuous T-periodic functions with the supremum norm. Define $P: B_T \to B_T$ by $\phi \in B_T$ implies that

$$(P\phi)(t) = \int_{-\infty}^{t} G(t,s)\phi(s)ds + \int_0^t p(s)ds.$$

If $P\phi = \phi$ then ϕ satisfies (24) and (22). By (25) we see that P is a contraction.

REMARK. If $G(t,s) = G(t-s)$ and if p is almost periodic with bounded integral, then $p: AP \to AP$ and (22) has an AP solution.

We could also show that if (14) is given an $L^1[0,\infty)$ perturbation then a companion of Theorem 3 would give a bounded solution. It would be interesting to show that if (14) were given a perturbation $p(t)$ for which $\int_{t-1}^t |p(s)|ds \to 0$ as $t \to \infty$, then there is still a bounded solution. All of these results offer a measure of the stability in case $Q(t) = 0$.

4b. The case $\int_0^t Q(s)ds \to -\infty$ as $t \to \infty$.

We return to

$$x' = A(t)x + \int_0^t C(t,s)x(s)ds \tag{14}$$

and

$$x' = Q(t)x + (d/dt)\int_0^t G(t,s)x(s)ds \tag{15}$$

where we suppose that for any fixed $T > 0$ then

$$\int_0^T |G(t,v)|dv \to 0 \text{ and } \int_0^t Q(s)ds \to -\infty \text{ as } t \to \infty. \tag{26}$$

Write (15) as

$$x(t) = x_0 e^{\int_0^t Q(s)ds} + \int_0^t e^{\int_s^t Q(u)du}(d/ds)\int_0^s G(s,v)x(v)dvds$$

$$= x_0 e^{\int_0^t Q(s)ds} + e^{\int_s^t Q(u)du}\int_0^s G(s,v)x(v)dv\Big|_0^t - \int_0^t e^{\int_s^t Q(u)du}Q(s)\int_0^s G(s,v)x(v)dvds$$

or

$$x(t) = x_0 e^{\int_0^t Q(s)ds} + \int_0^t G(t,v)x(v)dv - \int_0^t Q(s)e^{\int_s^t Q(u)du} \int_0^s G(s,v)x(v)dvds. \tag{27}$$

THEOREM 6. Suppose that (26) holds and that there is an $\alpha < 1$ with

$$\int_0^t [|G(t,v)| + |Q(v)|e^{\int_v^t Q(u)du} \int_0^v |G(v,u)|du]dv \le \alpha. \tag{28}$$

Then for each x_0 there is a unique bounded solution $x(t,0,x_0)$ of (14) which tends to 0 as $t \to \infty$.

We define a mapping from (27) and the proof proceeds just as before.

Recall that in Theorem 5 we needed $Q(t) < 0$.

EXAMPLE 6. Let

$$x' = (\sin t)x - (1.1)(n-1)\int_0^t (1+t-s)^{-n}x(s)ds$$

where $n > 4.2$. Then (26) and (28) are satisfied and the conclusion of Theorem 6 holds.

Proof. The equation can be written as

$$x' = (\sin t - 1.1)x + (d/dt)(1.1)\int_0^t (1+t-s)^{1-n}x(s)ds$$

with

$$G(t-s) = (1.1)(1+t-s)^{1-n}.$$

Then

$$\int_0^v |G(v,u)|du = \int_0^v (1.1)(1+v-u)^{1-n}du \le 1.1/(n-2).$$

A calculation then shows that (28) holds when

$$2(1.1)/(n-2) < 1$$

as required.

Compare this to the result using a Liapunov functional in Example 5. In fact, the fixed point condition of Theorem 6 yields a stability condition at least 5 times as good as that of the Liapunov condition (18) in Theorem 3.

EXAMPLE 7. If $n > 4$, then the zero solution of

$$x'(t) = (\sin t)x - (n-1)\int_0^t (1+t-s)^{-n}x(s)ds$$

is asymptotically stable.

To prove this, we mainly have to show that (28) is satisfied. We can write the equation as

$$x' = (\sin t - 1)x + (d/dt)\int_0^t (1+t-s)^{1-n}x(s)ds$$

so that

$$Q(t) = \sin t - 1 \text{ and } G(t-s) = (1+t-s)^{1-n}.$$

Substituting these values into (28) yields

$$\int_0^t (1+t-v)^{1-n}dv + \int_0^t \left[(1-\sin v)e^{\int_v^t (\sin u - 1)du}\left|\int_0^v (1+v-u)^{1-n}du\right|\right]dv$$

$$\leq [1/(n-2)]\left[1 + \int_0^t (1-\sin v)e^{-[t+\cos t]+[v+\cos v]}dv\right]$$

$$\leq ([1/(n-2)][1+1] < 1$$

provided that $4 < n$.

Liapunov theory is now more than one hundred years old and it has been a very fruitful area. The fixed point theory used in stability seems in its very early stages. The investigator will get better results by using both methods than by using only one of them.

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Appendix added Sept. 2003. This is work from fpver.tex I will have to change the R(t,s) and equation numbers. Also, I need to actually write out P. These are details for Theorem 6.

Proposition 2. Let M be defined in (29) and let (2), (12), (13), (14), (15), (24) hold. If P is defined in (26) and if $\phi \in M$ then $(P\phi)(t) \to 0$ as $t \to \infty$.

Proof. Let ϕ be fixed and let $q(t) = A(t)[h(\phi(t)) - \phi(t)]$ so that $q(t) \to 0$ as $t \to \infty$ by (24) and (2) since $|x - h(x)| \leq |K|x|$ implies that $h(x) \to 0$ as $x \to 0$. Then for $0 < T < t$ we have

$$\begin{aligned}\left|\int_0^t R(t,s)q(s)ds\right| &\leq \left|\int_0^T R(t,s)q(s)ds\right| + \left|\int_T^t R(t,s)q(s)ds\right| \\ &\leq \|q\|q\int_0^T |R(t,s)|ds + \int_T^t |R(t,s)|ds\|q\|_{[T,\infty)}.\end{aligned}$$

where $\|\cdot\|$ will represent the supremum on $[0,\infty)$, while $\|\cdot\|_{[T,\infty)}$ will represent the supremum on $[T,\infty)$. Use (5) and fix T so that for a given $\epsilon > 0$ we have $\|q\|_{[T,\infty)} < \frac{\epsilon}{2W}$, making the last term bounded by $\epsilon/2$. Then use (7) and take t so large that $\|q\| \int_0^T |R(t,s)|ds < \epsilon/2$.

Next, let $p(t) = g(x(t)) - x(t)$ and for $0 < T^* < T < t$ write

$$\begin{aligned}
&\int_0^t Q(s)e^{\int_s^t Q(u)du} \int_0^s G(s,v)p(v)dvds \\
&= \int_0^T Q(s)e^{\int_s^t Q(u)du} \int_0^s G(s,v)p(v)dvds \\
&\int_T^t Q(s)e^{\int_s^t Q(u)du} \int_0^{T^*} G(s,v)p(v)dvds \\
&+ \int_T^t Q(s)e^{\int_s^t Q(u)du} \int_{T^*}^s G(s,v)p(v)dvds.
\end{aligned}$$

To show that this can be made small, first pick T^* so that $\|p(v)\|_{[T^*,\infty)}$ is small; thus we can make the last term as small as we please since

$$\begin{aligned}
&\int_T^t Q(s)e^{\int_s^t Q(u)du} \int_{T^*}^s G(s,v)p(v)dvds \\
&\leq \int_T^t |Q(s)|e^{\int_s^t Q(u)du} \int_{T^*}^s |G(s,v)|dvds\|p(v)\|_{[T^*,\infty)} \\
&\leq \alpha\|p(v)\|_{[T^*,\infty)}.
\end{aligned}$$

Next, with T^* fixed, pick $T > T^*$ so that using (14) we can make the second term as small as we please. Finally, with T and T^* fixed and $t > T$ we use (12) and (15) to show tht the first term is bounded by

$$\int_0^T |Q(s)|e^{\int_s^T Q(u)du} \int_0^s |G(s,v)|dvds\|p\|e^{\int_T^t Q(u)ds} \leq \alpha\|p\|e^{\int_T^t Q(u)du}$$

which can be made small by taking t large. This completes the proof.