STABILITY, FIXED POINTS, AND INVERSES OF DELAYS

Leigh C. Becker and T. A. Burton

Abstract. The scalar equation (1) $x'(t) = -\int_{t-r(t)}^{t} a(t,s)g(x(s))\,ds$ with variable delay $r(t) \geq 0$ is investigated, where $t - r(t)$ is increasing and $xg(x) > 0$ ($x \neq 0$) in a neighborhood of $x = 0$. We find conditions for r, a, and g so that for a given continuous initial function ψ a mapping P for (1) can be defined on a complete metric space C_ψ and in which P has a unique fixed point. The end result is not only conditions for the existence and uniqueness of solutions of (1) but also for the stability of the zero solution. We also find conditions ensuring the zero solution is asymptotically stable by changing to an exponentially weighted metric on a closed subset of C_ψ. Finally, we parlay the methods for (1) into results for (2) $x'(t) = -\int_{t-r(t)}^{t} a(t,s)g(s,x(s))\,ds$ and (3) $x'(t) = -a(t)g(x(t-r(t)))$.

1. Introduction

For more than 100 years, Liapunov's direct method has been the main tool for investigating stability properties of ordinary, functional, and partial differential equations, often going under other names such as energy methods. Nevertheless, difficulties have persisted. This is one in a series of recent papers [5, 6, 7] looking into ways of circumventing those difficulties by means of fixed point theory, the focus of which has been on specific equations of historical importance and having significant applications. Other investigators have also studied stability using fixed point theory; for example, see [19] and [20].

Any investigation of the stability of an equation, or system of equations, using Liapunov's direct method first requires the construction of a suitable Liapunov function, or functional as the case may be. This can be an arduous task. But once a viable function(al) has been found, investigators may end up working with it for decades deriving more and more information about stability. A prime example of that is the Liapunov function

$$V(x,y) = y^2 + 2\int_0^x g(s)\,ds$$

for Liénard's system of equations

$$\begin{aligned} x' &= y \\ y' &= -f(x,y)y - g(x), \end{aligned} \tag{1.1}$$

a subject of study ever since a special case of (1.1) appeared in [17] in 1928.

In a parallel way, an investigation of the stability of an equation using fixed point theory involves the construction of a suitable fixed point mapping. That too can be an arduous task. But as has been the case

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with Liapunov function(al)s, once an appropriate mapping has been found, we believe that investigators will use it for decades in hopes of obtaining many new and interesting stability results.

Certain integral equations with variable delays have long been of keen interest to theoreticians and applied scientists alike. A prime example is the scalar equation

$$x'(t) = -\int_{t-r(t)}^{t} a(t,s)g(x(s))\,ds, \tag{1.2}$$

where $r(t)$ is a variable delay. One of our objectives is to construct a new fixed point mapping for (1.2) in order to establish sufficient conditions for the existence and uniqueness of solutions and to provide a framework for investigating stability. The mapping will depend on the initial condition for (1.2), so we presume there is a given continuous initial function at the outset of the construction. We begin by transforming (1.2) to a more tractable, but equivalent, equation, which we then invert to obtain an equivalent integral equation from which to define the mapping. Then we look for a complete metric space, which will be defined in terms of the initial function, so that the mapping is a contraction. Using Banach's Contraction Mapping Principle, we prove that a unique fixed point of the mapping exists, and in so doing, obtain a unique solution of (1.2) satisfying the given initial condition. A by-product of this procedure is a new stability result. In addition, by prudently choosing a different complete metric space, we obtain an asymptotic stability result for (1.2).

The fixed point methods that we use to obtain results for (1.2) will also facilitate our investigation of

$$x'(t) = -\int_{t-r(t)}^{t} a(t,s)g(s,x(s))\,ds \tag{1.3}$$

in Section 4, where $r(t) \geq 0$, $t - r(t)$ is strictly increasing, and $xg(t,x) > 0$ if $x \neq 0$ is sufficiently small. We are particularly interested in overcoming the difficulties presented by (i) a variable delay $r(t)$, (ii) a nonconvolution type of kernel $a(t,s)$, and (iii) a nonlinear function g that depends directly on t as well as on x. These aspects of (1.3) have been obstacles for other methods, such as Liapunov's direct method and transform methods. However, as we will see in Sections 3 and 4, a fixed point approach enables us to obtain stability results for (1.2) and (1.3) without assuming $r(t)$ is constant, or that it is bounded above by a constant, as has been the case in previous investigations of less general forms of (1.2) and related equations, such as (2.4) and (2.5). See Section 2. Finally, it is worth pointing out that our results allow $a(t,s)$ to change sign. Even though it is critical that g has the same sign as x near 0, it is a real wonder that some sort of stricture on the sign of $a(t,s)$ is not required.

2. Some history

Much of the literature has dealt with simpler forms of (1.2). The linear convolution equation

$$x'(t) = -\int_{t-r}^{t} a(t-s)x(s)\,ds, \tag{2.1}$$

with constant delay $r > 0$, was studied by Volterra [23] in 1928 with a biological application in mind. He suggested a way in which it might be possible to construct a Liapunov functional. In 1954, Brownell and

Ergen [1, 8] modeled the relation between the power generated by a circulating-fuel nuclear reactor and the reactor's temperature with a variant of equation (2.1). In 1963, Levin [15, p. 535] took Volterra's suggestion and constructed a Liapunov functional for

$$x'(t) = -\int_0^t a(t-s)g(x(s))\,ds \tag{2.2}$$

and revised it (jointly with Nohel [16, p. 35]) to include the nonconvolution equation

$$x'(t) = -\int_{t-r}^t a(t,s)g(x(s))\,ds. \tag{2.3}$$

Hale [12] studied these problems in terms of limit sets; they were central in the study of positive kernels initiated by Halanay [11] and MacCamy and Wong [18]. In 1993, Burton [4] presented a model for certain neural networks involving equations that are variants of (2.2) and (2.3).

Burton [6] recently constructed a fixed point mapping for (2.3) but at the time had not yet discovered how to handle an equation with variable delay and with g depending on both t and x. In this paper, we construct a fixed point mapping for (1.2)—and for (1.3) as well. In this regard, our work seems entirely new. The adjustments that have to made to the mapping for (2.3) to obtain comparable mappings for (1.2) and (1.3) and the techniques for doing so seem to be significant. Moreover, the stability results that we derive with fixed point mappings underscore their utility for investigating the stability of functional differential equations with variable delays.

Our work also paves the way for another mode of investigation. When an equation has no ordinary differential equation part (non-integral term), it may be very difficult to construct a fixed point mapping or Liapunov functional. However, by transforming the equation to a neutral form (cf. (3.2) and (3.3) below), we can make this task easier. Even though the modus operandi in this paper for obtaining stability results for an equation is through a fixed point mapping, we could also construct a Liapunov functional from the neutral form of the equation. This approach, which will be explored in a future paper, can lead to essentially different and complementary stability results.

The results of this paper will also be applied to

$$x'(t) = -a(t)g(x(t-r(t))) \tag{2.4}$$

in Section 5. An enormous number of papers have been written about (2.4). The most well-known of them involve the so-called $\frac{3}{2}$-stability theorems, which are found in recent works by Yoneyama [21], Krisztin [14], Graef et al. [9], and Gyori and Hartnung [10], just to name a few, but which date back to 1951 (cf. Yorke [22] and the references therein). One of the most basic of the $\frac{3}{2}$-stability theorems pertains to

$$x'(t) = -a(t)x(t-r(t)). \tag{2.5}$$

In a nutshell, it states that if there are positive constants α and q such that $0 \le a(t) \le \alpha$ and $0 \le r(t) \le q$ with $\alpha q \le \frac{3}{2}$, then the zero solution of (2.5) is uniformly stable. Moreover, the constant $\frac{3}{2}$ is the best possible upper bound in the sense that there are equations with unbounded solutions when $\alpha q > \frac{3}{2}$. In the Yoneyama paper, we find the less restrictive condition

$$\sup_{t\geq 0} \int_t^{t+q} a(s)\,ds \leq \frac{3}{2}. \tag{2.6}$$

This condition motivates the discussions in Sections 5 and 6.

Finally, we point out that the present work requires that $t - r(t)$ be strictly increasing. It is at the heart of the methods used in this paper; nevertheless, finding a way of relaxing this condition would be of interest.

3. Levin-Nohel's equation with variable delay

In this section, we provide the details for the procedure outlined in Section 1 for constructing a fixed point mapping for equation (1.2), which we hereafter refer to as (3.1), and for deriving conditions for existence, uniqueness, and stability of solutions. Specifically, we investigate the scalar equation

$$x'(t) = -\int_{t-r(t)}^{t} a(t,s)g(x(s))\,ds \tag{3.1}$$

for $t \geq 0$, where $r\colon [0,\infty) \to [0,\infty)$, $a\colon [0,\infty) \times [-r(0),\infty) \to R$, and $g\colon R \to R$ are continuous functions. We assume:

- (A_1) $r(t)$ is differentiable;
- (A_2) the function $t - r(t)\colon [0,\infty) \to [-r(0),\infty)$ is strictly increasing;
- (A_3) $t - r(t) \to \infty$ as $t \to \infty$.

By (A_2), $t - r(t) \geq -r(0)$ for all $t \geq 0$. In order that (3.1) have a solution for $t \geq 0$, an initial function must be specified on the interval $[-r(0), 0]$. By (A_2) and (A_3), $t_1 - r(t_1) = 0$ for a unique $t_1 \geq 0$ and $r(t) < t$ for $t > t_1$. Assumption (A_3) is crucial in asymptotic arguments. Finally, we note from (A_2) that $t - r(t)$ is 1–1; so it has an inverse which we denote hereafter by $f(t)$. For ease of communication, especially oral, we refer to $f(t)$ as the "inverse of the delay" even though strictly speaking $r(t)$ is the delay.

The absence of other terms in (3.1) makes it difficult to obtain a fixed point mapping. To make (3.1) more tractable for this purpose, we transform it into a neutral functional differential equation of the form

$$x'(t) = \lambda(t)g(x(t)) + \frac{d}{dt}h(t, g(\cdot)). \tag{3.2}$$

The motivation for this is that (3.2) looks like a linear equation when $g(x) = x$ except for the nonhomogeneous term. This suggests that the variation of parameters formula might somehow be used, at least for the linear case, to obtain an integral equation from which we might be able to define a fixed point mapping.

Generally speaking, finding a transformation that does not change the basic structure and properties of an equation tends to be difficult. Nevertheless, having such a transformation is central to the methods employed in this paper. The proof of the following theorem shows how to transform (3.1) into the form of (3.2), the result of which is (3.3) below. Although it is easy to verify that (3.3) reduces to (3.1) by merely carrying out the differentiation, the proof delineates the method that was used to obtain (3.3) in the first place. This method has been effectively employed in the past to obtain stability results for (2.3) in [6] and for Volterra equations in [2, p. 133] and [3, p. 123].

Theorem 3.1. *Let the function $f\colon [-r(0),\infty) \to [0,\infty)$ denote the inverse of $t - r(t)$. Then (3.1) is equivalent to*

$$x'(t) = -\int_t^{f(t)} a(u,t)\,du\,g(x(t)) + \frac{d}{dt}\int_{t-r(t)}^{t}\int_t^{f(s)} a(u,s)\,du\,g(x(s))\,ds. \tag{3.3}$$

Proof. Since the objective is to make (3.1) look like (3.2), we seek a function G so that (3.1) is equivalent to

$$x'(t) = -G(t,t)g(x(t)) + \frac{d}{dt}\int_{t-r(t)}^{t} G(t,s)g(x(s))\,ds. \tag{3.4}$$

Differentiating the integral, (3.4) becomes

$$\begin{aligned} x'(t) = &-G(t,t)g(x(t)) + G(t,t)g(x(t)) \\ &- G(t,t-r(t))g(x(t-r(t)))(1-r'(t)) \\ &+ \int_{t-r(t)}^{t} \frac{\partial G(t,s)}{\partial t} g(x(s))\,ds. \end{aligned}$$

It follows that (3.4) is equivalent to (3.1) provided G satisfies the following conditions:

$$G(t,t-r(t)) = 0 \tag{3.5}$$

and

$$\frac{\partial G(t,s)}{\partial t} = -a(t,s). \tag{3.6}$$

Now (3.6) implies that

$$G(t,s) = -\int_0^t a(u,s)\,du + \phi(s), \tag{3.7}$$

for some function ϕ. And for (3.5) to hold, (3.7) must satisfy

$$G(t,t-r(t)) = -\int_0^t a(u,t-r(t))\,du + \phi(t-r(t)) = 0. \tag{3.8}$$

Since f denotes the inverse of $t-r(t)$, we can replace the upper limit t with $f(t-r(t))$. Then (3.8) becomes

$$\phi(t-r(t)) = \int_0^{f(t-r(t))} a(u,t-r(t))\,du. \tag{3.9}$$

Finally, if (3.9) is to hold for all differentiable functions $r\colon [0,\infty) \to [0,\infty)$, it must in particular hold for $r(t) \equiv 0$. Thus,

$$\phi(s) = \int_0^{f(s)} a(u,s)\,du. \tag{3.10}$$

Substituting this into (3.7), we obtain

$$G(t,s) = -\int_0^t a(u,s)\,du + \int_0^{f(s)} a(u,s)\,du = \int_t^{f(s)} a(u,s)\,du \tag{3.11}$$

for $t \ge 0$ and $t - r(t) \le s \le t$. (For future reference, we point out that $f(s) \ge t$ [cf. Remark 3.1 below].)

In short, the function G defined by (3.11) satisfies both (3.5) and (3.6). Consequently, (3.4), which is (3.3) when written out fully, is equivalent to (3.1). □

Remark 3.1. The function f in (3.11) is strictly increasing since it is the inverse of $t - r(t)$ which is strictly increasing by (A_2). So for $t - r(t) \le s$, $f(s) \ge f(t - r(t))$. Thus in (3.11), $f(s) \ge t$.

The linear equation. Theorem 3.1 is a crucial result because the integrated form of (3.3) is the integral equation from which a fixed point mapping can be defined. The precise details in carrying out the integration depend on g. First, we tackle the simplest case: $g(x) = x$. Then (3.4), the abbreviated form of (3.3), simplifies to

$$x'(t) = -G(t,t)x(t) + \frac{d}{dt}\int_{t-r(t)}^{t} G(t,s)x(s)\,ds, \tag{3.12}$$

which is in fact the linear equation

$$x'(t) = -\int_{t-r(t)}^{t} a(t,s)x(s)\,ds. \tag{3.1$'$}$$

We begin by inverting (3.12), the result of which is (3.13) in the next theorem. From now on, $\psi(t)$ denotes any real-valued continuous function with domain $[-r(0), 0]$.

Theorem 3.2. *If $x(t)$ is a solution of (3.1′) on an interval $[0,T)$ and satisfies the initial condition $x(t) = \psi(t)$ for $-r(0) \le t \le 0$, then $x(t)$ is a solution of the integral equation*

$$\begin{aligned} x(t) = {} & e^{-\int_0^t G(s,s)\,ds}\psi(0) - e^{-\int_0^t G(u,u)\,du}\int_{-r(0)}^{0} G(0,u)\psi(u)\,du \\ & + \int_{t-r(t)}^{t} G(t,u)x(u)\,du - \int_0^t e^{-\int_s^t G(u,u)\,du}\,G(s,s)\int_{s-r(s)}^{s} G(s,u)x(u)\,du\,ds \end{aligned} \tag{3.13}$$

on $[0,T)$. Conversely, if a continuous function $x(t)$ is equal to $\psi(t)$ for $-r(0) \le t \le 0$ and is a solution of (3.13) on an interval $[0,\tau)$, then $x(t)$ is a solution of (3.1′) on $[0,\tau)$.

Proof. Applying the variation of parameters formula to (3.12), or multiplying it by the factor $e^{\int_0^t G(s,s)\,ds}$ and integrating from 0 to any $t \in [0,T)$, we obtain

$$x(t) = e^{-\int_0^t G(s,s)\,ds}\psi(0) + \int_0^t e^{-\int_s^t G(u,u)\,du}\,\frac{d}{ds}\int_{s-r(s)}^{s} G(s,u)x(u)\,du\,ds.$$

Then an integration by parts yields the integral equation

$$\begin{aligned} x(t) = {} & e^{-\int_0^t G(s,s)\,ds}\psi(0) + e^{-\int_s^t G(u,u)\,du}\int_{s-r(s)}^{s} G(s,u)x(u)\,du\,\Big|_0^t \\ & - \int_0^t e^{-\int_s^t G(u,u)\,du}\,G(s,s)\int_{s-r(s)}^{s} G(s,u)x(u)\,du\,ds, \end{aligned}$$

which simplifies to (3.13). Conversely, suppose that a continuous function $x(t)$ is equal to $\psi(t)$ on $[-r(0), 0]$ and satisfies (3.13) on an interval $[0,\tau)$. Then it is differentiable on $[0,\tau)$. Differentiating (3.13) with the aid of Leibniz's rule, we obtain (3.12). □

In the next theorem, a mapping will be defined directly from (3.13). A fixed point of that map will be a solution of (3.13) and, hence, of (3.1′) by Theorem 3.2. Because we are seeking stability results, we want solutions to be bounded, one of which is the desired fixed point. Thus, we need the mapping defined by

(3.13) to map bounded functions into bounded functions. For that reason, we assume constants $k \geq 0$ and $\alpha > 0$ exist so that

$$-\int_0^t G(s,s)\,ds \leq k \tag{3.14}$$

and

$$\int_{t-r(t)}^t |G(t,u)|\,du + \int_0^t e^{-\int_s^t G(u,u)du}\,|G(s,s)| \int_{s-r(s)}^s |G(s,u)|\,du\,ds \leq \alpha \tag{3.15}$$

for $t \geq 0$. A natural place to look for these fixed point mappings is the Banach space $(C, \|\cdot\|)$ of real-valued bounded continuous functions on $[-r(0), \infty)$ with the supremum norm $\|\cdot\|$; that is, for $\phi \in C$,

$$\|\phi\| := \sup\{\, |\phi(t)| : t \in [-r(0), \infty)\,\}.$$

In other words, we carry out our investigations in the complete metric space (C, ρ), where ρ denotes the supremum (uniform) metric: for $\phi_1, \phi_2 \in C$, $\rho(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|$. For a given continuous initial function $\psi\colon [-r(0), 0] \to R$, define the set $C_\psi \subset C$ by

$$C_\psi := \{\, \phi\colon [-r(0), \infty) \to R \mid \phi \in C,\ \phi(t) = \psi(t) \text{ for } -r(0) \leq t \leq 0\,\}.$$

We will also use $\|\cdot\|$ to denote the supremum norm of an initial function—it will be obvious from the function to which it is applied whether the norm denotes the supremum on $[-r(0), 0]$ or on $[-r(0), \infty)$. Finally, note that (C_ψ, ρ) is itself a complete metric space since C_ψ is a closed subset of C.

Subject to conditions (3.14) and (3.15), the mapping suggested by (3.13) and explicitly defined in the next theorem maps C_ψ into itself.

Theorem 3.3. *Let G be defined by* (3.11) *where f is the inverse of $t - r(t)$. Let P be a mapping on C_ψ defined as follows: for $\phi \in C_\psi$,*

$$(P\phi)(t) = \psi(t)$$

if $-r(0) \leq t \leq 0$, while

$$\begin{aligned}(P\phi)(t) = {}& e^{-\int_0^t G(s,s)\,ds}\psi(0) - e^{-\int_0^t G(u,u)\,du}\int_{-r(0)}^0 G(0,u)\psi(u)\,du \\ &+ \int_{t-r(t)}^t G(t,u)\phi(u)\,du - \int_0^t e^{-\int_s^t G(u,u)\,du}\,G(s,s)\int_{s-r(s)}^s G(s,u)\phi(u)\,du\,ds\end{aligned} \tag{3.16}$$

if $t > 0$. If (3.14) *and* (3.15) *hold, then $P\colon C_\psi \to C_\psi$.*

Proof. For $\phi \in C_\psi$, $P\phi$ is continuous and agrees with ψ on $[-r(0), 0]$ by virtue of the definition of P. For $t > 0$, it follows from (3.14) and (3.15) that

$$|(P\phi)(t)| \leq e^k|\psi(0)| + e^k\int_{-r(0)}^0 |G(0,u)||\psi(u)|\,du + \alpha\|\phi\|.$$

Consequently,

$$\|P\phi\| \leq e^k \|\psi\| \left(1 + \int_{-r(0)}^{0} |G(0,u)|\, du\right) + \alpha\|\phi\| < \infty. \tag{3.17}$$

Thus, $P\phi \in C_\psi$. $\square$

Theorems 3.2 and 3.3 give us the means to prove an existence and uniqueness result. Under the conditions stated in the next theorem, we prove that for every continuous initial function $\psi\colon [-r(0), 0] \to R$ there exists a unique continuous function x that is the solution of (3.1$'$) on $[0, \infty)$ and which satisfies the initial condition $x(t) = \psi(t)$ on $[-r(0), 0]$. We also prove every such solution is bounded. Furthermore, we prove the zero solution of (3.1$'$) has the property defined by the following statement:

Definition 3.1. The zero solution of (3.1$'$) is said to be *stable at* $t = 0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\psi\colon [-r(0), 0] \to (-\delta, \delta)$ implies $|x(t)| < \epsilon$ for $t \geq -r(0)$.

Finally, we give a condition that drives all solutions to zero.

Theorem 3.4. *Suppose constants $k \geq 0$ and $\alpha \in (0, 1)$ exist such that (3.14) and (3.15) hold for $t \geq 0$. Then for each continuous function $\psi\colon [-r(0), 0] \to R$, there is a unique continuous function $x\colon [-r(0), \infty) \to R$ with $x(t) = \psi(t)$ on $[-r(0), 0]$ that satisfies (3.1$'$) on $[0, \infty)$. Moreover, $x(t)$ is bounded on $[-r(0), \infty)$. Furthermore, the zero solution of (3.1$'$) is stable at $t = 0$. If, in addition,*

$$\int_0^t G(s, s)\, ds \to \infty \tag{3.18}$$

as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$.

Proof. Let $\psi\colon [-r(0), 0] \to R$ be a continuous function. We first show that there is a unique continuous function x that satisfies (3.1$'$) on $[0, \infty)$ and the initial condition: $x(t) = \psi(t)$ on $[-r(0), 0]$. The initial function ψ defines a space C_ψ. The mapping P, defined in Theorem 3.3, has a unique fixed point since it is a contraction on the complete metric space (C_ψ, ρ). To see this, consider any $\phi, \eta \in C_\psi$. For $t > 0$,

$$\begin{aligned} |(P\phi)(t) - (P\eta)(t)| &\leq \int_{t-r(t)}^{t} |G(t,u)|\, |\phi(u) - \eta(u)|\, du \\ &\quad + \int_0^t e^{-\int_s^t G(u,u)\, du} |G(s,s)| \int_{s-r(s)}^{s} |G(s,u)|\, |\phi(u) - \eta(u)|\, du\, ds \\ &\leq \left(\int_{t-r(t)}^{t} |G(t,u)|\, du + \int_0^t e^{-\int_s^t G(u,u)du} |G(s,s)| \int_{s-r(s)}^{s} |G(s,u)|\, du\, ds\right) \|\phi - \eta\|. \end{aligned} \tag{3.19}$$

By (3.15), $|(P\phi)(t) - (P\eta)(t)| \leq \alpha\|\phi - \eta\|$ for all $t > 0$. Trivially, this inequality also holds on $[-r(0), 0]$; consequently, $\|P\phi - P\eta\| \leq \alpha\|\phi - \eta\|$. Therefore, the mapping P is a contraction since $\alpha < 1$. By Banach's Contraction Mapping Principle, P has a unique fixed point x. The fixed point x is a bounded continuous function as $x \in C_\psi$. By Theorem 3.2, it is a solution of (3.1$'$) on $[0, \infty)$. It follows that x is the only bounded continuous function satisfying (3.1$'$) on $[0, \infty)$ and the initial condition.

Under the stated hypotheses, (3.1′) does not have any unbounded continuous solutions. Suppose to the contrary that $\tilde{x}$ is such a solution with $\tilde{x}(t) = \psi(t)$ on $[-r(0), 0]$ for some continuous function ψ. By the previous argument, there is also a unique bounded continuous solution x agreeing with ψ on $[-r(0), 0]$. Then on any interval $[-r(0), b]$ on which $\tilde{x}$ is defined, it follows from the first inequality of (3.19) and from (3.15) that

$$|(Px)(t) - (P\tilde{x})(t)| \le \alpha \|x - \tilde{x}\|_{[-r(0),b]},$$

where $\|x - \tilde{x}\|_{[-r(0),b]}$ denotes the supremum of $|x(t) - \tilde{x}(t)|$ on $[-r(0), b]$. Thus,

$$\|Px - P\tilde{x})\|_{[-r(0),b]} \le \alpha \|x - \tilde{x}\|_{[-r(0),b]}.$$

By Theorem 3.2, $Px = x$ and $P\tilde{x} = \tilde{x}$; so

$$\|x - \tilde{x}\|_{[-r(0),b]} \cdot (1 - \alpha) \le 0. \tag{3.20}$$

Therefore, $\tilde{x}(t) \equiv x(t)$ on $[-r(0), b]$ since $\alpha < 1$. This implies, contrary to the supposition, that $\tilde{x}$ is bounded.

The stability assertion is an immediate consequence of (3.17). If $x(t)$ is a solution with the initial function ψ, then

$$(1 - \alpha)\|x\| \le e^k \|\psi\| \left(1 + \int_{-r(0)}^{0} |G(0, u)|\, du\right).$$

Then there clearly exists a $\delta > 0$ for each $\epsilon > 0$ such that $|x(t)| < \epsilon$ for all $t \ge -r(0)$ if $\|\psi\| < \delta$.

In order to prove that solutions of (3.1′) tend to zero when (3.18) holds, we examine the mapping P when it is applied to functions $\phi \in C_\psi$ that tend to 0 as $t \to \infty$. Let C_ψ^0 denote this particular set; that is,

$$C_\psi^0 := \{\, \phi\colon [-r(0), \infty) \to R \mid \phi \in C,\ \phi(t) = \psi(t) \text{ for } -r(0) \le t \le 0,\ \phi(t) \to 0 \text{ as } t \to \infty \,\}. \tag{3.21}$$

Since C_ψ^0 is a closed subset of C_ψ and (C_ψ, ρ) is complete, the metric space (C_ψ^0, ρ) is also complete. We begin by proving that $(P\phi)(t) \to 0$ as $t \to \infty$ when $\phi \in C_\psi^0$. By (3.15) and (3.16), we have

$$|(P\phi)(t)| \le e^{-\int_0^t G(s,s)\, ds} \left(|\psi(0)| + \int_{-r(0)}^{0} |G(0, u)||\psi(u)|\, du\right) + \alpha \|\phi\|_{[t-r(t),t]} + |\Omega(t)|$$

for $t > 0$, where $\Omega(t)$ denotes the last term of (3.16). Now consider the asymptotic behavior of each of the above terms as $t \to \infty$. The first term tends to 0 by (3.18). Because $\phi(t) \to 0$ as $t \to \infty$, the same is true of the second term since $t - r(t) \to \infty$ by (A_3). Now we prove that $\Omega(t) \to 0$—much in the same way that one can prove that the convolution of an $L^1[0, \infty)$ function with a function tending to 0 as $t \to \infty$ is itself a function tending to 0. For any $T \in (0, t)$,

$$\begin{aligned}
|\Omega(t)| \le &\int_0^T |G(s, s)| e^{-\int_s^T G(u,u)\, du} \int_{s-r(s)}^{s} |G(s, u)|\, du\, ds \cdot \|\phi\|\, e^{-\int_T^t G(u,u)\, du} \\
&+ \int_T^t |G(s, s)| e^{-\int_s^t G(u,u)\, du} \int_{s-r(s)}^{s} |G(s, u)|\, du\, ds \cdot \|\phi\|_{[T-r(T),\infty)} \\
\le &\ \alpha \|\phi\| e^{-\int_T^t G(u,u)\, du} + \alpha \|\phi\|_{[T-r(T),\infty)}.
\end{aligned}$$

For a given $\epsilon > 0$, there exists a $T > 0$ such that $\|\phi\|_{[T-r(T),\infty)} < \epsilon/2\alpha$ by (A$_3$). For this T, there exists $\tau \geq T$ such that $t > \tau$ implies that $\|\phi\| e^{-\int_T^t G(u,u)\,du} < \epsilon/2\alpha$. Thus, for every $\epsilon > 0$, there exists a $\tau > 0$ such that $t > \tau$ implies that $|\Omega(t)| < \epsilon$. Consequently, $(P\phi)(t) \to 0$ as $t \to \infty$. So, $P\colon C_\psi^0 \to C_\psi^0$. Therefore, P is a contraction on C_ψ^0 with a unique fixed point x. By Theorem 3.2, x is a solution of (3.1$'$) on $[0,\infty)$. With the same kind of reasoning that led to (3.20), we conclude that $x(t)$ is the only continuous solution of (3.1$'$) agreeing with the initial function ψ. As $x \in C_\psi^0$, $x(t) \to 0$ as $t \to \infty$. $\square$

Remark 3.2. It is evident from (3.16) and (3.19) that whether or not the mapping P is a contraction on (C_ψ, ρ) depends on the part of $P\phi$ involving

$$(I\phi)(t) := \int_{t-r(t)}^t G(t,u)\phi(u)\,du - \int_0^t e^{-\int_s^t G(u,u)\,du}\, G(s,s) \int_{s-r(s)}^s G(s,u)\phi(u)\,du\,ds.$$

Note that if $G(s,s) \geq 0$ for $s \geq 0$, then

$$|(I\phi)(t)| \leq 2 \sup_{t\geq 0} \int_{t-r(t)}^t |G(t,u)||\phi(u)|\,du$$

for $t \geq 0$. This inequality is sharp (cf. Section 6) in the sense that it will not hold in general if the factor 2 is replaced by anything smaller. Also note that (3.15) reduces to

$$2 \sup_{t\geq 0} \int_{t-r(t)}^t |G(t,u)|\,du \leq \alpha \tag{3.22}$$

when $G(s,s) \geq 0$ insofar as (3.22) implies (3.15). Moreover, the "2" can be dropped when $G(s,s) \equiv 0$ because (3.15) then simplifies to

$$\int_{t-r(t)}^t |G(t,u)|\,du \leq \alpha \tag{3.22$'$}$$

for $t \geq 0$. By (3.19), the mapping P is a contraction if $\alpha < 1$.

The nonlinear equation. We now turn our attention back to (3.1), or equivalently (3.3). In terms of the function G defined in (3.11), equation (3.3) is

$$x'(t) = -G(t,t)g(x(t)) + \frac{d}{dt}\int_{t-r(t)}^t G(t,s)g(x(s))\,ds. \tag{3.4}$$

A nonlinear g imposes difficulties that do not exist for $g(x) = x$. However, our investigation above of (3.12) sheds some light on how to proceed. For instance, observe that the variation of parameters formula can still be used if (3.4) is rewritten in the form

$$x'(t) = -G(t,t)x(t) + G(t,t)[x(t) - g(x(t))] + \frac{d}{dt}\int_{t-r(t)}^t G(t,s)g(x(s))\,ds. \tag{3.23}$$

Of course, this reduces to (3.12) when $g(x) = x$. As we will see in the proof of Theorem 3.6, it is crucial to treat $x - g(x)$ as a single quantity; for that will allow us to handle it in the same way as we did $g(x)$.

Since construction of a fixed point mapping for (3.1) is the objective, we invert (3.23) with the variation of parameters formula. The result is (3.24) below. Aside from an additional term, the details are exactly like those in the proof of Theorem 3.2.

Theorem 3.5. *Let $\psi\colon [-r(0),0] \to R$ be a given continuous initial function. If $x(t)$ is a solution of* (3.1) *on an interval $[0,T)$ with $x(t)=\psi(t)$ for $-r(0)\le t\le 0$, then $x(t)$ is a solution of the integral equation*

$$\begin{aligned} x(t) &= e^{-\int_0^t G(s,s)ds}\psi(0) - e^{-\int_0^t G(u,u)du}\int_{-r(0)}^0 G(0,u)g(\psi(u))\,du \\ &\quad + \int_{t-r(t)}^t G(t,u)g(x(u))\,du - \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)\int_{s-r(s)}^s G(s,u)g(x(u))\,du\,ds \\ &\quad + \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)[x(s)-g(x(s))]\,ds. \end{aligned} \tag{3.24}$$

Conversely, if a continuous function $x(t)$ is equal to $\psi(t)$ for $-r(0)\le t\le 0$ and is a solution of (3.24) *on an interval $[0,\tau)$, then $x(t)$ is a solution of* (3.1) *on $[0,\tau)$.*

Equation (3.24) specifies how to define a mapping for the nonlinear equation (3.1) paralleling the mapping (3.16) for the linear equation (3.1′). It is the mapping P defined by (3.26) below. The challenge now is to find the appropriate space and conditions so that P is a contraction mapping. First, consider the difference between the images of two points in C_ψ mapped by P. For $\phi,\eta\in C_\psi$,

$$\begin{aligned} |(P\phi)(t)-(P\eta)(t)| &\le \int_{t-r(t)}^t |G(t,u)|\,|g(\phi(u))-g(\eta(u))|\,du \\ &\quad + \int_0^t e^{-\int_s^t G(u,u)du}|G(s,s)|\int_{s-r(s)}^s |G(s,u)|\,|g(\phi(u))-g(\eta(u))|\,du\,ds \\ &\quad + \int_0^t e^{-\int_s^t G(u,u)du}|G(s,s)|\,|\phi(s)-g(\phi(s))-[\eta(s)-g(\eta(s))]|\,ds. \end{aligned}$$

Suppose we impose a Lipschitz condition on g and let L denote a common Lipschitz constant for it and $x-g(x)$. If we also add the condition $G(t,t)\ge 0$, then the foregoing simplifies to

$$\begin{aligned} |(P\phi)(t)-(P\eta)(t)| \le \bigg(&\int_{t-r(t)}^t |G(t,u)|\,du + \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)\int_{s-r(s)}^s |G(s,u)|\,du\,ds \\ &+ \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)\,ds\bigg)L\|\phi-\eta\|. \end{aligned}$$

If $g(x)=x$, the last integral drops out and $L=1$. In that case, we obtain a practical contraction condition, namely (3.15), which reduces to (3.22) since $G(t,t)\ge 0$. But if g is nonlinear, we do not have a viable contraction condition because of the last integral—generally L will not be sufficiently small to offset that $\int_0^t e^{-\int_s^t G(u,u)du}G(s,s)\,ds\approx 1$ for relatively large t. We resolve this in Theorem 3.6 by changing from the supremum norm to one with exponential weight that is defined on a space C_ψ^l defined as follows: for a given continuous function $\psi\colon[-r(0),0]\to[-l,l]$, let

$$C_\psi^l := \{\,\phi\colon[-r(0),\infty)\to R \mid \phi\in C,\ \phi(t)=\psi(t)\text{ for }-r(0)\le t\le 0,\ |\phi(t)|\le l\,\}. \tag{3.25}$$

Theorem 3.6. *Let a mapping P be defined on a given space C_ψ^l as follows: for $\phi\in C_\psi^l$,*

$$(P\phi)(t)=\psi(t)\quad\text{for } -r(0)\le t\le 0;$$

while for $t > 0$,

$$
\begin{aligned}
(P\phi)(t) &= e^{-\int_0^t G(s,s)ds}\psi(0) - e^{-\int_0^t G(u,u)du}\int_{-r(0)}^{0} G(0,u)g(\psi(u))\,du \\
&\quad + \int_{t-r(t)}^{t} G(t,u)g(\phi(u))\,du - \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)\int_{s-r(s)}^{s} G(s,u)g(\phi(u))\,du\,ds \\
&\quad + \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)[\phi(s) - g(\phi(s))]\,ds.
\end{aligned}
\tag{3.26}
$$

Suppose

(i) *a constant $l > 0$ exists such that g satisfies a Lipschitz condition on $[-l, l]$, and*

(ii) *$G(t,t) \geq 0$ for $t \geq 0$.*

Then there is a metric d for C_ψ^l such that

(iii) *the metric space (C_ψ^l, d) is complete, and*

(iv) *P is a contraction on (C_ψ^l, d) if P maps C_ψ^l into itself.*

Proof. We must find a metric d for C_ψ^l so that the space (C_ψ^l, d) is not only complete but also one on which P is a contraction. For $s \in [-r(0), \infty)$, let

$$H(s) := \int_s^{f(s)} |a(u,s)|\,du,$$

where, as in Theorem 3.1, $f(s)$ denotes the inverse of $s - r(s)$. By (A_2), $s - r(s)$ is strictly increasing; hence so is $f(s)$. Consequently, $H(s) \geq 0$ since $s = f(s - r(s)) \leq f(s)$. Let L denote a common Lipschitz constant for $g(x)$ and $x - g(x)$ on $[-l, l]$. For $t \in [0, \infty)$ and a constant $\kappa > 3$, define $h\colon [0,\infty) \to [0,\infty)$ by

$$h(t) := \kappa L \int_0^t [G(u,u) + H(u)]\,du.$$

Now let S be the space of all continuous functions $\phi\colon [-r(0), \infty) \to R$ such that

$$|\phi|_h := \sup\{\, |\phi(t)|e^{-h(t)} \colon t \in [-r(0), \infty)\,\} < \infty.$$

Then $(S, |\cdot|_h)$ is a Banach space, which can be verified with Cauchy's criterion for uniform convergence. Thus, (S, d) is a complete metric space, where d denotes the induced metric: $d(\phi, \eta) = |\phi - \eta|_h$ for $\phi, \eta \in S$. With this metric, C_ψ^l is a closed subset of S. Therefore, the metric space (C_ψ^l, d) is complete, which concludes the proof of (iii).

As for (iv), suppose $P\colon C_\psi^l \to C_\psi^l$. Let $\phi, \eta \in C_\psi^l$. Since g satisfies a Lipschitz condition on $[-l, l]$, it follows that

$$
\begin{aligned}
&|(P\phi)(t) - (P\eta)(t)|e^{-h(t)} \\
&\qquad \leq \int_{t-r(t)}^{t} |G(t,u)|\, L|\phi(u) - \eta(u)|e^{-h(t)}\,du \\
&\qquad + \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)\int_{s-r(s)}^{s} |G(s,u)|\, L|\phi(u) - \eta(u)|e^{-h(t)}\,du\,ds \\
&\qquad + \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)L|\phi(s) - \eta(s)|e^{-h(t)}\,ds
\end{aligned}
\tag{3.27}
$$

for $t > 0$. For $u \le s \le t$,

$$-h(t) = -h(u) + [h(u) - h(t)] = -h(u) - \kappa L \int_u^t [G(v,v) + H(v)]\, dv$$
$$\le -h(u) - \kappa L \int_u^t H(v)\, dv \le -h(u) - \kappa L \int_u^s H(v)\, dv.$$

Using these inequalities in the first two integrals of (3.27) and

$$-h(t) \le -h(s) - \kappa L \int_s^t G(v,v)\, dv$$

in the third integral, we obtain

$$\begin{aligned}
&|(P\phi)(t) - (P\eta)(t)|e^{-h(t)} \\
&\quad\le \int_{t-r(t)}^t e^{-\kappa L \int_u^t H(v)dv} |G(t,u)|\, L|\phi(u) - \eta(u)|e^{-h(u)}\, du \\
&\quad\quad+ \int_0^t e^{-\int_s^t G(u,u)du} G(s,s) \int_{s-r(s)}^s e^{-\kappa L \int_u^s H(v)dv} |G(s,u)|\, L|\phi(u) - \eta(u)|e^{-h(u)}\, du\, ds \\
&\quad\quad+ \int_0^t e^{-(1+\kappa L)\int_s^t G(u,u)du} G(s,s) L|\phi(s) - \eta(s)|e^{-h(s)}\, ds.
\end{aligned}$$

In view of Remark 3.1,

$$|G(t,u)| \le \int_t^{f(u)} |a(v,u)|\, dv \le \int_u^{f(u)} |a(v,u)|\, dv = H(u)$$

for $t - r(t) \le u \le t$. Consequently,

$$\begin{aligned}
&|(P\phi)(t) - (P\eta)(t)|e^{-h(t)} \\
&\quad\le \Bigg[\int_{t-r(t)}^t e^{-\kappa L \int_u^t H(v)dv} H(u)\, du \\
&\quad\quad+ \int_0^t e^{-\int_s^t G(u,u)du} G(s,s) \int_{s-r(s)}^s e^{-\kappa L \int_u^s H(v)dv} H(u)\, du\, ds \\
&\quad\quad+ \int_0^t e^{-(1+\kappa L)\int_s^t G(u,u)du} G(s,s)\, ds\Bigg] L|\phi - \eta|_h.
\end{aligned} \tag{3.28}$$

Carrying out the integrations in (3.28), we find

$$\int_{t-r(t)}^t e^{-\kappa L \int_u^t H(v)dv} H(u)\, du \le \frac{1}{\kappa L} \quad \text{and} \quad \int_0^t e^{-b\int_s^t G(u,u)du} G(s,s)\, ds \le \frac{1}{b} \tag{3.29}$$

for $b = 1$ and $b = 1 + \kappa L$. As a result,

$$|(P\phi)(t) - (P\eta)(t)|e^{-h(t)} \le \left[\frac{1}{\kappa L} + \frac{1}{\kappa L} + \frac{1}{\kappa L + 1}\right] L|\phi - \eta|_h \le \frac{3}{\kappa}|\phi - \eta|_h \tag{3.30}$$

for all $t > 0$. In fact, this holds for all $t \ge -r(0)$ since $P\phi$ and $P\eta$ agree on $[-r(0), 0]$. Therefore, $d(P\phi, P\eta) \le \frac{3}{\kappa} d(\phi, \eta)$. As $\kappa > 3$, we conclude P is a contraction on (C_ψ^l, d). $\square$

We now establish the existence and uniqueness of solutions by showing that $P\colon C_\psi^l \to C_\psi^l$ if $\|\psi\|$ is sufficiently small.

Theorem 3.7. *Suppose g and G satisfy conditions* (i) *and* (ii) *in Theorem 3.6 and further suppose that*

(i) *g is odd and strictly increasing on $[-l, l]$;*

(ii) *$x - g(x)$ is nondecreasing on $[0, l]$;*

(iii) *there is an $\alpha \in (0,1)$ such that $2\int_{t-r(t)}^{t} |G(t,u)|\,du \le \alpha$ for $t \ge 0$.*

Then a $\delta \in (0, l)$ exists such that for each continuous function $\psi\colon [-r(0), 0] \to (-\delta, \delta)$, there is a unique continuous function $x\colon [-r(0), \infty) \to R$ with $x(t) = \psi(t)$ on $[-r(0), 0]$ that is a solution of (3.1) *on* $[0, \infty)$. *Moreover, $x(t)$ is bounded by l on $[-r(0), \infty)$. Furthermore, the zero solution of* (3.1) *is stable at $t = 0$.*

Proof. Let $\psi\colon [-r(0), 0] \to (-\delta, \delta)$ be a continuous function, where $\delta > 0$ satisfies the inequality

$$\delta + g(\delta)\int_{-r(0)}^{0} |G(0,u)|\,du \le (1-\alpha)g(l). \tag{3.31}$$

Such a δ exists since $g(0) = 0$ and g is (uniformly) continuous on $[-l, l]$ by the Lipschitz condition. Note that (3.31) implies $\delta < l$ since $g(l) \le l$ by (ii). Thus, $|\psi(t)| < l$ for $-r(0) \le t \le 0$. For such a ψ, $P\colon C_\psi^l \to C_\psi^l$. To see this, consider (3.26) for an arbitrary $\phi \in C_\psi^l$. Conditions (i) and (ii) imply

$$\begin{aligned}
|(P\phi)(t)| \le \delta + g(\delta)\int_{-r(0)}^{0} |G(0,u)|\,du + g(l)\int_{t-r(t)}^{t} |G(t,u)|\,du \\
+ g(l)\int_0^t e^{-\int_s^t G(u,u)du} G(s,s)\int_{s-r(s)}^{s} |G(s,u)|\,du\,ds \\
+ (l - g(l))\int_0^t e^{-\int_s^t G(u,u)du} G(s,s)\,ds
\end{aligned}$$

for $t > 0$. By (iii), (3.29), and (3.31), this reduces to

$$\begin{aligned}
|(P\phi)(t)| &\le \delta + g(\delta)\int_{-r(0)}^{0} |G(0,u)|\,du + \alpha g(l) + l - g(l) \\
&\le (1-\alpha)g(l) + (\alpha - 1)g(l) + l = l.
\end{aligned}$$

Hence, $|(P\phi)(t)| \le l$ for $t \in [-r(0), \infty)$ since $|(P\phi)(t)| = |\psi(t)| < l$ for $-r(0) \le t \le 0$. Therefore, $P\phi \in C_\psi^l$. As a result, P is a contraction on the complete metric space (C_ψ^l, d) by Theorem 3.6. Consequently, P has a unique fixed point $x \in C_\psi^l$. Thus $|x(t)| \le l$ for all $t \ge -r(0)$, and it is a solution of (3.1) on $[0, \infty)$ by Theorem 3.5. From (3.30) we can argue, much as we did in obtaining (3.20), that (3.1) has no other continuous solution, bounded or unbounded, agreeing with ψ on $[-r(0), 0]$. In short, $x(t)$ is the only continuous function satisfying (3.1) for $t \ge 0$ and with $x(t) = \psi(t)$ for $-r(0) \le t \le 0$.

In order to prove stability at $t = 0$, let $\epsilon > 0$ be given. Then choose $m > 0$ so that $m < \min\{\epsilon, l\}$. Replacing l with m beginning with (3.31), we see there is a $\delta > 0$ such that $\|\psi\| < \delta$ implies that the unique continuous solution x agreeing with ψ on $[-r(0), 0]$ satisfies $|x(t)| \le m < \epsilon$ for all $t \ge -r(0)$. □

Remark 3.3. Observe that both of the last two terms in (3.26) are equal to zero when $G(t,t) \equiv 0$. In that case, $\int_{t-r(t)}^{t} |G(t,u)|\,du \le \alpha$ implies

$$|(P\phi)(t)| \le \delta + g(\delta)\int_{-r(0)}^{0} |G(0,u)|\,du + g(l)\int_{t-r(t)}^{t} |G(t,u)|\,du \le (1-\alpha)g(l) + g(l)\alpha \ \le l.$$

The upshot is that the factor 2 can be dropped from (iii) of Theorem 3.7 if $G(t,t) = 0$ for $t \geq 0$.

It seems that we should also be able to obtain an asymptotic stability result in the framework of (C_ψ^l, d) by focusing on the subset of functions in C_ψ^l that tend to zero as $t \to \infty$. But there is a problem. This particular subset with the metric d is not a complete metric space. However, there is a way to get around this: the set C_ψ^0 defined by (3.21) with the supremum metric ρ is complete. Supposing that the conditions in Theorems 3.6 and 3.7 hold for some $l > 0$, we investigate asymptotic stability by shifting our attention to the subset of functions in C_ψ^0 that are bounded by l; namely,

$$C_\psi^{l,0} := \{\, \phi\colon [-r(0), \infty) \to R \mid \phi \in C,\, \phi(t) = \psi(t) \text{ for } -r(0) \leq t \leq 0,\, |\phi(t)| \leq l,\, \phi(t) \to 0 \text{ as } t \to \infty \,\}.$$

Since $C_\psi^{l,0}$ is a closed subset of C_ψ^0, the metric space $(C_\psi^{l,0}, \rho)$ is complete. Under the conditions of the next theorem, the zero solution of (3.1) is asymptotically stable in the sense given by Definition 3.2.

Definition 3.2. The zero solution of (3.1) is *asymptotically stable* if it is stable at $t = 0$ and a $\delta > 0$ exists such that for any continuous function $\psi\colon [-r(0), 0] \to (-\delta, \delta)$, the solution $x(t)$ with $x(t) = \psi(t)$ on $[-r(0), 0]$ tends to zero as $t \to \infty$.

Theorem 3.8. *Suppose all of the conditions in Theorems 3.6 and 3.7 hold. Furthermore, suppose g is continuously differentiable on $[-l, l]$ and $g'(0) \neq 0$. If $\int_0^t G(s,s)\,ds \to \infty$ as $t \to \infty$, then the zero solution of* (3.1) *is asymptotically stable.*

Proof. For $\xi \in [0, l]$, define

$$q(\xi) := \min\{g'(x) : |x| \leq \xi\} \quad \text{and} \quad Q(\xi) := \max\{g'(x) : |x| \leq \xi\}.$$

As we shall see, the mapping P defined by (3.26) will be a contraction on $(C_\psi^{l,0}, \rho)$ provided

$$2\beta Q(l) < q(l), \tag{3.32}$$

where

$$\beta := \sup_{t \geq 0} \int_{t-r(t)}^t |G(t,u)|\,du.$$

We may assume (3.32) holds—for if it does not, we merely decrease the value of $l > 0$ until it does. To see this, first notice from (iii) in Theorem 3.7 that $2\beta < 1$; thus,

$$2\beta = 1 - \epsilon \tag{3.33}$$

for some $\epsilon \in (0,1)$. Clearly, $\lim_{\xi \to 0} q(\xi) = \lim_{\xi \to 0} Q(\xi) = g'(0)$. Since g is (strictly) increasing, $g'(0) \neq 0$, and g' is continuous, a neighborhood of $x = 0$ exists in which $g'(x) > 0$. Consequently, $Q(\xi) > 0$ for $0 \leq \xi \leq l$. This and $\lim_{\xi \to 0} Q(\xi) \neq 0$ imply

$$\lim_{\xi \to 0} \frac{q(\xi)}{Q(\xi)} = \frac{\displaystyle\lim_{\xi \to 0} q(\xi)}{\displaystyle\lim_{\xi \to 0} Q(\xi)} = 1.$$

Hence, there is a $\gamma \in (0, l]$ such that

$$\left| \frac{q(\xi)}{Q(\xi)} - 1 \right| < \epsilon$$

for $0 < \xi < \gamma$. Choosing a value for ξ from $(0, \gamma)$, we have $(1-\epsilon)Q(\xi) < q(\xi)$. This, along with (3.33), yields

$$2\beta Q(\xi) = (1-\epsilon)Q(\xi) < q(\xi).$$

Replacing the original value of l with $l = \xi$, we obtain (3.32). Note that the conditions in Theorems 3.6 and 3.7 will still hold with this smaller l.

Now we argue that $P\colon C_\psi^{l,0} \to C_\psi^{l,0}$ when $l > 0$ satisfies (3.32), the conditions of Theorems 3.6 and 3.7 hold, and $\|\psi\|$ is sufficiently small. For $\delta > 0$ satisfying (3.31), let $\psi\colon [-r(0), 0] \to (-\delta, \delta)$ be a continuous function. Let $\phi \in C_\psi^{l,0}$. The proof of Theorem 3.7 shows that $\delta < l$ and $|(P\phi)(t)| \le l$ for $t \in [-r(0), \infty)$. Hence, $(P\phi)(t) \to 0$ would imply that P maps $C_\psi^{l,0}$ into itself. To show that this is the case, consider $|(P\phi)(t)|$. But first note for any $\phi \in C_\psi^{l,0}$ that

$$|g(\phi(t))| \le L|\phi(t)| \quad \text{and} \quad |\phi(t) - g(\phi(t))| \le L|\phi(t)|$$

since $g(x)$ and $x - g(x)$ satisfy a Lipschitz condition on $[-l, l]$ with a common Lipschitz constant L and $g(0) = 0$. Because of this and (iii) of Theorem 3.7, it follows from (3.26) that

$$\begin{aligned} |(P\phi)(t)| = {} & e^{-\int_0^t G(s,s)ds}\left(|\psi(0)| + \int_{-r(0)}^0 |G(0,u)|\,|g(\psi(u))|\,du\right) \\ & + \frac{1}{2}L\alpha\|\phi\|_{[t-r(t),t]} + L\int_0^t e^{-\int_s^t G(u,u)du}G(s,s)\int_{s-r(s)}^s |G(s,u)|\,|\phi(u)|\,du\,ds \\ & + L\int_0^t e^{-\int_s^t G(u,u)du}G(s,s)|\phi(s)|\,ds. \end{aligned}$$

As $t \to \infty$, the first two terms tend to 0 since $\int_0^t G(s,s)\,ds \to \infty$, $\phi(t) \to 0$, and $t - r(t) \to \infty$. The third term tends to 0 by the argument given after (3.21) for $\Omega(t)$. The last term tends to 0 by a similar argument. Thus, $(P\phi)(t) \to 0$ as $t \to \infty$.

Finally, we verify that P is a contraction on $(C_\psi^{l,0}, \rho)$. In the ensuing argument, bounds on the derivatives of $g(x)$ and $b(x) := x - g(x)$ on the interval $[-l, l]$ are used. By (i) in Theorem 3.7 and the definition of Q, $0 \le g'(x) \le Q(l)$. By (i) and (ii) in Theorem 3.7, b is nondecreasing on $[-l, l]$. This and $g'(x) \ge q(l)$ imply $0 \le b'(x) \le 1 - q(l)$. Let $\phi, \eta \in C_\psi^{l,0}$. By (3.26) and the Mean Value Theorem, we have

$$\begin{aligned} |(P\phi)(t) - {} & (P\eta)(t)| \\ & \le \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)|b(\phi(s)) - b(\eta(s))|ds + \int_{t-r(t)}^t |G(t,u)(g(\phi(u)) - g(\eta(u))|\,du \\ & \quad + \int_0^t e^{-\int_s^t G(u,u)du}G(s,s)\int_{s-r(s)}^s |G(s,u)(g(\phi(u)) - g(\eta(u))|\,du\,ds \\ & \le \left[(1-q(l)) + 2\sup_{t\ge 0}\int_{t-r(t)}^t |G(t,u)|\,du\,Q(l)\right]\|\phi - \eta\| \\ & = [(1-q(l)) + 2\beta Q(l)]\,\|\phi - \eta\| \\ & = \mu\,\|\phi - \eta\|, \end{aligned}$$

for all $t > 0$, where $\mu := (1 - q(l)) + 2\beta Q(l)$. This implies $\|P\phi - P\eta\| \le \mu\|\phi - \eta\|$. Note $\mu \in (0, 1)$. Consequently, for a continuous $\psi\colon [-r(0), 0] \to (-\delta, \delta)$, P has a unique fixed point $x \in C_\psi^{l,0}$. By Theorem 3.5, x is the unique continuous solution of (3.1) with $x(t) = \psi(t)$ on $[-r(0), 0]$. By virtue of $x \in C_\psi^{l,0}$, $x(t)$ tends to 0 as $t \to \infty$. By Theorem 3.7, the zero solution is stable at $t = 0$. $\square$

Example 3.1. Consider (3.1) with the delay $r(t) = t/2$. Then $t - r(t) = t/2$ and assumptions (A_1)–(A_3) are satisfied. Since the inverse of $t - r(t)$ is $f(t) = 2t$, the function G defined in (3.11) is

$$G(t, s) = \int_t^{2s} a(u, s)\, du$$

for $t \ge 0$ and $t/2 \le s \le t$. Consequently, condition (iii) of Theorem 3.7 holds if

$$2 \sup_{t\ge0} \int_{t/2}^{t} \left| \int_t^{2u} a(v, u)\, dv \right| du < 1.$$

For example, let $a(t, s) := \gamma e^{-(t-s)}$, where γ is a positive constant. A straightforward calculation shows the above condition holds if $\gamma < 1/2$. Since $G(t, t) = \gamma(1 - e^{-t})$, condition (ii) of Theorem 3.6 holds and $\int_0^t G(s, s)\, ds \to \infty$ as $t \to \infty$. Therefore, if the conditions pertaining to g in Theorems 3.6, 3.7, and 3.8 are satisfied, the zero solution of

$$x'(t) = -\gamma \int_{t/2}^{t} e^{-(t-s)} g(x(s))\, ds$$

is asymptotically stable.

Stability via nonincreasing functions dominating delays. To evaluate the integral in condition (iii) of Theorem 3.7, we need an explicit formula for the inverse function f. But, unlike Example 3.1, it is frequently a technical problem, or may not even be possible, to find such a formula. However, if $a(t, s) \ge 0$, it may be possible to surmount this technicality. Suppose the delay $r(t) \ge 0$ is dominated by a nonincreasing function. For such an r, a sketch of an increasing $t - r(t)$ along with its reflection $f(t)$ about the line $y = t$, suggests:

Lemma 3.1. *If $r(t)$ is dominated by a nonincreasing function $\theta(t)$ for $t \ge -r(0)$, then*

$$f(t) \le t + \theta(t). \tag{3.34}$$

Proof. Note that $\theta(t) \ge 0$ since $r(t) \ge 0$. Hence, as θ is nonincreasing,

$$\theta(t + \theta(t)) \le \theta(t).$$

Since $f(t)$ denotes the inverse of $t - r(t)$, let $f^{-1}(t)$ denote $t - r(t)$ itself. Then

$$f^{-1}(t + \theta(t)) = (t + \theta(t)) - r(t + \theta(t)) \ge t + \theta(t + \theta(t)) - r(t + \theta(t)) \ge t.$$

By (A_2), $f^{-1}(t)$ is strictly increasing; so f is strictly increasing. Therefore,

$$f(f^{-1}(t + \theta(t))) \ge f(t),$$

which simplifies to (3.34). $\square$

Under these circumstances, the function G is dominated by an integral with an upper limit given by an explicit formula:

$$G(t,s) = \int_t^{f(s)} a(v,s)\,dv \le \int_t^{s+\theta(s)} a(v,s)\,dv.$$

As a result, condition (iii) of Theorem 3.7 will be satisfied if

$$2\sup_{t\ge 0}\int_{t-r(t)}^{t}\int_t^{u+\theta(u)} a(v,u)\,dv\,du < 1. \tag{3.35}$$

The following example illustrates the efficacy of (3.35).

Example 3.2. Consider the nonlinear nonconvolution equation

$$x'(t) = -\frac{1}{3}\int_{t-\cos^2 t}^{t} te^{-s}x^3(s)\,ds \tag{3.36}$$

for $t \ge 0$. Since $r(t) = \cos^2 t$, assumptions (A_1) and (A_3) are satisfied. As for (A_2), $t - r(t) = t - \cos^2 t$ is nondecreasing since its derivative is $1 + \sin(2t) \ge 0$. In fact, it follows from the Mean Value Theorem that $t - \cos^2 t$ is strictly increasing on $[0,\infty)$ since $1 + \sin(2t) > 0$ for $t \ne \frac{3\pi}{4} + n\pi$ ($n = 0, 1, 2, \ldots$). In the notation of (3.1), take $a(t,s) = te^{-s}$ and $g(x) = \frac{1}{3}x^3$. In Theorems 3.6 and 3.7, let $l = 1$. Since $|g'(x)| \le 1$ for $|x| \le 1$, g satisfies a Lipschitz condition on $[-1,1]$. Thus, condition (i) of Theorem 3.6 is satisfied. Clearly, (i) and (ii) of Theorem 3.7 are also satisfied. Since $a(t,s) \ge 0$, condition (ii) of Theorem 3.6 is satisfied. Finally, condition (iii) of Theorem 3.7 will be satisfied if we can show that (3.35) holds. Since $\cos^2 t \le 1$, we take $\theta(t) \equiv 1$. Then

$$\begin{aligned} 2\int_{t-r(t)}^{t}\int_t^{u+\theta(u)} a(v,u)\,dv\,du &= 2\int_{t-\cos^2 t}^{t}\int_t^{u+1} ve^{-u}\,dv\,du \\ &= e^{-(t-\cos^2 t)}\big(5 - 4\cos^2 t + \cos^4 t + 4t - 2t\cos^2 t\big) - e^{-t}(5+4t). \end{aligned} \tag{3.37}$$

With the aid of the computer algebra software Maple$^{\mathrm{TM}}$, we find that (3.37) has an absolute maximum value of $0.6627\ldots$ at $t = 0.4380\ldots$ on the interval $[0,\infty)$. Hence,

$$2\sup_{t\ge 0}\int_{t-\cos^2 t}^{t}\int_t^{u+1} ve^{-u}\,dv\,du \le .663 < 1. \tag{3.38}$$

Therefore, the zero solution of (3.36) is stable at $t = 0$.

The convolution case. Equation (3.1) becomes

$$x'(t) = -\int_{t-r}^{t} a(t-s)g(x(s))\,ds \tag{3.39}$$

when $a(t,s) = a(t-s)$ and $r(t) \equiv r$, a positive constant. This equation has been studied widely in the literature, mainly under the Levin-Nohel [16, p. 35] conditions: $a\colon [0,r] \to R$ is twice continuously differentiable and

$$a(r) = 0, \qquad a(t) \ge 0, \qquad a'(t) \le 0, \qquad a''(t) \ge 0. \tag{3.40}$$

Hale [13, pp. 120–21] used a Liapunov functional to show all solutions of (3.39) are bounded when (3.40) holds and $\int_0^x g(s)\,ds \to \infty$ as $|x| \to \infty$. Levin and Nohel [16] also studied (3.38) extensively under these conditions. This raises the question: Are the conditions in (3.40) necessary for boundedness of solutions or stability? We will show in the next example that they are not. It is to be noted that none of the conditions in (3.40) hold for equation (3.42) below. And yet we will conclude that its zero solution is stable.

First observe that for any convolution equation (3.39) with constant delay $r > 0$, the function G defined in (3.11) simplifies to

$$G(t,s) = \int_t^{s+r} a(u-s)\,du = \int_{t-s}^{r} a(v)\,dv \tag{3.41}$$

for $t \geq 0$ and $t - r \leq s \leq t$ since $f(s) = s + r$.

Example 3.3. Consider the equation

$$x'(t) = \int_{t-r}^{t} \cos\frac{\pi(t-s)}{r}\, g(x(s))\,ds, \tag{3.42}$$

where g satisfies the three conditions stated in Theorems 3.6 and 3.7. The kernel is $a(t) = -\cos(\pi t/r)$. In contrast to (3.40), $a(r) \neq 0$ and both $a(t)$ and $a''(t)$ change sign on $[0,r]$ while $a'(t) > 0$ on $(0,r)$. By (3.41),

$$G(t,s) = -\int_{t-s}^{r} \cos\frac{\pi v}{r}\,dv = -\frac{r}{\pi}\sin\frac{\pi v}{r}\Big|_{t-s}^{r} = \frac{r}{\pi}\sin\frac{\pi(t-s)}{r}.$$

In particular, $G(t,t) = 0$ for $t \geq 0$. So condition (ii) of Theorem 3.6 holds. Also, in view of Remark 3.3, condition (iii) of Theorem 3.7 holds if $\alpha := 2(r/\pi)^2 < 1$ since

$$\int_{t-r}^{t} |G(t,u)|\,du = \frac{r}{\pi}\int_{t-r}^{t}\left|\sin\frac{\pi(t-u)}{r}\right| du = \left(\frac{r}{\pi}\right)^2 \int_0^{\pi} \sin v\,dv = 2\left(\frac{r}{\pi}\right)^2.$$

It follows that the zero solution of (3.42) is stable at $t = 0$ if $r < \pi/\sqrt{2}$. However, as Remark 3.4 shows, it is not asymptotically stable.

Remark 3.4. A necessary condition that the zero solution of (3.39) be asymptotically stable is that the mean value of $a(t)$ on $[0,r]$ not be zero. To see this, suppose that it is. By (3.41), $G(s,s) = \int_0^r a(v)\,dv = 0$. Then (3.24) simplifies to

$$x(t) = \psi(0) - \int_{-r}^{0} G(0,u)g(\psi(u))\,du + \int_{-r}^{0} G(0,u)g(x(t+u))\,du.$$

Note that every function that is constant on $[-r,\infty)$ satisfies this equation, thereby making it a solution of (3.39) on $[0,\infty)$. This conclusion is also immediate from (3.39) itself when it is rewritten as

$$x'(t) = -\int_0^r a(u)g(x(t-u))\,du.$$

In sum, if the mean value of $a(t)$ on $[0,r]$ is zero, the zero solution of (3.39) is not asymptotically stable.

Levin and Nohel show that if (3.40) holds and if $a''(t)$ is not identically zero, then solutions of (3.39) tend to zero. But, as we show next with Example 3.4, even if none of the conditions in (3.40) hold and $a''(t) \equiv 0$, the zero solution may still be asymptotically stable.

Example 3.4. Suppose all of the conditions pertaining to g in Theorem 3.8 are satisfied. Let $a(t) = b+t-\frac{r}{2}$, where $r > 0$ and b is an arbitrarily small positive number satisfying $br^2 + \frac{r^3}{6} < 1$. By (3.41),

$$G(t,s) = \int_{t-s}^{r} \big(b + v - \frac{r}{2}\big)\, dv = br - b(t-s) - \frac{1}{2}(t-s)^2 + \frac{r}{2}(t-s)$$

for $t \geq 0$ and $t - r \leq s \leq t$. Hence, $G(s,s) = br > 0$ and $\int_0^t G(s,s)\, ds \to \infty$ as $t \to \infty$. Since $t - s \leq r$, $G(t,s) \geq 0$. Consequently,

$$\begin{aligned} 2\int_{t-r}^{t} |G(t,s)|\, ds &= 2\int_{t-r}^{t} \big[br - b(t-s) - \frac{1}{2}(t-s)^2 + \frac{r}{2}(t-s)\big]\, ds \\ &= \int_0^r (2br - 2bu - u^2 + ru)\, du = br^2 + \frac{r^3}{6} \end{aligned}$$

for $t \geq 0$. Thus, if $\alpha := br^2 + \frac{r^3}{6} < 1$, all of the conditions in Theorem 3.8 for asymptotic stability are met.

We can also employ Theorem 3.8 to prove that the zero solution of (3.39) is asymptotically stable when the Levin-Nohel conditions hold—even if $a''(t) \equiv 0$—provided $a(0)$ satisfies a boundedness condition.

Example 3.5. Suppose g satisfies the conditions in Theorem 3.8 and a satisfies those in (3.40). Then, if $0 < a(0) < 3/r^2$, the zero solution of (3.39) is asymptotically stable.

Proof. Since $a''(t) \geq 0$ for $0 \leq t \leq r$, a is convex on $[0,r]$. This implies that the graph of a lies on or below the chord with endpoints $(0, a(0))$ and $(r, a(r))$. Hence, as $a(r) = 0$, $a(t) \leq -\frac{a(0)}{r}t + a(0)$ for $0 \leq t \leq r$. Since $a(t) \geq 0$, we see from (3.41) that $G(t,s) \geq 0$. Moreover, as $a(0) > 0$,

$$G(s,s) = \int_0^r a(v)\, dv > 0.$$

Therefore, condition (ii) of Theorem 3.6 holds and $\int_0^t G(s,s)\, ds \to \infty$ as $t \to \infty$. Furthermore,

$$2\int_{t-r}^{t} G(t,s)\, ds = 2\int_{t-r}^{t}\int_{t-s}^{r} a(v)\, dv\, ds \leq 2a(0)\int_{t-r}^{t}\int_{t-s}^{r} \big(1 - \frac{v}{r}\big)\, dv\, ds = a(0)\frac{r^2}{3}.$$

Therefore, condition (iii) of Theorem 3.7 holds with $\alpha := a(0)r^2/3 < 1$. □

Remark 3.5. In both Examples 3.4 and 3.5, if $g(x) = x$, the zero solution is in fact globally asymptotically stable. In other words, since $g(x) = x$ satisfies all of the conditions stated in Theorems 3.6, 3.7, and 3.8 for every $l > 0$, all solutions of

$$x'(t) = -\int_{t-r}^{t} a(t-s)x(s)\, ds$$

approach zero as $t \to \infty$ if a satisfies the conditions given in those examples. This conclusion may also be drawn from Theorem 3.4.

4. A General Form of the Levin-Nohel Equation

We can obtain existence and stability results comparable to those obtained for (3.1) for

$$x'(t) = -\int_{t-r(t)}^{t} a(t,s)g(s,x(s))\,ds, \tag{4.1}$$

by slightly modifying the proofs in Section 3. Assume $r\colon [0,\infty) \to [0,\infty)$, $a\colon [0,\infty)\times[-r(0),\infty) \to R$, and $g\colon [-r(0),\infty)\times R \to R$ are continuous functions; and, as before, $r(t)$ has properties (A_1), (A_2), and (A_3). From a perusal of the proofs of Theorems 3.1 and 3.5, it becomes evident that if all occurrences of $g(x(t))$ are replaced with $g(t,x(t))$ that both theorems hold for (4.1). Making this replacement in (3.26) as well, we see that Theorem 3.6 also holds for (4.1) provided the Lipschitz condition (i) is modified by assuming that $g(t,x)$ satisfy a Lipschitz condition with respect to x. In other words, suppose there are positive constants l, L such that

$$|g(t,x) - g(t,y)| \le L|x-y| \tag{4.2}$$

for $t \ge -r(0)$ and all $x, y \in [-l, l]$. The validity of these theorems for (4.1) permits the following generalization of Theorem 3.7.

Theorem 4.1. *For $t \ge -r(0)$, suppose*

(i) *$g(t,-x) = -g(t,x)$;*
(ii) *an $l > 0$ exists such that g satisfies a Lipschitz condition with respect to x on $[-r(0),\infty)\times[-l,l]$;*
(iii) *there are functions w and W that are continuous, odd, and strictly increasing on $[-l,l]$ such that $w(x) \le g(t,x) \le W(x)$ for $x \in [0,l]$;*
(iv) *$x - w(x)$ is nondecreasing on $[0,l]$;*
(v) *$|x - g(t,x)| \le l - w(l)$ for $x \in [-l,l]$.*

Furthermore, for $t \ge 0$, suppose

(vi) *$G(t,t) \ge 0$;*
(vii) *a positive number $\alpha < \frac{w(l)}{W(l)}$ exists such that $2\int_{t-r(t)}^{t} |G(t,u)|\,du \le \alpha$.*

Then the conclusions of Theorem 3.7 hold for (4.1).

Proof. By (iii) and (vii), $w(l) - \alpha W(l) > 0$. Hence, there is a $\delta > 0$ such that

$$\delta + W(\delta)\int_{-r(0)}^{0} |G(0,u)|\,du \le w(l) - \alpha W(l). \tag{4.3}$$

Let $\psi\colon [-r(0),0] \to (-\delta,\delta)$ be a continuous function. For any $\phi \in C_\psi^l$, let $P\phi$ be as defined in (3.26) but with $g(u,\phi(u))$ replacing $g(\phi(u))$. Then $P\phi \in C_\psi^l$. To see this, first observe $w(l) \le l$ by (iv). Thus, by (4.3), $|(P\phi)(t)| = |\psi(t)| < l$ for $-r(0) \le t \le 0$. Now consider $(P\phi)(t)$ for $t > 0$. By (i) and (iii), $|g(t,x)| \le W(l)$

for $x \in [-l, l]$ and $t \geq -r(0)$. Using this, (iii), and (v), we obtain

$$\begin{aligned}
|(P\phi)(t)| \leq \delta + W(\delta) \int_{-r(0)}^{0} |G(0,u)|\, du + W(l) \int_{t-r(t)}^{t} |G(t,u)|\, du \\
+ W(l) \int_{0}^{t} e^{-\int_s^t G(u,u)du} G(s,s) \int_{s-r(s)}^{s} |G(s,u)|\, du\, ds \\
+ (l - w(l)) \int_{0}^{t} e^{-\int_s^t G(u,u)du} G(s,s)\, ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|(P\phi)(t)| &\leq \delta + W(\delta) \int_{-r(0)}^{0} |G(0,u)|\, du + \alpha W(l) + l - w(l) \\
&\leq w(l) - \alpha W(l) + \alpha W(l) + l - w(l) = l
\end{aligned}$$

by (vii), (3.29), and (4.3). Thus, $|(P\phi)(t)| \leq l$ for $t \in [-r(0), \infty)$. To sum up, $P\phi \in C_\psi^l$ for $\phi \in C_\psi^l$. It follows from the version of Theorem 3.6 for equation (4.1) that P is a contraction on the complete space (C_ψ^l, d). Therefore, P has a unique fixed point. The proof that the zero solution of (4.1) is stable at $t = 0$ is exactly like the one given for Theorem 3.7. $\square$

Example 4.1. Consider the equation

$$x'(t) = -\int_{t-\cos^2 t}^{t} kte^{-s} \left(\tfrac{1}{3} + \sin^2 t\right) x^3(s)\, ds, \tag{4.4}$$

where $k > 0$. Note its resemblance to (3.36). The difference between the two equations is the presence of k in the kernel $a(t,s)$ and the replacement of $g(x) = \frac{1}{3}x^3$ by $g(t,x) = \left(\frac{1}{3} + \sin^2 t\right)x^3$. Condition (i) of Theorem 4.1 clearly holds for this g. Letting $w(x) := \frac{1}{3}x^3$ and $W(x) := \frac{4}{3}x^3$, we see that (iii) holds for any $l > 0$. Setting $l = 1$, condition (iv) is met. For $x \in [-1, 1]$, $|x - g(t,x)| \leq \frac{2}{3}$. Consequently, (v) holds since $1 - w(1) = \frac{2}{3}$. Condition (ii) holds because $\partial g/\partial x$ is bounded for $t \geq -r(0)$ and $x \in [-1, 1]$. Since $a(t,s) = kte^{-s} \geq 0$ and $f(t) \geq t$, $G(t,t) = \int_t^{f(t)} a(u,t)\, du \geq 0$. Thus, (vi) holds. Letting $\theta(u) \equiv 1$ in (3.35), (vii) holds if

$$2 \sup_{t \geq 0} \int_{t-\cos^2 t}^{t} \int_{t}^{u+1} kve^{-u}\, dv\, du \leq .663k < \frac{w(1)}{W(1)}$$

by (3.38). Since $w(1)/W(1) = 0.25$, we conclude the zero solution of (4.4) is stable at $t = 0$ if $.663k < .25$.

5. A Differential Equation with Variable Delay

In Section 2, we listed some papers with stability results for the scalar delay equation

$$x'(t) = -a(t)g(x(t - r(t))). \tag{5.1}$$

These results were obtained using Liapunov, Razumikhin, and various ad hoc methods. In Theorem 5.1, we present comparable results based on the fixed point methods of this paper. We assume $a\colon [0,\infty) \to [0,\infty)$ and $g\colon R \to R$ are continuous functions. We also assume the delay $r\colon [0,\infty) \to [0,\infty)$ is continuously differentiable and satisfies assumptions (A_1)–(A_3) in Section 3.

Theorem 5.1. *Suppose g is odd, strictly increasing, and satisfies a Lipschitz condition on an interval $[-l, l]$ and that $x - g(x)$ is nondecreasing on $[0, l]$. If*

$$\sup_{t \geq t_1} \int_{t-r(t)}^{t} a(u)\, du < \frac{1}{2}, \tag{5.2}$$

where t_1 is the unique solution of $t - r(t) = 0$, and if a continuous function $\tilde{a}\colon [0, \infty) \to R$ exists such that

$$a(t) = \tilde{a}(t)(1 - r'(t)) \tag{5.3}$$

on $[0, \infty)$, then the zero solution of (5.1) is stable at $t = 0$. Furthermore, if g is continuously differentiable on $[-l, l]$ with $g'(0) \neq 0$ and

$$\int_0^t a(u)\, du \to \infty \quad \text{as } t \to \infty, \tag{5.4}$$

then the zero solution of (5.1) is asymptotically stable.

Proof. Once again, let $f\colon [-r(0), \infty) \to [0, \infty)$ denote the inverse of $t - r(t)$. If there is a continuous function $\tilde{a}$ such that (5.3) holds, then we can write (5.1) as

$$x'(t) = -\tilde{a}(f(t))g(x(t)) + \frac{d}{dt} \int_{t-r(t)}^{t} \tilde{a}(f(s))g(x(s))\, ds \tag{5.5}$$

since

$$\begin{aligned} x'(t) &= -\tilde{a}(f(t))g(x(t)) + \tilde{a}(f(t))g(x(t)) - \tilde{a}(f(t - r(t)))g(x(t - r(t)))(1 - r'(t)) \\ &= -\tilde{a}(t)(1 - r'(t))g(x(t - r(t))) \\ &= -a(t)g(x(t - r(t))). \end{aligned}$$

Comparing (5.5) to (3.4), we see $\tilde{a} \circ f$ takes on the role of G in Section 3. The conditions for g given before (5.2) are precisely conditions (i) of Theorem 3.6 and (i), (ii) of Theorem 3.7. By (A_2), $1 - r'(t) \geq 0$ on $[0, \infty)$. This and $a(t) \geq 0$ imply $\tilde{a}(t) \geq 0$ on $[0, \infty)$. It follows that $\tilde{a}(f(t)) \geq 0$ for all $t \geq 0$; hence, (ii) of Theorem 3.6 holds. The integral corresponding to the one in (iii) of Theorem 3.7 is $\int_{t-r(t)}^{t} \tilde{a}(f(s))\, ds$. Defining u by $s = u - r(u)$, we have

$$\int_{t-r(t)}^{t} \tilde{a}(f(s))\, ds = \int_{t}^{f(t)} \tilde{a}(f(u - r(u)))(1 - r'(u))\, du = \int_{t}^{f(t)} a(u)\, du$$

by (5.3). By (5.2), there is an $\alpha \in (0, 1)$ such that $2 \int_{t-r(t)}^{t} a(u)\, du \leq \alpha$ for $t \geq t_1$. One can show that this is equivalent to

$$2 \int_{t}^{f(t)} a(u)\, du \leq \alpha \tag{5.6}$$

for $t \geq 0$. Thus, (iii) of Theorem 3.7 holds. Since all of the conditions in Theorems 3.6 and 3.7 are met, it follows that the zero solution of (5.1) is stable at $t = 0$.

As regards asymptotic stability, suppose g is also continuously differentiable on $[-l, l]$ with $g'(0) \neq 0$. The analogue of the integral in Theorem 3.8 is

$$\int_0^t \tilde{a}(f(s))\,ds = \int_{t_1}^{f(t)} a(u)\,du.$$

If (5.4) is true, $\int_{t_1}^{f(t)} a(u)\,du \to \infty$ as $t \to \infty$ since $f(t) \geq t$. $\square$

Remark 5.1. By (A_2), $r'(t) \leq 1$ on $[0, \infty)$. Hence, the function $\tilde{a}$ in (5.3) exists if $r'(t) < 1$ on $[0, \infty)$ or if $a(t) = 0$ whenever $r'(t) = 1$.

6. Sharpness of the Inequality in Remark 3.2

In this paper (cf. (3.16) and Remark 3.2) and others (cf. [5] and [6]), variants of the mapping

$$(Ix)(t) := \int_{t-r(t)}^t G(t,u)x(u)\,du - \int_0^t e^{-\int_s^t G(u,u)\,du} G(s,s) \int_{s-r(s)}^s G(s,u)x(u)\,du\,ds \tag{6.1}$$

are encountered, where x, G, r are continuous functions, x is bounded, $G(t,t) \geq 0$, and $r(t) \geq 0$. The inequality

$$|(Ix)(t)| \leq 2 \sup_{t \geq 0} \int_{t-r(t)}^t |G(t,u)||x(u)|\,du \tag{6.2}$$

brings to mind the $\frac{3}{2}$-stability theorems mentioned in Section 2. In particular, the integral condition (2.6) suggests that (6.2) could be improved upon: perhaps

$$|(Ix)(t)| \leq k \sup_{t \geq 0} \int_{t-r(t)}^t |G(t,u)||x(u)|\,du \tag{6.3}$$

for $k = 2/3$, or at least for a value of k less than 2. Of course, if more is known about G and r than what is stated above, then (6.3) may indeed hold for $k < 2$. For example, if $G(t,t) \equiv 0$, then (6.3) holds when $k = 1$.

The purpose of this section is to demonstrate that for any $k < 2$ it is always possible to find functions x, G, r having the properties described after (6.1) and a $t = \tau$ so that

$$|(Ix)(\tau)| > k \sup_{t \geq 0} \int_{t-r(t)}^t |G(t,u)||x(u)|\,du. \tag{6.4}$$

That is to say, unless more is known about G and r, we show that the smallest possible positive value for k so that (6.3) is always valid is in fact $k = 2$. To this end, let $r(t) \equiv r$, where r is a positive constant, and $G(t,u) = c(u+r)$, where $c\colon [0,\infty) \to (0,\infty)$ is continuous. Then (6.1) simplifies to

$$(Ix)(t) := \int_{t-r}^t c(u+r)x(u)\,du - \int_0^t e^{-\int_s^t c(u+r)du} c(s+r) \int_{s-r}^s c(u+r)x(u)\,du\,ds. \tag{6.5}$$

Lemma 6.1. *Let $[m, M]$ be a closed bounded interval with $m > 0$. Let $c\colon [0,\infty) \to [m, M]$ be continuous. For every $A \in (0,1)$, there are positive constants r and τ such that*

$$2e^{-\int_{t-r}^{t} c(u+r)du} - 1 - e^{-\int_0^t c(u+r)du} > A \tag{6.6}$$

for all $t \geq \tau$.

Proof. For a given $A \in (0,1)$, define $\gamma > 0$ by $A + 2\gamma = 1$. Since $c(t) \leq M$, there is an $r > 0$ such that $2e^{-\int_{t-r}^{t} c(u+r)du} > 2 - \gamma$ for all $t \geq 0$. And since $c(t) \geq m$, there is a $\tau \geq 0$, independent of r, such that $e^{-\int_0^t c(u+r)du} < \gamma$ for all $t \geq \tau$. Thus,

$$2e^{-\int_{t-r}^{t} c(u+r)\,du} - 1 - e^{-\int_0^t c(u+r)\,du} > 2 - \gamma - 1 - \gamma = 1 - 2\gamma = A.$$

for all $t \geq \tau$. $\square$

Lemma 6.2. *As in Lemma 6.1, let $c\colon [0,\infty) \to [m, M]$ be continuous. For every $A \in (0,1)$, there are positive constants r, τ and a bounded piecewise continuous function $x\colon [-r,\infty) \to (-\infty,\infty)$ such that*

$$|(Ix)(\tau)| > (1+A)\sup_{t\geq 0}\int_{t-r}^{t} c(u+r)|x(u)|\,du. \tag{6.7}$$

Proof. For $A \in (0,1)$, let r, τ be positive constants so that (6.6) holds. Define $x\colon [-r,\infty) \to (-\infty,\infty)$ by

$$x(u)c(u+r) = \begin{cases} -1 & \text{if } -r \leq u \leq \tau - r \\[2mm] 1 & \text{if } u > \tau - r. \end{cases} \tag{6.8}$$

Let Ex denote the second term of Ix in (6.5). Then for $t = \tau$, we have

$$\begin{aligned}(Ex)(\tau) = &-\int_0^{\tau-r} e^{-\int_s^\tau c(u+r)du} c(s+r)\int_{s-r}^{s} c(u+r)x(u)\,du\,ds \\ &-\int_{\tau-r}^{\tau} e^{-\int_s^\tau c(u+r)du} c(s+r)\int_{s-r}^{s} c(u+r)x(u)\,du\,ds.\end{aligned}$$

Since $\int_{s-r}^{s} c(u+r)x(u)\,du = -r$ and $\left|\int_{s-r}^{s} c(u+r)x(u)\,du\right| \leq r$ in the first and second terms, respectively, it follows that

$$\begin{aligned}(Ex)(\tau) &\geq re^{-\int_s^\tau c(u+r)\,du}\Big|_0^{\tau-r} - re^{-\int_s^\tau c(u+r)du}\Big|_{\tau-r}^{\tau} \\ &= r\left[e^{-\int_{\tau-r}^{\tau} c(u+r)\,du} - e^{-\int_0^\tau c(u+r)\,du} - 1 + e^{-\int_{\tau-r}^{\tau} c(u+r)du}\right] \\ &= r\left[2e^{-\int_{\tau-r}^{\tau} c(u+r)du} - 1 - e^{-\int_0^\tau c(u+r)du}\right].\end{aligned}$$

By (6.6), $(Ex)(\tau) > rA$. Therefore,

$$(Ix)(\tau) > r + rA = (1+A)r = (1+A)\int_{\tau-r}^{\tau} c(u+r)|x(u)|\,du, \tag{6.9}$$

which implies (6.7). $\square$

It follows that the statement for the operator I in (6.4) is true for bounded piecewise continuous functions. Finally, we show that it is also true for bounded continuous functions.

Theorem 6.1. *Let $c\colon [0,\infty) \to [m, M]$ be a continuous function, where $[m, M]$ is a closed bounded interval with $m > 0$. Then, for every $A \in (0, 1)$, there are positive constants r and τ and a bounded continuous function $\hat{x}\colon [-r,\infty) \to (-\infty,\infty)$ such that*

$$|(I\hat{x})(\tau)| > (1 + A) \sup_{t\geq 0} \int_{t-r}^{t} c(u + r)|\hat{x}(u)|\, du. \tag{6.10}$$

Proof. For a given continuous function $c\colon [0,\infty) \to [m, M]$ and $A \in (0, 1)$, let r, τ be positive constants and let $x(u)$ be the function given in (6.8) so that (6.7) holds. Observe that by means of the unit step function

$$\mathbf{1}_{+}(u) := \begin{cases} 0 & \text{if } u \leq 0 \\ 1 & \text{if } u > 0, \end{cases}$$

we can rewrite (6.8) as

$$x(u)c(u + r) = 2\mathbf{1}_{+}(u - \tau + r) - 1 \tag{6.8$'$}$$

for $u \in [-r,\infty]$.

For an $\epsilon > 0$, the function $\hat{x}$ will be expressed in terms of

$$\lambda_\epsilon(t) := \int_{-\infty}^{t} \gamma_\epsilon(s)\, ds,$$

where γ_ϵ is defined by

$$\gamma_\epsilon(t) := \frac{\zeta(t/\epsilon)}{\int_{-\infty}^{\infty} \zeta(t/\epsilon)\, dt}$$

and ζ is

$$\zeta(t) := \begin{cases} 0 & \text{if } |t| \geq 1 \\ \exp \dfrac{1}{t^2 - 1} & \text{if } |t| < 1. \end{cases}$$

These are well-known functions from the theory of distributions: ζ is infinitely differentiable; γ_ϵ behaves like the Dirac delta functional δ for ϵ sufficiently small; and $\lambda_\epsilon(t)$ converges to $\mathbf{1}_{+}(t)$ as $\epsilon \to 0$. For details, see Zemanian [24, pp. 2–13]. Furthermore, (i) $\lambda_\epsilon(t) = 0$ for $t \leq -\epsilon$; (ii) $\lambda_\epsilon(t) = 1$ for $t \geq \epsilon$; (iii) $0 < \lambda_\epsilon(t) \leq \frac{1}{2}$ for $-\epsilon < t \leq 0$; (iv) $\frac{1}{2} < \lambda_\epsilon(t) < 1$ for $0 < t < \epsilon$. It follows that

$$|\mathbf{1}_{+}(t) - \lambda_\epsilon(t)| \leq \frac{1}{2} \tag{6.11}$$

for $t \in (-\infty,\infty)$.

Now define $x_\epsilon\colon [-r,\infty) \to (-\infty,\infty)$ by

$$x_\epsilon(u)c(u + r) = 2\lambda_\epsilon(u - \tau + r) - 1. \tag{6.12}$$

For $\epsilon > 0$ sufficiently small, we will prove

$$(Ix_\epsilon)(\tau) > (1 + A)r \tag{6.13}$$

by showing that $(Ix_\epsilon)(\tau)$ converges to $(Ix)(\tau)$ as $\epsilon \to 0$. This then would prove (6.10) as

$$\sup_{t\geq 0} \int_{t-r}^{t} c(u+r)|x_\epsilon(u)|\,du = r.$$

To this end, consider $(Ix)(\tau) - (Ix_\epsilon)(\tau)$. Comparing (6.12) and (6.8′), we note that $x_\epsilon(u) = x(u)$ except for $\tau - r - \epsilon < u < \tau - r + \epsilon$. Hence,

$$\begin{aligned}(Ix)(\tau) - (Ix_\epsilon)(\tau) = &\int_{\tau-r}^{\tau-r+\epsilon} c(u+r)(x(u) - x_\epsilon(u))\,du \\ &- \int_{\tau-r-\epsilon}^{\tau-r+\epsilon} e^{-\int_s^\tau c(u+r)du} c(s+r) \int_{s-r}^{s} c(u+r)(x(u) - x_\epsilon(u))\,du\,ds.\end{aligned}$$

By (6.8′), (6.11), and (6.12), we have

$$|c(u+r)(x(u) - x_\epsilon(u))| = 2|\mathbf{1}_+(u-\tau+r) - \lambda_\epsilon(u-\tau+r)| \leq 1.$$

Consequently,

$$\begin{aligned}|(Ix)(\tau) - (Ix_\epsilon)(\tau)| &\leq \int_{\tau-r}^{\tau-r+\epsilon} du + r\int_{\tau-r-\epsilon}^{\tau-r+\epsilon} e^{-\int_s^\tau c(u+r)du} c(s+r)\,ds \\ &= \epsilon + r\left(e^{-\int_{\tau-r+\epsilon}^{\tau} c(u+r)\,du} - e^{-\int_{\tau-r-\epsilon}^{\tau} c(u+r)\,du}\right).\end{aligned}$$

Thus, $(Ix_\epsilon)(\tau) \to (Ix)(\tau)$ as $\epsilon \to 0$. In view of (6.9), define $\sigma > 0$ by $\sigma := (Ix)(\tau) - (1+A)r$. Then there exists an $\epsilon > 0$ such that $|(Ix_\epsilon)(\tau) - (Ix)(\tau)| < \sigma$. This implies

$$(Ix_\epsilon)(\tau) > (Ix)(\tau) - \sigma = (1+A)r,$$

which is (6.13). Therefore, (6.10) holds with $\hat{x} = x_\epsilon$. □

Remark 6.1. It follows from (6.12) that if the function c in Theorem 6.1 is differentiable, then so is $\hat{x}$.

References

1. F. H. Brownell and W. K. Ergen, *A theorem on rearrangements and its application to certain delay differential equations*, J. Rat. Mech. Anal. **3** (1954), 565–579.
2. T. A. Burton, *Volterra Integral and Differential Equations*, Mathematics in Science and Engineering, vol. 167, Academic Press, Orlando, 1983.
3. ———, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Mathematics in Science and Engineering, vol. 178, Academic Press, Orlando, 1985.
4. ———, *Averaged neural networks*, Neural Networks **6** (1993), 667–680.
5. ———, *Stability by fixed point theory or Liapunov theory: A comparison*, Fixed Point Theory **4** (2003), 15–32.
6. ———, *Fixed Points and Stability of a Nonconvolution Equation*, Proc. Amer. Math. Soc. **132** (2004), 3679–3687.
7. T. A. Burton and Tetsuo Furumochi, *Fixed points and problems in stability theory for ordinary and functional differential equations*, Dynamical Systems and Appl. **10** (2001), 89–116.
8. W. K. Ergen, *Kinetics of the circulating-fuel nuclear reactor*, J. Appl. Phys. **25** (1954), 702–711.
9. J. R. Graef, Chuanxi Qian, and Bo Zhang, *Asymptotic behavior of solutions of differential equations with variable delays*, Proc. London Math. Soc. (3) **81** (2000), 72–92.
10. Istvan Gyori and Ferenc Hartung, *On equi-stability with respect to parameters in functional differential equations*, Nonlinear Funct. Anal. and Appl. **7** (2002), 329–351.
11. A. Halanay, *On the asymptotic behavior of the solutions of an integro-differential equation*, J. Math. Anal. Appl. **10** (1965), 319–324.
12. Jack Hale, *Sufficient conditions for stability and instability of autonomous functional-differential equations*, J. Differential Equations **1** (1965), 452–482.
13. Jack Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

14. T. Krisztin, *On the stability properties for one-dimensional functional differential equations*, Funkcial. Ekvac. **34** (1991), 241–256.
15. J. J. Levin, *The asymptotic behavior of the solution of a Volterra equation*, Proc. Amer. Math. Soc. **14** (1963), 534–541.
16. J. J. Levin and J. A. Nohel, *On a nonlinear delay equation*, J. Math. Anal. Appl. **8** (1964), 31–44.
17. A. Liénard, *Étude des oscillations entretenues*, Revue Générale de l'Électricité **23** (1928), 901–946.
18. R. C. MacCamy and J. S. W. Wong, *Stability theorems for some functional equations*, Trans. Amer. Math. Soc. **164** (1972), 1–37.
19. Seiji Saito, *Global stability of solutions for quasilinear ordinary differential systems*, Math. Japonica **34** (1989)), 821–929.
20. Marcel-Adrian Serban, *Global asymptotic stability for some difference equations via fixed point technique*, Seminar on Fixed Point Theory, Cluj-Napoca **2** (2002), pp. 87–96.
21. T. Yoneyama, *On the 3/2 stability theorem for one-dimensional delay-differential equations*, J. Math. Anal. Appl. **125** (1987), 161–173.
22. J. A. Yorke, *Asymptotic stability for one dimensional differential-delay equations*, J. Differential Equations **7** (1970), 189–202.
23. V. Volterra, *Sur la théorie mathématique des phénomènes héréditaires*, J. Math. Pures Appl. **7** (1928), 249–298.
24. A. H. Zemanian, *Distribution Theory and Transform Analysis: An Introduction to Generalized Functions, with Applications*, Dover, New York, 1987.

Christian Brothers University, 650 E. Parkway South, Memphis, TN 38104
E-mail address: lbecker@cbu.edu

Northwest Research Institute, 732 Caroline St., Port Angeles, WA 98362
E-mail address: taburton@olypen.com