STABILITY AND FIXED POINTS: ADDITION OF TERMS

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Abstract. In applying Perron's linear approximation theorem to the stability study of $x' = x - 2\sin x$ we understand that we can replace $\sin x$ by x and study stability of $x' = -x$. The reason reduces to the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. In another vein, it is an easy application of Liapunov's direct method to see that in the study of stability of $x' = -(1 - 100\sin t)x^3 - 100\sin t \sin^3 x$ we may simply replace $\sin^3 x$ by x^3 and study the stability of $x' = -x^3$. The reason reduces to the fact that $\lim_{x\to 0} \frac{\sin^3 x}{x^3} = 1$. In this paper we use contraction mappings to study stability of three prominant classes of functional differential equations involving two functionals, $g(x_t)$ and $G(x_t)$, having the property that $\lim_{x\to 0} \frac{G(x)}{g(x)} = 1$. We show that we can replace G with g and study the stability of the resulting equation.

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1. INTRODUCTION

While the literature on stability theory of ordinary differential equations is massive, when the investigator encounters a stability problem, the first tool that comes to mind is Perron's theorem [17]. Under generous conditions a nonlinear function $f(x)$ can be replaced by $x \lim_{x\to 0} \frac{f(x)}{x}$ in the stability study. Can an idea like that be advanced to classical nonlinear functional differential equations? We illustrate a general fixed point technique which does exactly that.

In this paper we study stability properties of three classical types of functional differential equations by means of fixed point theory. In a series of earlier papers [2-7], we have developed fixed point criteria for determining stability of equations like

$$x'(t) = -a(t)g(x(t-r)), \tag{i}$$

$$x'(t) = -\int_{t-r}^{t} a(t,s)g(x(s))ds, \tag{ii}$$

and

$$x'(t) = -\int_0^t a(t,s)g(x(s))ds. \qquad (iii)$$

Here, we look at

$$x'(t) = -a(t)g(x(t-r)) - b(t)G(x(t-r)),$$

$$x'(t) = -\int_{t-r}^t [a(t,s)g(x(s)) + b(t,s)G(x(s))]ds,$$

and

$$x'(t) = -\int_0^t [a(t,s)g(x(s)) + b(t,s)G(x(s))]ds$$

and ask the following question. If $\lim_{x\to 0}\frac{g(x)}{G(x)} = 1$, can we determine stability of these equations by studying stability of the equations obtained when G is replaced by g?

Conditions are given under which the answer is affirmative. We also study the question when that limit is not 1 and when there are several delays.

We would point out that these equations are vastly different. Yet, the reader will see that the stability proofs are almost identical. This illustrates that the fixed point techniques used here have wide application.

All of these are important and nontrivial problems, even when $G = 0$. The literature is immense in that case. For (i) one may consult the references in [2, 7, 8, 12]. Both (ii) and (iii) have been discussed extensively, as may be seen in Brownell and Ergen [1], Hale [9,10], Levin [13-14], Levin and Nohel [15], Nohel [16], and Volterra [18], as well as many others.

In most of the proofs of stability we deal only with $t_0 = 0$ since a general t_0 is handled in the same way. When a function is written without its argument, then that argument is t.

Part I: THE DELAY EQUATION

1. THE FIXED POINT MAPPING

We begin with the equation

$$x'(t) = -a(t)g(x(t-r)) - b(t)G(x(t-r)) \qquad (1)$$

with a continuous initial function

$$\psi : [-r, 0] \to R. \qquad (2)$$

Several of the steps we take are unnecessary if either $b(t) = 0$, if $r = 0$, or if $g(x) = x$. But those cases can be readily distinguished from our final mapping. Let

$$c(t) := a(t) + b(t). \qquad (3)$$

THEOREM 2.1. A solution $x(t, 0, \psi)$ of (1) can be expressed as

$$\begin{aligned} x(t) &= \psi(0)e^{-\int_0^t c(s+r)ds} - e^{-\int_0^t c(u+r)du} \int_{-r}^0 c(u+r)g(\psi(u))du \\ &+ \int_0^t e^{-\int_s^t c(u+r)du} c(s+r)[x(s) - g(x(s))]ds \\ &+ \int_0^t e^{-\int_s^t c(u+r)du} b(s)[g(x(s-r) - G(x(s-r))]ds \\ &+ \int_{t-r}^t c(u+r)g(x(u))du \\ &- \int_0^t e^{-\int_s^t c(u+r)du} c(s+r) \int_{s-r}^s c(u+r)g(x(u))duds. \end{aligned} \tag{4}$$

Proof. We write

$$\begin{aligned} &x'(t) \\ &= -a(t)g(x(t-r)) - b(t)g(x(t-r)) + b(t)[g(x(t-r)) - G(x(t-r))] \\ &= -c(t+r)g(x(t)) + \frac{d}{dt}\int_{t-r}^t c(s+r)g(x(s))ds + b(t)[g(x(t-r)) - G(x(t-r))] \\ &= -c(t+r)x(t) + c(t+r)[x(t) - g(x(t))] + b(t)[g(x(t-r)) - G(x(t-r))] \\ &+ \frac{d}{dt}\int_{t-r}^t c(s+r)g(x(s))ds. \end{aligned}$$

By the variation of parameters formula

$$\begin{aligned} x(t) &= \psi(0)e^{-\int_0^t c(s+r)ds} + \int_0^t e^{-\int_s^t c(u+r)du} c(s+r)[x(s) - g(x(s))]ds \\ &+ \int_0^t e^{-\int_s^t c(u+r)du} b(s)[g(x(s-r)) - G(x(s-r))]ds \\ &+ \int_0^t e^{-\int_s^t c(u+r)du} \frac{d}{ds}\int_{s-r}^s c(u+r)g(x(u))du. \end{aligned} \tag{6}$$

If we integrate the last term by parts we obtained the desired form.

3. REVIEW OF THE LINEAR CASE

This section can be found in [2], but it is fundamental to see as a stepping stone to later sections. We will see conclusions here not found in later sections, but can sometimes be added by the interested reader with additional assumptions.

Let r be a positive constant, $a : [0, \infty) \to R$ be continuous, and consider the equation

$$x'(t) = -a(t)x(t-r) \tag{6}$$

with continuous initial function

$$\psi : [-r, 0] \to R. \tag{7}$$

Let $(M, \|\cdot\|)$ be the complete metric space of bounded continuous functions $\phi : [-r, \infty) \to R$ with the supremum metric satisfying $\phi(t) = \psi(t)$ on $[-r, 0]$. Define $P : M \to M$ by $(P\phi)(t) = \psi(t)$ for $t \in [-r, 0]$, and use (4) to define

$$\begin{aligned}(P\phi)(t) &= \psi(0)e^{-\int_0^t a(s+r)ds} - e^{-\int_0^t a(s+r)ds}\int_{-r}^0 a(s+r)\psi(s)ds \\ &\quad + \int_{t-r}^t a(s+r)\phi(s)ds - \int_0^t e^{-\int_s^t a(u+r)du}a(s+r)\int_{s-r}^s a(u+r)\phi(u)du.\end{aligned} \tag{8}$$

THEOREM 3.1. Suppose that for each $t_0 \geq 0$ there exists a $J > 0$ with

$$-\int_{t_0}^t a(s+r)ds \leq J \qquad \text{for} \qquad t \geq t_0 \tag{9}$$

and that there exists $\alpha < 1$ with

$$\sup_{t\geq 0}\int_{t-r}^t |a(s+r)|ds + \sup_{t\geq 0}\int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|\int_{s-r}^s |a(u+r)|du \leq \alpha. \tag{10}$$

(i) Then the zero solution of (6) is stable.

(ii) If J is independent of $t_0 \geq 0$, then the zero solution is uniformly stable.

(iii) If

$$\int_0^t a(s+r)ds \to \infty \qquad \text{as} \qquad t \to \infty \tag{11}$$

then the zero solution of (6) is asymptotically stable.

(iv) If J is independent of t_0 and if

$$\int_{t_0}^t a(s+r)ds \to \infty \qquad \text{as} \qquad t - t_0 \to \infty \qquad \text{uniformly in} \qquad t_0 \tag{12}$$

then $x = 0$ is uniformly asymptotically stable.

Proof. To prove (i) and (ii) we note that if $\phi, \eta \in M$, then for $\|\phi - \eta\| = \sup_{t\geq 0}|\phi(t) - \eta(t)|$ we have

$$\begin{aligned}&|(P\phi)(t) - (P\eta)(t)| \\ &\leq \left(\int_{t-r}^t |a(s+r)|ds + \int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|\int_{s-r}^s |a(u+r)|du\right)\|\phi - \eta\| \\ &\leq \alpha\|\phi - \eta\|\end{aligned}$$

so P is a contraction. Certainly, $P : M \to M$. Thus, there is a unique $\phi \in M$ such that $P\phi = \phi$. Moreover, if we also denote $\|\psi\| = \sup_{-r \le t \le 0} |\psi(t)|$ we have

$$|(P\phi)(t)| \le \|\psi\| e^J + e^J \alpha \|\psi\| + \alpha \|\phi\|$$

so if $P\phi = \phi$ we have

$$\|\phi\|(1-\alpha) \le \|\psi\| e^J (1+\alpha), \tag{13}$$

a suitable stability relation at $t_0 = 0$. The proof for general t_0 is parallel. Relation (ii) follows in the same way.

To prove (iii) we add to M the condition that $\phi(t) \to 0$ as $t \to \infty$. We can then use the classical proof that the convolution of an L^1 function with a function tending to zero does, itself, tend to zero to show that $P : M \to M$.

The proof of (iv) is similar and will not be given.

4. REVIEW OF THE NONLINEAR CASE

This section can be found in [3], but, like the last one, it is a fundamental stepping stone to future sections. When trying to extend these results to asymptotic stability pay careful attention to the warning after the proof of Theorem 4.1.

Let r be a positive constant, $a : [0, \infty) \to R$ be continuous, and consider

$$x'(t) = -a(t) g(x(t-r)) \tag{14}$$

with continuous initial function

$$\psi : [-r, 0] \to R, \tag{15}$$

where g is continuous, locally Lipschitz, and odd, while $x - g(x)$ is nondecreasing and $g(x)$ is increasing on an interval $[0, L]$ for some $L > 0$. Then define a mapping P by $(P\phi)(t) = \psi(t)$ on $[-r, 0]$ and from (4) define

$$\begin{aligned}(P\phi)(t) &= \psi(0) e^{-\int_0^t a(s+r)ds} - e^{-\int_0^t a(s+r)ds} \int_{-r}^0 a(s+r) g(\psi(s)) ds \\ &\quad + \int_0^t e^{-\int_s^t a(u+r)du} a(s+r)[\phi(s) - g(\phi(s))] ds + \int_{t-r}^t a(u+r) g(\phi(u)) du \\ &\quad - \int_0^t e^{-\int_s^t a(u+r)du} a(s+r) \int_{s-r}^s a(u+r) g(\phi(u)) du ds \end{aligned} \tag{16}$$

Let $\|\psi\| < L$ and define a mapping set

$$M = \{\phi : [-r, \infty) \to R | \phi \in C, \phi_0 = \psi, |\phi(t)| \le L\}.$$

THEOREM 4.1. Let g be odd, increasing on $[0, L]$, satisfy a Lipschitz condition, and let $x - g(x)$ be nondecreasing on $[0, L]$. Suppose also that for each $L_1 \in (0, L]$ we have

$$|L_1 - g(L_1)| \sup_{t \ge 0} \int_0^t e^{-\int_s^t a(u+r)du} |a(s+r)| ds + g(L_1) \sup_{t \ge 0} \int_{t-r}^t |a(u+r)| du$$

$$+g(L_1)\sup_{t\geq 0}\int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|\int_{s-r}^s |a(u+r)|duds < L_1 \tag{17}$$

and there exists $J > 0$ such that

$$-\int_0^t a(s+r)ds \leq J \qquad \text{for} \qquad t \geq 0. \tag{18}$$

Then the zero solution of (14) is stable.

Proof. By (17) there is an $\alpha < 1$ such that if $\phi \in M$ then

$$\begin{aligned}
&|(P\phi)(t)| \\
&\leq \|\psi\|e^J + e^J\|g(\psi)\|\int_{-r}^0 |a(s+r)|ds + |L - g(L)|\sup_{t\geq 0}\int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|ds \\
&+ g(L)\int_{t-r}^t |a(u+r)|du + g(L)\int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|\int_{s-r}^s |a(u+r)|duds \\
&\leq e^J[\|\psi\| + \|g(\psi)\|\int_{-r}^0 |a(s+r)|ds] + \alpha L.
\end{aligned}$$

Choose $\delta > 0$ so that $\|\psi\| < \delta$ and K the Lipschitz constant for g on $[0, L]$ implies that

$$e^J[\delta + K\delta\int_{-r}^0 |a(s+r)|ds] < (1-\alpha)L. \tag{19}$$

Then $|(P\phi)(t)| \leq L$ so we can show that $P : M \to M$. Since the mapping is given by integrals and the functions are Lipschitz, it is known (cf. Hale [11; p. 20], [3], and [4]) that we can then adopt an exponentially weighted metric making P a contraction. In this problem, an appropriate metric is that induced by the norm

$$|\phi|_K = \sup_{t\geq 0} e^{-(3K+2)\int_0^t |a(s+r)|ds}|\phi(t)|.$$

(At this point, if (17) holds only for L, itself, then we have a boundedness result.) For a given $\epsilon > 0$, $\epsilon < L$, substitute ϵ for L and obtain the usual stability proof.

Warning. If we were to add the condition to M that $\phi(t) \to 0$ as $t \to \infty$ with a view to proving asymptotic stability, we should note that M would not be complete under the weighted metric.

5. THE CASE FOR $\lim_{x\to 0}\frac{G(x)}{g(x)} = 1$

We will have expressions of the form

$$a(t,s)g(x) + b(t,s)G(x) \tag{20}$$

with an $L > 0$ so that $g, G : [-L, L] \to R$ are continuous, a, b are continuous for $0 \leq s \leq t < \infty$,

$$g \qquad \text{and} \qquad G \qquad \text{are odd and satisfy a Lipschitz condition,} \tag{21}$$

and

$$xg(x) > 0 \qquad \text{and} \qquad xG(x) > 0 \qquad \text{if} \qquad x \neq 0. \tag{22}$$

Condition (22) is often necessary for stability in the problems considered. But now we come to three conditions which seem unusual. We want to show that they can be substantially reduced. We will ask that

$$0 \leq x - g(x) \qquad \text{and} \qquad x - g(x) \qquad \text{is nondecreasing on} \qquad [0, L]. \tag{23}$$

Note 1. If $|\frac{d}{dx}g(x)| \leq D$ on $[0, L]$, for some $D > 0$, then write (20) as

$$[Da(t, s)][g(x)/D] + b(t, s)G(x)$$

and we then see that $0 \leq x - g(x)/D$ and $x - g(x)/D$ is nondecreasing. Thus, grouping terms accomplishes (23). Next, we need

$$\lim_{x \to 0} \frac{G(x)}{g(x)} = 1. \tag{24}$$

Note 2. If $\lim_{x\to 0} \frac{DG(x)}{g(x)} = \xi \neq 0$, then rewrite (20) as

$$Da(t, s)g(x)/D + \xi b(t, s)G(x)/\xi$$

and rename $G(x)/\xi$ as G. Thus, our two critical conditions (23) and (24) can both be avoided if $\xi \neq 0$ and if $\frac{d}{dx}g$ exists.

Finally, we need conditions we can not so easily avoid:

$$g, G \quad \text{odd and Lipschitz,} \quad g(x), |g(x) - G(x)| \quad \text{nondecreasing on} \quad [0, L]. \tag{25}$$

Consider the equation

$$x'(t) = -a(t)g(x(t - r)) - b(t)G(x(t - r)) \tag{26}$$

in which g and G are both odd, while g, $x - g(x)$, and $|g(x) - G(x)|$ are all nondecreasing and satisfy a Lipschitz condition for a constant K, all on the interval $[0, L]$ for some $L > 0$.

THEOREM 5.1. Suppose that $c(t) := a(t) + b(t) \geq 0$ and that (23), (24), and (25) hold. If, in addition,

$$\sup_{t \geq 0} \int_{t-r}^{t} c(s + r)ds < 1/2, \tag{27}$$

and

$$\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds \qquad \text{is bounded} \tag{28}$$

then the zero solution of (26) is stable.

Proof. By (27) there is a $\beta < 1$ with

$$2\sup_{t\geq 0}\int_{t-r}^{t} c(u+r)du \leq \beta. \tag{29}$$

Take

$$\alpha = \frac{1+\beta}{2} \tag{30}$$

and fix $L_1 > 0$ such that $0 < L < L_1$ implies that

$$\frac{|g(L)-G(L)|}{g(L)}\sup_{t\geq 0}\int_0^t e^{-\int_s^t c(u+s)du}|b(s)|ds + 2\sup_{t\geq 0}\int_{t-r}^{t} c(u+r)du \leq \alpha. \tag{31}$$

Find $\delta > 0$ such that $\|\psi\| < \delta$ implies that

$$\|\psi\| + \|g(\psi)\|\int_{-r}^{0} c(u+r)du \leq g(L)(1-\alpha). \tag{32}$$

Define

$$M = \{\phi : [-r,\infty) \to R | \phi_0 = \psi, \phi \in C, |\phi(t)| \leq L\}$$

and use (4) to define P as we have done before. For $\phi \in M$ we have

$$\begin{aligned}|(P\phi)(t)| &\leq \|\psi\| + \|g(\psi)\|\int_{-r}^{0} c(u+r)du + L - g(L)\\ &\quad + |g(L)-G(L)|\sup_{t\geq 0}\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds + 2g(L)\sup_{t\geq 0}\int_{t-r}^{t} c(s+r)ds\\ &\leq g(L)(1-\alpha) + L - g(L) + \alpha g(L) \leq L.\end{aligned}$$

We have shown that $P : M \to M$ and we can change the metric to one with an exponential weight so that P is a contraction with fixed point ϕ. As the argument works for a given $\epsilon = L < L_1$ this will prove stability at $t_0 = 0$. The proof for a general t_0 is completely parallel.

EXAMPLE 5.1. Let $0 < r < 1/2$, $g(x) = x$, $G(x) = (\operatorname{sgn} x)\ln(1+x)$, $a(t) = 1 - 2\sin t$, $b(t) = 2\sin t$. Then $c(t) = 1$, (24) and (28) are satisfied, while $\int_{t-r}^{t} c(s+r)ds = r < 1/2$ satisfies (29). Thus,

$$x'(t) = -(1-2\sin t)x(t-r) - 2\sin t(\operatorname{sgn} x(t-r))\ln(1+|x(t-r)|) \tag{33}$$

is stable for

$$r < 1/2. \tag{34}$$

6. THE COST OF BORROWING

Condition (24) was critical in the last section. Yet, without it we can still borrow the $b(t)$ and add it to $a(t)$, but there is a cost. This material was treated in [3] for a less general equation. We now consider

$$x'(t) = -a(t)g(x(t-r)) - b(t)G(x(t-r)) \tag{35}$$

with a, b continuous and with a continuous initial function

$$\psi : [-r, 0] \to R. \tag{36}$$

Using Theorem 2.1 we define the mapping equation $(P\phi)(t) = \psi(t)$ for $-r \leq t \leq 0$ and

$$\begin{aligned}(P\phi)(t) &= \psi(0)e^{-\int_0^t c(s+r)ds} - e^{-\int_0^t c(u+r)du}\int_{-r}^0 c(u+r)g(\psi(u))du \\ &+ \int_0^t e^{-\int_s^t c(u+r)du}c(s+r)[\phi(s) - g(\phi(s))]ds \\ &+ \int_0^t e^{-\int_s^t c(u+r)du}b(s)[g(\phi(s-r) - G(\phi(s-r))]ds \\ &+ \int_{t-r}^t c(u+r)g(\phi(u))du \\ &- \int_0^t e^{-\int_s^t c(u+r)du}c(s+r)\int_{s-r}^s c(u+r)g(\phi(u))duds. \end{aligned} \tag{37}$$

THEOREM 6.1. Let (23) and (25) hold. Define

$$H(L) := \sup_{0 \leq x \leq L} |g(x) - G(x)| \tag{38}$$

and

$$c(t) := a(t) + b(t) \geq 0. \tag{39}$$

Assume that

$$\frac{H(L)}{g(L)} \sup_{t \geq 0} \int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds + 2\sup_{t \geq 0}\int_{t-r}^t c(u+r)du < 1. \tag{40}$$

Under these conditions there is a $\delta > 0$ such that if ψ is a continuous initial function with $\|\psi\| < \delta$ then the unique solution $x(t, 0, \psi)$ of (35) satisfies $|x(t, 0, \psi)| < L$ for $t \geq 0$.

Proof. Define

$$M = \{\phi : [-r, \infty) \to R | \phi \in C, \phi_0 = \psi, |\phi(t)| \leq L\}.$$

Now $\phi \in M$ so $|\phi(t)| \le L$ and by our assumptions we will get

$$
\begin{aligned}
|(P\phi)(t)| \le \|\psi\| + \|g(\psi)\| \int_{-r}^{0} c(u+r)du \\
+ L - g(L) + H(L) \sup_{t\ge 0} \int_0^t e^{-\int_s^t c(u+r)du} |b(s)| ds \\
+ 2g(L) \sup_{t\ge 0} \int_{t-r}^{t} c(u+r)du \le L
\end{aligned}
$$

upon application of (40), provided that $\|\psi\| < L_1$ and L_1 is small enough. This will yield $P : M \to M$. Because $P : M \to M$, because the functions are Lipschitz, and because the mapping is represented by integrals, it is possible to define an exponentially weighted metric under which P is a contraction. The unique fixed point is a solution of (35).

REMARK. If we compare (40) with (28) we see that the first term in (40)

$$
\frac{H(L)}{g(L)} \sup_{t\ge 0} \int_0^t e^{-\int_s^t c(u+r)du} |b(s)| ds \tag{41}
$$

may be called the "cost" of borrowing $b(t)$ and adding it to $a(t)$. When (24) holds, there is no cost. We now want the reader to see an example obtained in [3] which goes with (24) to show that there is more than one way for the cost to become small.

Consider the equation

$$
x'(t) = -a(t)x^{2n+1}(t-r) - b(t)x^{2n+3}(t-r) \tag{42}
$$

where n is a positive integer, while a, b are continuous. Let

$$
c(t) := a(t) + b(t) \ge 0.
$$

Then (40) will reduce to

$$
(1 - L^2) \sup_{t\ge 0} \int_0^t e^{-\int_s^t c(u+r)du} |b(s)| ds + 2 \sup_{t\ge 0} \int_{t-r}^{t} c(s+r) ds < 1 \tag{43}
$$

where

$$
1 - L^2 = 1 - \frac{2n+1}{2n+3} = \frac{2}{2n+3}. \tag{44}
$$

If (43) holds then

$$
\int_0^t e^{-\int_s^t c(u+r)du} |b(s)| ds \qquad \text{is bounded.} \tag{45}
$$

We see that the cost tends to zero as $n \to \infty$. Yet,

$$G(x)/g(x) = x^2$$

which has limit zero as x approaches zero, as opposed to the condition

$$\lim_{x \to 0} \frac{G(x)}{g(x)} = 1. \tag{24}$$

Thus, two entirely different forces are at work. For a particular case, let

$$a(t) = 1 - 2\sin t, \qquad b(t) = 2\sin t, \qquad c(t) = 1.$$

Then

$$\int_0^t e^{-\int_s^t 1du} |b(s)| ds \leq 2$$

so to satisfy (43) we need

$$2\frac{2}{2n+3} + 2r < 1$$

or

$$r < \frac{2n-1}{4n+6}.$$

It is evident that if $n = 0$, then $2\sin t$ is too large to successfully borrow to stabilize the equation. If we replace $2\sin t$ by $k\sin t$ where $1 < k < 3/2$ then we can borrow $b(t)$ even with $n = 0$. As mentioned before, this example appeared in [3] and contrasts with (24).

7. MULTIPLE DELAYS

Consider the scalar equation

$$x'(t) = -a(t)x(t-r) - b(t)x(t-h) - d(t)x(t/2) \tag{46}$$

where a, b, d are continuous with $0 \leq h \leq r$, where

$$c(t) := a(t+r) + b(t+h) + 2d(2t), \tag{47}$$

and where

$$-\int_0^t c(s)ds \qquad \text{is bounded above.} \tag{48}$$

THEOREM 7.1 If (48) holds and if there is an $\alpha < 1$ with

$$\begin{aligned}&\int_{t-r}^t |a(u+r)|du + \int_{t-h}^t |b(u+h)|du + \int_{t/2}^t 2|d(2u)|du \\ &+ \int_0^t e^{-\int_s^t c(u)du} |c(s)| [\int_{s-r}^s |a(u+r)|du + \int_{s-h}^s |b(u+h)|du + \int_{s/2}^s 2|d(2u)|du] ds \\ &\leq \alpha\end{aligned} \tag{49}$$

then for each continuous initial function $\psi : [-r,0] \to R$ the unique solution $x(t,0,\psi)$ of (46) is bounded; the zero solution of (46) is stable. If, in addition,

$$\int_0^t c(u)du \to \infty \qquad \text{as} \qquad t \to \infty, \tag{50}$$

then the zero solution of (46) is asymptotically stable.

Proof. We can write (46) as

$$\begin{aligned}
&x'(t)\\
&= -a(t+r)x(t) - b(t+h)x(t) - 2d(2t)x(t)\\
&+ \frac{d}{dt}\left(\int_{t-r}^t a(s+r)x(s)ds + \int_{t-h}^t b(s+h)x(s)ds + \int_{t/2}^t 2d(2s)x(s)ds\right)
\end{aligned}$$

and for a continuous initial function $\psi : [-r,0] \to R$ we have

$$\begin{aligned}
&x(t) = \psi(0)e^{-\int_0^t c(s)ds} + \int_0^t e^{-\int_s^t c(u)du} \times\\
&\frac{d}{ds}\left(\int_{s-r}^s a(u+r)x(u)du + \int_{s-h}^s b(u+h)x(u)du + \int_{s/2}^s 2d(2u)x(u)du\right).
\end{aligned}$$

Upon integration by parts we obtain

$$\begin{aligned}
&x(t)\\
&= \psi(0)e^{-\int_0^t c(s)ds} - e^{-\int_0^t c(u)du}\left(\int_{-r}^0 a(u+r)\psi(u)du + \int_{-h}^0 b(u+h)\psi(u)du\right)\\
&+ \int_{t-r}^t a(u+r)x(u)du + \int_{t-h}^t b(u+h)x(u)du + \int_{t/2}^t 2d(2u)x(u)du\\
&- \int_0^t e^{-\int_s^t c(u)du}c(s)\times\\
&\left(\int_{s-r}^s a(u+r)x(u)du + \int_{s-h}^s b(u+h)x(u)du + \int_{s/2}^s 2d(2u)x(u)du\right)ds.
\end{aligned} \tag{51}$$

We can use this equation to define a mapping P as we have done before. The conditions of the theorem will show that P will map bounded continuous functions into bounded continuous functions. Moreover, P is a contraction because of (49). This can be used to show the stability. With the addition of (50) we can take our mapping set to be the complete metric (supremum metric) space of bounded continuous functions (agreeing with the initial function on the initial interval) which tend to zero as $t \to \infty$. We can use the classical proof that the convolution of an L^1 function with a function which tends to zero does itself tend to zero in order to prove that $P\phi$ is in the set whenever ϕ is.

This example is linear, but it works equally well for nonlinear problems. There are simply more terms in the calculations. We can consider

$$x'(t) = -a(t)g(x(t-r)) - b(t)G(x(t-h)) - d(t)Q(x(t/2)), \tag{52}$$

obtain ode terms by the introduction of the neutral integrals, obtain linear terms by adding and subtracting, and proceed just as we did in Theorem 2.1.

Here are the details. We ask that $|x - g(x)|, |x - G(x)|, |x - Q(x)|$ are all nondecreasing on an interval $[0, L]$ and that g, G, Q are all odd on $[-L, L]$. Write (52) as

$$\begin{aligned} x'(t) &= -a(t)x(t-r) + a(t)[x(t-r) - g(x(t-r))] \\ &\quad - b(t)x(t-h) + b(t)[x(t-h) - G(x(t-h))] \\ &\quad - d(t)x(t/2) + d(t)[x(t/2) - Q(x(t/2))]. \end{aligned}$$

Thus, in the variation of parameters formula (51) we will need to add the term

$$\begin{aligned} &\int_0^t e^{-\int_s^t c(u)du}\Big(a(s)[x(s-r) - g(x(s-r))] \\ &+ b(s)[x(s-h) - G(x(s-h))] + d(s)[x(s/2) - Q(x(s/2))]\Big)\, ds. \end{aligned} \tag{53}$$

We set up the mapping set as we did in the proof of Theorem 7.1 and the mapping P using (51), augmented by (53). Thus, for $|\phi(t)| \le L$ we will have

$$\begin{aligned} &|(P\phi)(t)| \le \psi(0)e^{-\int_0^t c(s)ds} \\ &- e^{-\int_0^t c(u)du}\left(\int_{-r}^0 a(u+r)\psi(u)du + \int_{-h}^0 b(u+h)\psi(u)du\right) \\ &+ L\int_{t-r}^t |a(u+r)|du + L\int_{t-h}^t |b(u+h)|du + L\int_{t/2}^t 2|d(2u)|du \\ &+ \int_0^t e^{-\int_s^t c(u)du}|c(s)|\times \\ &\left(L\int_{s-r}^s |a(u+r)|du + L\int_{s-h}^s |b(u+h)|du + L\int_{s/2}^s 2|d(2u)|du\right)ds \\ &+ |L - g(L)|\int_0^t e^{-\int_s^t c(u)du}|a(s)|ds + |L - G(L)|\int_0^t e^{-\int_s^t c(u)du}|b(s)|ds \\ &+ |L - Q(L)|\int_0^t e^{-\int_s^t c(u)du}|d(s)|ds. \end{aligned} \tag{54}$$

The condition for stability is that this quantity be bounded by L; that will mean that our usual set M will be mapped into itself by P. We then need to ask that g, G, Q satisfy a local Lipschitz condition so that the metric can be changed to make P a contraction. Three basic results will then follow from (54). The last three terms in (54) represent the cost of changing the nonlinear terms to linear terms; Theorem 7.4 shows how to make the cost go to zero.

THEOREM 7.2 Suppose there is an $L > 0$ such that g, G, Q are locally Lipschitz and odd on $[-L, L]$, that $|x-g(x)|, |x-G(x)|, |x-Q(x)|$ are nondecreasing on $[0, L]$, and that there is an $\alpha < 1$ with

$$L \int_{t-r}^{t} |a(u+r)|du + L \int_{t-h}^{t} |b(u+h)|du + L \int_{t/2}^{t} 2|d(2u)|du$$
$$+ L \int_0^t e^{-\int_s^t c(u)du} |c(s)| \left(\int_{s-r}^{s} |a(u+r)|du + \int_{s-h}^{s} |b(u+h)|du + \int_{s/2}^{s} 2|d(2u)|du \right) ds$$
$$+ |L - g(L)| \int_0^t e^{-\int_s^t c(u)du} |a(s)|ds + |L - G(L)| \int_0^t e^{-\int_s^t c(u)du} |b(s)|ds$$
$$+ |L - Q(L)| \int_0^t e^{-\int_s^t c(u)du} |d(s)|ds \le \alpha L. \tag{55}$$

If (48) holds then the zero solution of (52) is stable.

THEOREM 7.3. If $a(t), b(t), d(t)$ are all non-negative, then (55) in Theorem 7.2 can be replaced by

$$2L \int_{t-r}^{t} a(u+r)du + 2L \int_{t-h}^{t} b(u+h)du + 2L \int_{t/2}^{t} 2d(2u)du$$
$$+ |L - g(L)| \int_0^t e^{-\int_s^t c(u)du} a(s)ds + |L - G(L)| \int_0^t e^{-\int_s^t c(u)du} b(s)ds$$
$$+ |L - Q(L)| \int_0^t e^{-\int_s^t c(u)du} d(s)ds \le \alpha L. \tag{56}$$

THEOREM 7.4. If $a(t), b(t), d(t)$ are all non-negative and if

$$\lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} \frac{G(x)}{x} = \lim_{x \to 0} \frac{Q(x)}{x} = 1 \tag{57}$$

then (55) in Theorem 7.3 can be replaced by

$$2L \int_{t-r}^{t} a(u+r)du + 2L \int_{t-h}^{t} b(u+h)du + 2L \int_{t/2}^{t} 2d(2u)du \le \alpha L. \tag{58}$$

There are numerous other combinations. Not all of the limits in (57) need be 1, but (58) is then modified. In Theorems 7.3 and 7.4 we do not need to ask that all of the coefficients be positive, but that will demand other changes.

PART II: VOLTERRA TYPE EQUATIONS

8. THE AVERAGED DELAY EQUATION

Consider the scalar equation

$$x'(t) = -\int_{t-r}^{t} a(t,s)g(x(s))ds - \int_{t-r}^{t} b(t,s)G(x(s))ds \tag{59}$$

where r is a positive constant and $a, b : [0, \infty) \times [0, \infty) \to R$ are continuous. Write (59) as

$$x'(t) = -\int_{t-r}^{t} [a(t,s) + b(t,s)]g(x(s))ds + \int_{t-r}^{t} b(t,s)[g(x(s)) - G(x(s))]ds.$$

Define

$$A(t,s) := \int_{t-s}^{r} [a(u+s,s) + b(u+s,s)]du \tag{60}$$

and ask that

$$A(t,t) = \int_{0}^{r} [a(u+t,t) + b(u+t,t)]du \geq 0 \tag{61}$$

with

$$\beta := 2\sup_{t\geq 0} \int_{t-r}^{t} |A(t,u)|du < 1 \tag{62}$$

and

$$\int_{0}^{t} e^{-\int_s^t A(u,u)du} \int_{s-r}^{s} |b(s,u)|duds \qquad \text{is bounded.} \tag{63}$$

Theorem 8.1. Let (22)-(25) and (61)-(63) hold. Then the zero solution of (59) is stable.

Proof. We first define some constants. For

$$\alpha = \frac{1+\beta}{2}$$

fix $L > 0$ so that

$$\frac{|g(L) - G(L)|}{g(L)} \sup_{t\geq 0} \int_{0}^{t} e^{-\int_s^t A(u,u)du} \int_{s-r}^{s} |b(s,u)|duds + \beta < \alpha.$$

For this L, find $\delta > 0$ so that

$$\delta + g(\delta) \int_{-r}^{0} |A(0,u)|du < g(L)(1-\alpha).$$

Then let $\|\psi\| < \delta$.

We can now write (59) as

$$\begin{aligned}
x'(t) &= -A(t,t)g(x(t)) + \frac{d}{dt}\int_{t-r}^{t} A(t,s)g(x(s))ds \\
&\quad + \int_{t-r}^{t} b(t,s)[g(x(s)) - G(x(s))]ds \\
&= -A(t,t)x(t) + A(t,t)[x - g(x)] \\
&\quad + \int_{t-r}^{t} b(t,s)[g(x(s)) - G(x(s))]ds + \frac{d}{dt}\int_{t-r}^{t} A(t,s)g(x(s))ds.
\end{aligned}$$

To specify a solution we need a continuous initial function $\psi : [-r, 0] \to [-L, L]$. By the variation of parameters formula we obtain

$$\begin{aligned} x(t) =& \psi(0)e^{-\int_0^t A(s,s)ds} - e^{-\int_0^t A(u,u)du} \int_{-r}^0 A(0,u)g(\psi(u))du \\ &+ \int_0^t e^{-\int_s^t A(u,u)du} A(s,s)[x(s) - g(x(s))]ds \\ &+ \int_0^t e^{-\int_s^t A(u,u)du} \int_{s-r}^s b(s,u)[g(x(u)) - G(x(u))]duds \\ &+ \int_{t-r}^t A(t,u)g(x(u))du \\ &- \int_0^t e^{-\int_s^t A(u,u)du} A(s,s) \int_{s-r}^s A(s,u)g(x(u))duds. \end{aligned}$$

Next, we define a mapping set

$$M = \{\phi : [-r, \infty) \to R | \phi_0 = \psi, \phi \in C, |\phi(t)| \leq L\}.$$

Then we use the above formula for $x(t)$ to define a mapping $P : M \to M$ by $\phi \in M$ implies $(P\phi)(t) = \psi(t)$ on $[-r, 0]$ and for $t > 0$ define

$$\begin{aligned} (P\phi)(t) =& \psi(0)e^{-\int_0^t A(s,s)ds} - e^{-\int_0^t A(u,u)du} \int_{-r}^0 A(0,u)g(\psi(u))du \\ &+ \int_0^t e^{-\int_s^t A(u,u)du} A(s,s)[\phi(s) - g(\phi(s))]ds \\ &+ \int_0^t e^{-\int_s^t A(u,u)du} \int_{s-r}^s b(s,u)[g(\phi(u)) - G(\phi(u))]du \\ &+ \int_{t-r}^t A(t,u)g(\phi(u))du \\ &- \int_0^t e^{-\int_s^t A(u,u)du} A(s,s) \int_{s-r}^s A(s,u)g(\phi(u))duds. \end{aligned}$$

Clearly, if $\phi \in M$ then $P\phi$ is continuous. Also,

$$\begin{aligned} |(P\phi)(t)| \leq& \|\psi\| + \|g(\psi)\| \int_{-r}^0 |A(0,u)|du + L - g(L) \\ &+ |g(L) - G(L)| \sup_{t\geq 0} \int_0^t e^{-\int_s^t A(u,u)du} \int_{s-r}^s |b(s,u)|duds \\ &+ 2g(L) \sup_{t\geq 0} \int_{t-r}^t |A(t,u)|du \\ &\leq \delta + g(\delta) \int_{-r}^0 |A(0,u)|du + L - g(L) + g(L)\alpha \\ &\leq g(L)(1-\alpha) + L - g(L) + g(L)\alpha \\ &= L. \end{aligned}$$

It now follows that $P : M \to M$. We can define a metric with an exponential weight which makes P a contracton with fixed point, a solution of (59). For a given ϵ sufficiently small, we may substitute ϵ for L and have a stability relation.

REMARK. Equations (1) and (59) can be combined and a stability result proved in the same way.

9. A VOLTERRA EQUATION

Consider the scalar equation

$$x'(t) = -\int_0^t a(t,s)g(x(s))ds - \int_0^t b(t,s)G(x(s))ds \tag{64}$$

where $a, b : [0,\infty) \times [0,\infty) \to R$ is continuous. In addition, suppose that there is a function $D(t,s)$ with

$$\frac{\partial D(t,s)}{\partial t} = -[a(t,s) + b(t,s)] \tag{65}$$

and

$$D(t,t) \geq 0. \tag{66}$$

For example, it may be possible to select

$$D(t,s) = \int_t^\infty [a(v,s) + b(v,s)]dv \tag{67}$$

if the integral exists.

Next, we suppose that

$$\int_0^t e^{-\int_s^t D(u,u)du} \int_0^s |b(s,u)|duds \qquad \text{is bounded,} \tag{68}$$

and there is a $\beta < 1$ with

$$2\int_0^t |D(t,u)|du < \beta. \tag{69}$$

If (67) holds then we could write (69) as

$$2\int_0^t |\int_t^\infty [a(v,u) + b(v,u)]dv|du < \beta. \tag{70}$$

THEOREM 9.1. Let (22)-(25) hold. If (65), (66), (68), and (69) hold, then the zero solution of (64) is stable.

Proof. We will need some constants defined in order to show that we have a proper mapping. For

$$\alpha = \frac{1+\beta}{2}$$

fix $L > 0$ so that

$$\frac{|g(L) - G(L)|}{g(L)} \int_0^t e^{-\int_s^t D(u,u)du} \int_0^s |b(s,u)|duds + \beta < \alpha.$$

For this L, let

$$|x_0| < g(L)(1 - \alpha).$$

Next, write (64) as

$$\begin{aligned}
x'(t) &= -\int_0^t [a(t,s) + b(t,s)]g(x(s))ds + \int_0^t b(t,s)[g(x(s)) - G(x(s))]ds \\
&= -D(t,t)g(x(t)) + \frac{d}{dt}\int_0^t D(t,s)g(x(s))ds + \int_0^t b(t,s)[g(x(s)) - G(x(s))]ds \\
&= -D(t,t)x(t) + D(t,t)[x(t) - g(x(t))] + \frac{d}{dt}\int_0^t D(t,s)g(x(s))ds \\
&+ \int_0^t b(t,s)[g(x(s)) - G(x(s))]ds
\end{aligned}$$

By the variation of parameters formula and integration by parts we obtain

$$\begin{aligned}
x(t) =& x_0 e^{-\int_0^t D(s,s)ds} + \int_0^t e^{-\int_s^t D(u,u)du} D(s,s)[x(s) - g(x(s))]ds \\
&+ \int_0^t e^{-\int_s^t D(u,u)du} \int_0^s b(s,u)[g(x(u)) - G(x(u))]duds \\
&+ \int_0^t D(t,u)g(x(u))du - \int_0^t e^{-\int_s^t D(u,u)du} D(s,s) \int_0^s D(s,u)g(x(u))duds.
\end{aligned}$$

Define a set

$$M\{\phi : [0,\infty) \to R | \phi(0) = x_0, \phi \in C, |\phi(t)| \le L\}$$

and then use our equation for x to define a mapping $P : M \to M$ by $\phi \in M$ implies that

$$\begin{aligned}
(P\phi)(t) = & x_0 e^{-\int_0^t D(s,s)ds} + \int_0^t e^{-\int_s^t D(u,u)du} D(s,s)[\phi(s) - g(\phi(s))]ds \\
&+ \int_0^t e^{-\int_s^t D(u,u)du} \int_0^s b(s,u)[g(\phi(u)) - G(\phi(u))]duds \\
&+ \int_0^t D(t,u)g(\phi(u))du - \int_0^t e^{-\int_s^t D(u,u)du} D(s,s) \int_0^s D(s,u)g(\phi(u))duds.
\end{aligned}$$

To see that P does map $M \to M$, we note that if $\phi \in M$ then $P\phi$ is continuous and $(P\phi)(0) = x_0$. Moreover, by the constants defined above we see that

$$|(P\phi)(t)| \le g(L)(1 - \alpha) + L - g(L) + \alpha g(L) = L.$$

This shows that $P : M \to M$. It is now possible to define a metric with exponential weight showing that P is a contraction with unique fixed point $\phi \in M$. That fixed point is a solution of (1). To see that we have stability, for a given $\epsilon > 0$ with $\epsilon < L$ for which our constants were chosen, substitute ϵ for L and take $\delta < g(L)(1 - \alpha)$.

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