SINGULAR INTEGRAL EQUATIONS, LIAPUNOV FUNCTIONALS, AND RESOLVENTS

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Abstract. This paper, together with a recent paper by the second named author on convex singular kernels, establishes a base for further investigation of mildly singular equations with Liapunov theory. We study the two nonlinear scalar integral equations

$$x(t) = a(t) - \int_0^t D(t,s)[x(s) + G(s,x(s))]ds$$

and

$$z(t) = a(t) - \int_0^t D(t,s)g(s,z(s))ds$$

where D has a singularity at $t = s$. The first equation is decomposed into three other simpler equations. We then construct a Liapunov functional for each of the equations which will yield L^p properties of the solutions.

1. Introduction

We consider a scalar integral equation of the form

$$x(t) = a(t) - \int_0^t D(t,s)[x(s) + G(s,x(s))]ds \tag{1}$$

in which D is a mildly singular kernel in a sense described in Section 2, while $G : [0,\infty) \times \Re \to \Re$ and $a : [0,\infty) \to \Re$ are continuous. Various assumptions will be made on $a(t)$ depending on whether or not D is of convolution type.

It is known that (1) can be decomposed into

$$y(t) = a(t) - \int_0^t D(t,s)y(s)ds \tag{2}$$

and two variation of parameters formulae

$$y(t) = a(t) - \int_0^t R(t,s)a(s)ds$$

$$x(t) = y(t) - \int_0^t R(t,s)G(s,x(s))ds \tag{3}$$

1991 *Mathematics Subject Classification.* Primary: 45G05, 34D20.

Key words and phrases. integral equations, fractional differential equations, Liapunov functionals, singular kernels.

where x solves (1), y solves (2), and $R(t,s)$ is the resolvent solving

$$R(t,s) = D(t,s) - \int_s^t D(t,u)R(u,s)du. \tag{4}$$

For (3) see [10, p. 192] or [3, p. 164].

We will construct a Liapunov functional for a nonlinear form of (2), namely,

$$z(t) = a(t) - \int_0^t D(t,s)g(s,z(s))ds \tag{2a}$$

which will yield our first main result. It will be necessary to develop the properties of R in two steps owing to the fact that both terms in the integral in (4) are singular, making it difficult to interchange the order of integration in one crucial step. A similar problem is encountered in establishing (3). These are both solved by the Tonelli-Hobson test [1] or [11]. The solution we present will depend fundamentally on a recent result of Becker [2] which states that

$$R(t,s) = D(t,s) - R_1(t,s) \tag{5}$$

where $R_1(t,s)$ is the continuous solution of the equation

$$R_1(t,s) = D^*(t,s) - \int_s^t D(t,u)R_1(u,s)du \tag{6}$$

when

$$D^*(t,s) = \int_s^t D(t,u)D(u,s)du \tag{7}$$

is continuous. As $R_1(t,s)$ is continuous the aforementioned difficulty of interchange of order of integration will vanish upon application of the Tonelli-Hobson test. See Section 7 for integration.

We will construct a Liapunov functional for (6) yielding certain integrability properties of R_1. Those properties in (5) will yield properties of R since D is given. With this information in hand and with properties of y known from an earlier Liapunov functional, we will be able to offer important properties of the solution of (1). Finally, we will use R itself in a Liapunov functional for (3). For each integral equation, its kernel will be used in the same way to construct a Liapunov functional for the integral equation.

2. A Liapunov Functional for ($2a$)

Before we start with our analysis, we need to mention two problems.

Existence For Equation (2a) we suppose that $g : [0,\infty) \times \Re \to \Re$ is continuous, that there is a $K > 0$ with

$$|g(t,x)| \leq |x| \text{ and } |g(t,x) - g(t,y)| \leq K|x-y|, \tag{8}$$

and that whenever $\phi : [0,\infty) \to \Re$ is continuous, then both $\int_0^t |D(t,s)|ds$ and $\int_0^t D(t,s)\phi(s)ds$ are continuous. Moreover, we ask that $D(t,s)$ be continuous for $0 \le s < t < \infty$. With these assumptions we are set up to give a contraction mapping argument with weighted norm, following Becker [2]. But for the weight we need to also suppose that for each $T > 0$ there is a $\gamma > 0$ and an $\eta < 1$ with

$$\int_0^t e^{-\gamma(t-s)}|D(t,s)|ds \le \eta$$

for $0 \le t \le T$. This is enough to ensure the existence of a unique solution of (2), (2a), and (6) when (7) holds. The reader can consult [2] for details. However, (1) offers several additional difficulties. $G(t,x)$ is a perturbation which may represent uncertainties and it would be unsuitable to ask for a Lipschitz condition. Moreover, because of the singularity if we only ask continuity, then the classical existence proof of Tonelli (see [3, p. 178]) would be troublesome, although the difficulties might be overcome. What seems to be best is to use the fixed point theorem of Krasnoselskii in which we define a mapping from (1) by

$$(P\phi)(t) = \left(a(t) - \int_0^t D(t,s)\phi(s)ds\right) - \int_0^t D(t,s)G(s,\phi(s))ds$$

with the first term a contraction and the second a compact map. One can see the details of such an argument in [8] where we are discussing periodic solutions. These mappings are also used throughout [3].

Our work here is primarily an illustration of the use of a number of Liapunov functionals to solve a complex problem. It would be a distraction to develop that fixed point theory here. Instead, when we discuss (1) we will state that any solution existing on $[0,\infty)$ satisfies the conclusions stated in the theorem.

Interchange of order of integration When we decompose (1), when we integrate the derivative of any of our Liapunov functionals, and when we discuss the classical relations of C and R, we always need to interchange the order of integration. A problem arises in every case because of the singularity. We avoid those problems by using the Hobson-Tonelli test. For that, we need some of the assumptions in our **existence** discussion, as well as the condition that there is an $\epsilon_0 > 0$ so that if $0 \le \epsilon \le \epsilon_0$, then

$$\int_s^t |D(u+\epsilon,s)|du$$

exists. Notice that $\epsilon = 0$ is included. All of these problems were encountered in [5] and [8] where we constructed Liapunov functionals for integral equations with convex kernels containing similar singularities.

Theorem 2.1. *Let $xg(t,x) \geq 0$. Suppose there are positive numbers α, β and an even integer $p > 0$ so that*

$$\beta + (p-1)\alpha < p, \tag{9}$$

that for each $\epsilon > 0$ we have

$$\sup_{t \geq 0} \int_{\epsilon}^{\infty} |D(u+t,t)|du \leq \beta, \tag{10}$$

and that for $t \geq 0$ then

$$\int_0^{\infty} |D(t,s)|ds \leq \alpha. \tag{11}$$

Moreover, assume that there exists $\mu > 0$ with

$$\mu \in (0, p - \beta - (p-1)\alpha) \tag{12}$$

such that for all sufficiently small $\epsilon > 0$ then

$$\sup_{s \in [0,\infty)} \int_s^{\infty} |D(u+\epsilon, s) - D(u,s)|du < \mu. \tag{13}$$

If $a \in L^p[0,\infty)$ and if z solves (2a) on $[0,\infty)$ then $g(t,z(t)) \in L^p[0,\infty)$.

Proof. For $\epsilon > 0$ satisfying (13) and for $t \geq 0$ define

$$V(t,\epsilon) = \int_0^t \left[\int_{t-s+\epsilon}^{\infty} |D(u+s,s)|du\right] |g(s,z(s))|^p ds \tag{14}$$

so that $u \geq t - s + \epsilon \geq \epsilon$ since $0 \leq s \leq t$; that is, the integrand is continuous. In preparation for V' we derive two relations. First, since $z(t)$ is a solution of (2a), it is true that

$$pg^{p-1}(t,z(t))\left[a(t) - z(t) - \int_0^t D(t,s)g(s,z(s))ds\right] = 0.$$

Next, due to

$$-|D(t+\epsilon,s)| \leq -|D(t,s)| + |D(t+\epsilon,s) - D(t,s)|$$

we have

$$\begin{aligned}
V'(t,\epsilon) &= \int_{\epsilon}^{\infty} |D(u+t,t)|du|g(t,z)|^p - \int_0^t |D(t+\epsilon,s)||g(s,z(s))|^p ds \\
&\leq \beta|g(t,z)|^p - \int_0^t |D(t,s)||g(s,z(s))|^p ds \\
&+ \int_0^t |D(t,s) - D(t+\epsilon,s)||g(s,z(s))|^p ds \\
&+ pg^{p-1}(t,z(t))\left[a(t) - z(t) - \int_0^t D(t,s)g(s,z(s))ds\right].
\end{aligned}$$

Denote by H the last line in V'; that is,

$$
\begin{aligned}
H &= pg^{p-1}(t,z(t))\left[a(t) - z(t) - \int_0^t D(t,s)g(s,z(s))ds\right] \\
&= pg^{p-1}(t,z(t))a(t) - pg^{p-1}(t,z(t))z(t) \\
&- p\int_0^t D(t,s)g(s,z(s))g^{p-1}(t,z(t))ds
\end{aligned}
$$

and note that by (8) we have

$$-g^{p-1}(t,z(t))z(t) \le -g^p(t,z(t)).$$

Next, note that for $p \ge 2$ we have

$$\frac{1}{\frac{p}{p-1}} + \frac{1}{p} = 1$$

for use in Young's inequality:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

where $a \ge 0$, $b \ge 0$, and $q = p/(p-1)$. For

$$\gamma \in \left(0, \frac{p-(p-1)\alpha - \beta - \mu}{p-1}\right)$$

and for M satisfying

$$M^{\frac{1}{p}}\gamma^{\frac{p-1}{p}} \ge 1,$$

we apply the inequality to

$$M^{\frac{1}{p}}|a(t)| \cdot \gamma^{\frac{p-1}{p}}|g(t,z(t))|^{p-1}$$

obtaining

$$
\begin{aligned}
|g(t,z(t))|^{p-1}|a(t)| &\le M^{\frac{1}{p}}|a(t)| \cdot \gamma^{\frac{p-1}{p}}|g(t,z(t))|^{p-1} \\
&\le M\frac{a^p(t)}{p} + \gamma\frac{g^p(t,z(t))}{\frac{p}{p-1}}.
\end{aligned}
$$

Then this, along with Young's inequality also applied to the integrand below, yields

$$\begin{aligned}
H &\le p|g(t,z(t))|^{p-1}|a(t)| - pg^{p-1}(t,z(t))z(t) \\
&+ p\int_0^t |D(t,s)||g(s,z(s))||g(t,z(t))|^{p-1}ds \\
&\le pM\frac{a^p(t)}{p} + p\gamma\frac{g^p(t,z(t)}{\frac{p}{p-1}} - pg^p(t,z(t)) \\
&+ p\int_0^t |D(t,s)|\left(\frac{g^p(t,z(t))}{\frac{p}{p-1}} + \frac{g^p(s,z(s))}{p}\right)ds \\
&= Ma^p(t) + \gamma(p-1)g^p(t,z(t)) - pg^p(t,z(t)) \\
&+ (p-1)\int_0^t |D(t,s)|dsg^p(t,z(t)) + \int_0^t |D(t,s)|g^p(s,z(s))ds.
\end{aligned}$$

Putting this back into V' yields

$$\begin{aligned}
V'(t,\epsilon) &\le \beta g^p(t,z(t)) - \int_0^t |D(t,s)|g^p(s,z(s))ds \\
&+ \int_0^t |D(t+\epsilon,s) - D(t,s)|g^p(s,z(s))ds \\
&+ Ma^p(t) + \gamma(p-1)g^p(t,z(t)) - pg^p(t,z(t)) \\
&+ (p-1)\left[\int_0^t |D(t,s)|ds\right]g^p(t,z(t)) + \int_0^t |D(t,s)|g^p(s,z(s))ds \\
&\le \left[\beta + \gamma(p-1) - p + (p-1)\int_0^t |D(t,s)|ds\right]g^p(t,z(t)) \\
&+ Ma^p(t) + \int_0^t |D(t+\epsilon,s) - D(t,s)|g^p(s,z(s))ds.
\end{aligned}$$

Taking (11) into consideration

$$\begin{aligned}
V'(t,\epsilon) &\le [\beta + \gamma(p-1) - p + (p-1)\alpha]g^p(t,z(t)) \\
&+ Ma^p(t) + \int_0^t |D(t+\epsilon,s) - D(t,s)|g^p(s,z(s))ds. \qquad (15)
\end{aligned}$$

If we integrate the last term from 0 to t and interchange the order of integration we obtain

$$\begin{aligned}
&\int_0^t\int_0^u |D(u+\epsilon,s) - D(u,s)|g^p(s,z(s))dsdu \\
&= \int_0^t\left[\int_s^t |D(u+\epsilon,s) - D(u,s)|du\right]g^p(s,z(s))ds \\
&\le \mu\int_0^t g^p(s,z(s))ds.
\end{aligned}$$

Using (15) this yields

$$V(t,\epsilon)-V(0,\epsilon)\le[\beta+(p-1)\alpha-p+\gamma(p-1)]\int_0^t g^p(s,z(s))ds$$

$$+M\int_0^t a^p(s)ds+\int_0^t\int_0^u|D(u+\epsilon,s)-D(u,s)|g^p(s,z(s))dsdu$$

$$\le[\beta+(p-1)\alpha-p+\gamma(p-1)]\int_0^t g^p(s,z(s))ds+M\int_0^t a^p(s)ds$$

$$+\mu\int_0^t g^p(s,z(s))ds$$

$$\le[\beta+(p-1)\alpha-p+\gamma(p-1)+\mu]\int_0^t g^p(s,z(s))ds+M\int_0^t a^p(s)ds.$$

As

$$\begin{aligned}\mu^* &:= \beta+(p-1)\alpha-p+\gamma(p-1)+\mu\\ &< \beta+(p-1)\alpha-p+\frac{p-(p-1)\alpha-\beta-\mu}{p-1}(p-1)+\mu\\ &= \beta+(p-1)\alpha-p+p-(p-1)\alpha-\beta-\mu+\mu=0,\end{aligned}$$

it follows that $\mu^*<0$ and

$$0\le V(t,\epsilon)\le V(0,\epsilon)+\mu^*\int_0^t g^p(s,z(s))ds+M\int_0^t a^p(s)ds,$$

as required. □

CONTEXT

Everything we do here will center on variants of (10) and (11), with particular attention paid to the constants α and β when either of them (or both) is greater than or equal to 1. This paper is entirely about small kernels, that is, kernels satisfying variants of conditions such as (A) or (B) below; and it is restricted to such kernels because there are absolutely no sign restrictions on any of the functions in (1). Not only are such results interesting in their own right, but there is a pecularity about Liapunov functionals and integral equations demonstrated in Section 6. It is shown that we may add kernels and add Liapunov functionals. So that if we have an integral equation of interest and a Liapunov functional for it, then we can add one of our small kernels to cover uncertain perturbations and add our Liapunov functionals developed here to the aforementioned Liapunov functional and have a ready-made perturbation result. We have discussed equations with convex kernels and mild singularities using Liapunov functionals in [5]. Those kernels can be large, but there are very severe sign restrictions.

Early classical results for

$$y(t) = a(t) - \int_0^t D(t,s)y(s)ds \tag{2}$$

(and nonlinear analogs) ask

$$\sup_{0\le t<\infty} \int_0^t |D(t,s)|ds = \alpha < 1. \tag{A}$$

When (A) holds there are three central results (among many others):

(i) If $a \in L^\infty$, so is y. (See [9, p. 127] [3, p. 22], for example.)

(ii) Also, $\sup_{0\le t<\infty} \int_0^t |R(t,s)| \le \frac{\alpha}{1-\alpha}$. (See [3, p. 54].)

(iii) If for each $T > 0$ we have $\lim_{t\to\infty} \int_0^T |D(t,s)|ds = 0$, then the same is true for R. (See [12] for a discussion.)

Much later we find a parallel theory for integration of the first coordinate of D, as developed in [7] and [3, p. 152-163]. If there is a $\beta < 1$ with

$$\sup_{0\le s\le t<\infty} \int_s^t |D(u,s)|du \le \beta < 1 \tag{B}$$

then there are L^p results and a pleasant parallel to (ii) in the form

$$\sup_{0\le s\le t<\infty} \int_s^t |R(u,s)|du \le \frac{\beta}{1-\beta}.$$

Both (ii) and this last result depend on D being continuous because of a needed interchange of order of integration. However, in case D has mild singularities of the type discussed here, the equations (5), (6), and (7) will allow us to obtain good substitutes.

With this context, we have four claims. First, write (9) as

$$\frac{\beta}{p} + \left(1 - \frac{1}{p}\right)\alpha < 1. \tag{9a}$$

Claim 1. Inequality (9) cannot hold for $\alpha \ge 2$.

Here are the details. Since p is an even integer with $p > 0$, we see that

$$\frac{1}{p} \le \frac{1}{2}$$

and that

$$1 - \frac{1}{p} \ge 1 - \frac{1}{2} = \frac{1}{2};$$

hence

$$\frac{1}{2}\alpha \le \left(1 - \frac{1}{p}\right)\alpha < 1,$$

from which we have $\alpha < 2$, as required.

Claim 2. From (9a) we have

$$\frac{\beta-\alpha}{p} < 1-\alpha$$

and it follows that for any $\alpha \in (0,1)$ and any $\beta > 0$ we may always find an even integer $p \geq 2$ such that (9) holds true.

Claim 3. If $\alpha \in (1,2)$, then (9) holds for a positive even integer p only if $\beta < \alpha$. Furthermore, a necessary and sufficient condition for the existence of such an integer so that (9) holds is $\beta + \alpha < 2$.

To see this, we first note that (9), namely $\beta + (p-1)\alpha < p$ is equivalent to $\beta - \alpha < p(1-\alpha)$. Thus, $p > 0$ and $\alpha > 1$ implies $\beta < \alpha$.

If (9) holds for an even integer $p \geq 2$, then as $1-\alpha < 0$, it follows that $p(1-\alpha) \leq 2(1-\alpha)$. Hence, $\beta-\alpha < 2(1-\alpha)$, which implies $\beta+\alpha < 2$. Conversely, if $\beta + \alpha < 2$, then $\beta - \alpha < 2(1-\alpha)$. Consequently, (9) holds with $p = 2$.

Claim 4. If $\alpha = 1$ then a necessary and sufficient condition for the existence of some positive even integer such that (9) holds is $\beta < \alpha = 1$. In this case (9) holds for any positive even integer.

Finally, we have a corollary to Theorem 2.1.

Corollary 2.2. *Let $xg(t,x) \geq 0$ and assume that there exist α, β such that for each $\epsilon > 0$ we have*

$$\sup_{t\geq 0} \int_{\epsilon}^{\infty} |D(u+t,t)|du \leq \beta, \tag{10}$$

and for $t \geq 0$

$$\int_0^{\infty} |D(t,s)|ds \leq \alpha < 2. \tag{11}$$

Moreover, suppose that for some even integer p with

$$\beta + (p-1)\alpha < p \tag{9}$$

we have

$$\sup_{s\in[0,\infty)} \int_s^{\infty} |D(u+\epsilon,s) - D(u,s)|du < p - \beta - (p-1)\alpha. \tag{13}$$

If $a \in L^p[0,\infty)$ and if z solves (2a) on $[0,\infty)$ then $g(t,z(t)) \in L^p[0,\infty)$.

Note that if $\alpha \in (0,1)$ and if the integral in (13) is bounded then in view of the monotonicity of $p - \beta - (p-1)\alpha$ in p (or Claim 2) we may see that there always exists a (smallest) positive even integer p_0 such that (9) and (13) hold true for all $p \geq p_0$. It follows that if $\alpha \in (0,1)$, if $\beta < 1$ and p_0 is the smallest positive even integer such that (9) and (13) hold true, then $a \in L^p[0,\infty)$ implies that $g(t,z(t)) \in L^p[0,\infty)$ for any even integer $p \geq p_0$.

3. A Liapunov Functional for (2a) when $p = 1$

Important equations are missed when $p \geq 2$ and we also need to prepare for the resolvent equation where we will pick up the case of z bounded when $a(t)$ is bounded in the convolution case. Thus, we return to

$$z(t) = a(t) - \int_0^t D(t,s)g(s,z(s))ds, \quad t \geq 0, \tag{2a}$$

with the existence assumptions detailed in Section 2.

Theorem 3.1. *Let $z(t)$ be a solution of (2a) for $0 \leq t < \infty$. Assume that there exists a function $h : [0,\infty) \to [0,\infty)$ with*

$$|g(t,x)| \leq h(t)|x|, \text{ for all } (t,x) \in [0,\infty) \times \Re, \tag{8a}$$

a function $\beta : [0,\infty) \to [0,\infty)$, and an $\epsilon > 0$ such that for all $t \geq 0$ we have the convergent integral satisfying

$$\int_\epsilon^\infty |D(u+t,t)|du \leq \beta(t). \tag{10a}$$

Moreover, suppose that there is a positive constant T with

$$\sup_{t \geq T} h(t)[\beta(t) + \phi(t)] < 1 \tag{12a}$$

where the continuous function ϕ is defined by the convergent integral

$$\int_s^\infty |D(u+\epsilon,s) - D(u,s)|du =: \phi(s). \tag{13a}$$

Then $a \in L^1[0,\infty)$ implies $z \in L^1[0,\infty)$. In fact, the integral in (11) need not be bounded.

Proof. For the selected $\epsilon > 0$ we define

$$V(t,\epsilon) = \int_0^t \left[\int_{t-s+\epsilon}^\infty |D(u+s,s)|du\right] |g(s,z(s))|ds, \quad t \geq 0. \tag{14a}$$

Since $0 \leq s \leq t$ we have $u \geq t - s + \epsilon \geq \epsilon$, so V is well defined and the integrand is continuous.

The derivative of V yields

$$\begin{aligned}
V'(t,\epsilon) &= \int_{\epsilon}^{\infty} |D(u+t,t)|du|g(t,z(t))| \\
&\quad - \int_0^t |D(t+\epsilon,s)||g(s,z(s))|ds \\
&\leq \beta(t)|g(t,z(t))| - \int_0^t |D(t,s)||g(s,z(s))|ds \\
&\quad + \int_0^t |D(t+\epsilon,s)-D(t,s)||g(s,z(s))|ds \\
&\leq \beta(t)h(t)|z(t)| - |a(t)-z(t)| \\
&\quad + \int_0^t |D(t+\epsilon,s)-D(t,s)||g(s,z(s))|ds.
\end{aligned}$$

In the final lines of our computation below we will need to see that

$$\begin{aligned}
&\int_0^t \int_0^u |D(u+\epsilon,s)-D(u,s)||g(s,z(s))|dsdu \\
&\qquad = \int_0^t \left[\int_s^t |D(u+\epsilon,s)-D(u,s)|du\right]|g(s,z(s))|ds \\
&\qquad \leq \int_0^t \phi(s)|g(s,z(s))|ds.
\end{aligned}$$

Integrating this estimate for V' yields

$$\begin{aligned}
V(t,\epsilon)-V(0,\epsilon) &\leq \int_0^t [\beta(s)h(s)|z(s)| + |a(s)| - |z(s)|]ds \\
&\quad + \int_0^t \int_0^u |D(u+\epsilon,s)-D(u,s)||g(s,z(s))|dsdu \\
&= \int_0^t [\beta(s)h(s)|z(s)| + |a(s)| - |z(s)|]ds \\
&\quad + \int_0^t \int_s^t |D(u+\epsilon,s)-D(u,s)|du h(s)|z(s)|ds \\
&= \int_0^t [\beta(s)h(s)|z(s)| + \phi(s)h(s)|z(s)|]ds \\
&\quad + \int_0^t |a(s)|ds - \int_0^t |z(s)|ds.
\end{aligned}$$

In view of (12a) for $\mu = \sup_{t\geq T}\{h(t)[\beta(t)+\phi(t)]\} < 1$ we have for $t \geq T$

$$
\begin{aligned}
&V(t,\epsilon) - V(0,\epsilon) \leq \int_0^T h(s)[\beta(s)+\phi(s)]|z(s)|]ds \\
&+ \int_T^t h(s)[\beta(s)+\phi(s)]|z(s)|ds + \int_0^t |a(s)|ds - \int_0^t |z(s)|ds \\
&< \int_0^T h(s)[\beta(s)+\phi(s)]|z(s)|ds + \mu \int_T^t |z(s)|ds \\
&+ \int_0^t |a(s)|ds - \int_T^t |z(s)|ds \\
&= \int_0^T h(s)[\beta(s)+\phi(s)]|z(s)|ds + \int_0^t |a(s)|ds - (1-\mu)\int_T^t |z(s)|ds.
\end{aligned}
$$

As the solution exists for all $t \geq 0$, the third-to-last integral is a finite number, while the last integral yields the result. □

4. A Liapunov Functional for the Resolvent

Our focus here is on the resolvent equation

$$R_1(t,s) = D^*(t,s) - \int_s^t D(t,u)R_1(u,s)du \tag{6}$$

where $R_1(t,s)$ and

$$D^*(t,s) = \int_s^t D(t,u)D(u,s)du \tag{7}$$

are both continuous, while D satisfies the existence and interchange conditions in the first part of Section 2. We now present a result which is parallel to Theorem 3.1 with $h(t) = 1$, D^* replacing $a(t)$, and yielding the conclusion that there is a $\mu^* > 0$ with

$$\int_s^t |R_1(u,s)|du \leq \mu^* \int_s^t |D^*(u,s)|du. \tag{17}$$

From (3) and (5) we see that

$$y(t) = a(t) - \int_0^t [D(t,s) - R_1(t,s)]a(s)ds. \tag{18}$$

If D is of convolution type, so are R and R_1 so that when $\mu^* \int_s^t |D^*(u,s)|du \leq M$ for some finite value, then (17) becomes

$$\int_0^{t-s} |R_1(v)|dv = \int_s^t |R_1(u-s)|du \leq M.$$

In other words, $R_1 \in L^1[0,\infty)$.

If D is also in $L^1[0,\infty)$, then $R \in L^1[0,\infty)$ from (5) and so from (3) we have that

$$a \in L^\infty \implies y \in L^\infty.$$

Thus, in the convolution case we get both $a \in L^1$ and $a \in L^\infty$ imply the same for y.

Continuing in the same vein, if $R \in L^1[0,\infty)$ and if $a(t) \to 0$ as $t \to \infty$, from (3) we get that $y(t) \to 0$ as $t \to \infty$.

The rewards continue as we look at the other equation in (3).

It would be a real *coup* to advance that to the non-convolution case, but we will see parallel L^p results.

Theorem 4.1. *Suppose there is a constant $\beta < 1$ such that for each sufficiently small $\epsilon > 0$ we have*

$$\sup_{t\geq 0} \int_\epsilon^\infty |D(u+t,t)|du \leq \beta \tag{10b}$$

and

$$\sup_{s\in[0,\infty)} \int_s^\infty |D(u+\epsilon,s) - D(u,s)|du =: \mu < 1-\beta. \tag{13b}$$

Then for

$$\mu^* = \frac{1}{1-\beta-\mu}$$

we have

$$\int_s^t |R_1(u,s)|du \leq \mu^* \int_s^t |D^*(u,s)|du. \tag{19}$$

Proof. For $\epsilon > 0$ so small that (10b) and (13b) hold, define

$$V(t,\epsilon) = \int_s^t \int_{t-u+\epsilon}^\infty |D(v+u,u)|dv|R_1(u,s)|du$$

so that

$$\begin{aligned}
V'(t,\epsilon) = &\int_\epsilon^\infty |D(v+t,t)|dv|R_1(t,s)| \\
&- \int_s^t |D(t+\epsilon,u)||R_1(u,s)|du \\
&\leq \beta|R_1(t,s)| - \int_s^t |D(t,u)||R_1(u,s)|du \\
&+ \int_s^t |D(t+\epsilon,u) - D(t,u)||R_1(u,s)|du \\
&\leq \beta|R_1(t,s)| - |R_1(t,s)| + |D^*(t,s)| \\
&+ \int_s^t |D(t+\epsilon,u) - D(t,u)||R_1(u,s)|du.
\end{aligned}$$

Integration of the last term yields

$$\begin{aligned}
&\int_s^t \int_s^v |D(v+\epsilon,u) - D(v,u)||R_1(u,s)|dudv \\
&= \int_s^t \int_u^t |D(v+\epsilon,u) - D(v,u)|dv|R_1(u,s)|du \\
&\le \int_s^t \mu|R_1(u,s)|du.
\end{aligned}$$

Thus, if we integrate V' from s to t we have

$$\begin{aligned}
V(t,\epsilon) \le V(s,\epsilon) &- (1-\beta)\int_s^t |R_1(v,s)|dv \\
&+ \mu \int_s^t |R_1(v,s)|dv + \int_s^t |D^*(u,s)|du \\
&=: -\lambda \int_s^t |R_1(v,s)|dv + \int_s^t |D^*(u,s)|du.
\end{aligned}$$

Taking $\mu^* = 1/\lambda$ completes the proof. □

5. The Nonlinear Equation

We now consider (1) in the form of

$$x(t) = y(t) - \int_0^t R(t,s)G(s,x(s))ds. \tag{3}$$

Theorem 5.1. *Let $x(t)$ solve (1) on $[0,\infty)$. Suppose that $R(t,s) = R(t-s)$, $R \in L^1[0,\infty)$, and that $y \in L^\infty$. Assume also that*

$$|G(t,x)| \le \phi(t)|x| \tag{20}$$

where $\phi : [0,\infty) \to \Re$ is continuous. If $\phi(t) \to 0$ as $t \to \infty$, then $x \in L^\infty$. If $y(t) \to 0$ as $t \to \infty$ and if $\phi(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$.

Proof. Let $y \in L^\infty$, $R \in L^1[0,\infty)$, and let $\phi(t) \to 0$. If x is not bounded, then there is a sequence $\{t_n\} \uparrow \infty$ such that $|x(t)| \le |x(t_n)|$ if $0 \le t \le t_n$. Let $\|\cdot\|$ denote the supremum norm. Then for large n we have $|x(t_n)|$ unbounded and

$$|x(t_n)| \le \|y\| + |x(t_n)| \int_0^{t_n} |R(t_n - s)|\phi(s)ds \le \|y\| + (1/2)|x(t_n)|,$$

a contradiction.

Next, if $R \in L^1[0,\infty)$ and if $y(t) \to 0$ then we have x bounded and

$$|x(t)| \le |y(t)| + \|x\| \int_0^t |R(t-s)|\phi(s)ds \to 0.$$

□

In preparation for our next result, note from Theorem 4.1 that

$$\mu^* = \frac{1}{1-\beta-\mu}$$

where β and μ are defined in (10b) and (13b).

Theorem 5.2. *Let $y(t)$ solve (2) and let (10b) and (13b) hold. If*

$$\int_0^t \left[\int_s^t [|D(u,s)| + \mu^*|D^*(u,s)|]du\right]|a(s)|ds$$

is bounded for $t \geq 0$, then $a(t) \in L^1[0,\infty)$ implies that $y \in L^1[0,\infty)$.

Proof. Note that $y(t) = a(t) - \int_0^t R(t,s)a(s)ds$ and $R(t,s) = D(t,s) - R_1(t,s)$. From this we have

$$\begin{aligned}
&\int_0^t |y(u)|du - \int_0^t |a(u)|du \\
&\leq \int_0^t \int_0^u |D(u,s) - R_1(u,s)||a(s)|dsdu \\
&= \int_0^t \int_s^t |D(u,s) - R_1(u,s)||a(s)|duds \\
&\text{(but by (19)} \int_s^t |R_1(u,s)|du \leq \mu^* \int_s^t |D^*(u,s)|du \text{ so)} \\
&\leq \int_0^t \int_s^t (|D(u,s)| + \mu^*|D^*(u,s)|)du|a(s)|ds,
\end{aligned}$$

as required □

Notice that in Section 2 we were forced to take p larger than 1 to satisfy (9) and we obtained $g(t,z(t)) \in L^p$. Thus, suppose that in those results we have

$$g(t,z) = z + G(t,z)$$

and we have obtained $z \in L^2$. Can we force that back to $z \in L^1$? In [5] we studied a convex kernel with singularity and we obtained $g(t,x) \in L^2$ with no way to force it back into L^1. Moreover, using the properties of the Liapunov functional we also showed ways to get $x(t) - a(t)$ bounded.

There are many times when we want L^1 instead of L^2. There are "roundabout" theorems such as [4, pp. 62–3] featuring the properties of $z \in L^1$. Moreover, when we work with the resolvent then we definitely prefer the L^1 property. See also [3, pp. 134–8].

The next two theorems show us how to meet such needs. We do it here for $x \in L^2$, but using Hölder's inequality it can be extended, as we will note parenthetically in one of the proofs. We also show how to use x bounded for the same conclusion.

Theorem 5.3. *Let x solve (1), y solve (2), $y \in L^1[0,\infty)$, $x \in L^2[0,\infty)$, and let (10b), (13b), and (20) hold. If, in addition,*

$$(*) \qquad \sup_{t\geq 0} \int_0^t \left[\int_s^t (|D(u,s)| + \mu^* |D^*(u,s)|) du \right]^2 \phi^2(s) ds < \infty,$$

then $x \in L^1[0,\infty)$.

Proof. We have

$$\int_0^t |x(s)|ds - \int_0^t |y(s)|ds \leq \int_0^t \int_0^u |D(u,s) - R_1(u,s)|\phi(s)|x(s)|dsdu$$
$$= \int_0^t \int_s^t |D(u,s) - R_1(u,s)|du\phi(s)|x(s)|ds$$
$$\leq \int_0^t \int_s^t (|D(u,s)| + \mu^* |D(u,s)|) du\phi(s)|x(s)|ds$$

(we use x^2, but we could use x^p, Hölder's inequality, and change (*))

$$\leq \sqrt{\int_0^t \left[\int_s^t (|D(u,s)| + \mu^* |D^*(u,s)|) du \right]^2 \phi^2(s) ds \int_0^t x^2(s) ds},$$

as required. □

The same computation yields the following result.

Corollary 5.4. *Let (10b), (13b), (20), and (*) hold. Then $x \in L^2[0,\infty)$ implies $y - x \in L^1[0,\infty)$.*

Theorem 5.5. *Let x solve (1), y solve (2), $y \in L^1[0,\infty)$, x be bounded, and let (10b), (13b), and (20) hold. If, in addition,*

$$(**) \qquad \sup_{t\geq 0} \int_0^t \left[\int_s^t (|D(u,s)| + \mu^* |D^*(u,s)|) du \right] \phi(s) ds < \infty,$$

then $x \in L^1[0,\infty)$.

Proof. Follow the proof of Theorem 5.3 down to the relation

$$\int_0^t |x(s)|ds - \int_0^t |y(s)|ds \leq \int_0^t \int_s^t (|D(u,s)| + \mu^* |D^*(u,s)|) du\phi(s)|x(s)|ds.$$

As x is bounded, the conclusion is immediate. □

Corollary 5.6. *Let (10b), (13b), (20), and (**) hold. Then x bounded implies $y - x \in L^1[0,\infty)$.*

Corollary 5.7. *If (10b), (13b), (20) hold and if*

$$\sup_{0\leq s\leq t<\infty} \left[\int_s^t (|D(u,s)| + \mu^* |D^*(u,s)|) du\phi(s) \right] \leq \gamma < 1,$$

then $y \in L^1[0,\infty)$ implies $x \in L^1[0,\infty)$.

In the last proof we see that upon integration of that last display we obtain

$$\int_0^t |x(s)|ds \le \int_0^t |y(s)|ds + \int_0^t \gamma |x(s)|ds,$$

yielding the result. Corollary 5.7, with Theorem 5.2, yields the next result.

Corollary 5.8. *Let the conditions of Corollary 5.7 and*

$$\sup_{t\ge 0} \int_0^t \left[\int_s^t (|D(u,s)| + \mu^* |D^*(u,s)|)du\right] |a(s)|ds < \infty$$

hold. Then $a \in L^1[0,\infty)$ *implies* $x \in L^1[0,\infty)$.

6. A Liapunov Functional Based on $R(t,s)$

Equation (3) with (5) is

$$x(t) = y(t) - \int_0^t [D(t,s) - R_1(t,s)]G(s,x(s))ds.$$

If $R_1 = 0$ and if $y \in L^p$ we would define

$$V_1(t,\epsilon) = \int_0^t \int_{t-s+\epsilon}^\infty |D(u+s,s)|du|G(s,x(s))|ds.$$

If $D = 0$ we would define

$$V_2(t) = \int_0^t \int_{t-s}^\infty |R_1(u+s,s)|du|G(s,x(s))|ds.$$

One of the surprising aspects of Liapunov's direct method for investigators well acquainted with Liapunov theory for differential equations is that when we add kernels then we can add Liapunov functionals. That will be illustrated here.

In preparation for construction of a Liapunov functional based on the unknown function $R_1(t,s)$ we note two relations. First,

$$\int_0^\infty |R_1(v+t,t)|dv = \int_t^\infty |R_1(w,t)|dw$$

and if the conditions of Theorem 4.1 hold, then we consider (19)

$$\int_s^t |R_1(u,s)|du \le \mu^* \int_s^t |D^*(u,s)|du$$

and ask that there is a $\Lambda > 0$ with

$$\int_t^\infty |R_1(u,t)|du \le \mu^* \int_t^\infty |D^*(u,t)|du \le \Lambda. \tag{21}$$

Next, by a change of variable we see that

$$\int_{t-s}^\infty |R_1(u+s,s)|du \le \int_{t-t}^\infty |R_1(u+s,s)|du = \int_s^\infty |R_1(v,s)|dv \le \Lambda.$$

This means that when the conditions of Theorem 4.1 and (21) hold then

$$V(t,\epsilon)=\int_0^t\Big[\int_{t-s+\epsilon}^{\infty}|D(u+s,s)|du+\int_{t-s}^{\infty}|R_1(u+s,s)|du\Big]|G(s,x(s))|ds \tag{22}$$

is well-defined.

Theorem 6.1. *If the conditions of Theorem 4.1, (20), and (21) hold, if there is a* $\gamma<1$ *with*

$$[\beta+\Lambda+\mu]\phi(t)\le\gamma,\quad 0\le t<\infty, \tag{23}$$

then $y\in L^1[0,\infty)$ *implies* $x\in L^1[0,\infty)$.

Proof. With V defined in (22) we have

$$\begin{aligned}V'(t,\epsilon)&=\Big[\int_{\epsilon}^{\infty}|D(u+t,t)|du+\int_0^{\infty}|R_1(u+t,t)|du\Big]|G(t,x(t))\\&-\int_0^t[|D(t+\epsilon,s)|+|R_1(t,s)|]|G(s,x(s))|ds\\&\le[\beta+\Lambda]|G(t,x)|-\int_0^t[|D(t,s)|+|R_1(t,s)|]|G(s,x(s))|ds\\&+\int_0^t|D(t+\epsilon,s)-D(t,s)||G(s,x(s))|ds\end{aligned}$$

where β is defined in (10b) and Λ is defined in (21)

$$\begin{aligned}&\le[\beta+\Lambda]|G(t,x)|+|y(t)|-|x(t)|+\int_0^t|D(t+\epsilon,s)\\&-D(t,s)||G(s,x(s))|ds\\&\le\{[\beta+\Lambda]\phi(t)-1\}|x(t)|+|y(t)|+\int_0^t|D(t+\epsilon,s)-D(t,s)|\phi(s)|x(s)|ds.\end{aligned}$$

Integrate the last term to obtain

$$\begin{aligned}&\int_0^t\int_0^u|D(u+\epsilon,s)-D(u,s)|\phi(s)|x(s)|dsdu\\&=\int_0^t\int_s^t|D(u+\epsilon,s)-D(u,s)|du\phi(s)|x(s)|ds\\&\le\int_0^t\mu\phi(s)|x(s)|ds\end{aligned}$$

where μ is defined in (13b).

Thus, integration of V' yields

$$
\begin{aligned}
0 \le V(t,\epsilon) \le V(0,\epsilon) &+ \int_0^t \{[\beta+\Lambda+\mu]\phi(s)-1\}|x(s)|ds + \int_0^t |y(s)|ds \\
&\le -(1-\gamma)\int_0^t |x(s)|ds + \int_0^t |y(s)|ds,
\end{aligned}
$$

as required. □

Corollary 6.2. *Let the conditions of Theorem 4.1 and (20) hold with $\phi(t) \le 1$. If, in addition,*

$$\sup_{t\ge 0}\int_t^\infty |D^*(u,t)|du < (1-\beta-\mu)^2,$$

then $y \in L^1[0,\infty)$ implies that $x \in L^1[0,\infty)$.

Proof. Since $\phi(t) \le 1$, if we set

$$\psi(t) = \sup_{t\ge 0}\frac{|G(t,x)|}{|x|}$$

then $\psi(t) \le 1$ for $t \ge 0$. Moreover, (21) holds with

$$\Lambda = \mu^* \sup_{t\ge 0}\int_t^\infty |D^*(u,t)|du.$$

Thus, taking into consideration $\mu^*(1-\beta-\mu) = 1$ (from the definition of μ^*) we have for $t \ge 0$

$$
\begin{aligned}
[\beta+\Lambda+\mu]\psi(t) \le \gamma :&= \left[\beta + \mu^*\sup_{s\ge 0}\int_s^\infty |D^*(u,s)|du + \mu\right]\sup_{t\ge 0}\psi(t) \\
&\le \left[\beta + \mu^*\sup_{s\ge 0}\int_s^\infty |D^*(u,s)|du + \mu\right]\cdot 1 \\
&< \left[\beta + \mu^*(1-\beta-\mu)^2 + \mu\right] \\
&= [\beta + 1 - \beta - \mu + \mu] \\
&= 1;
\end{aligned}
$$

that is, (23) is satisfied. □

7. Integrations

This section was added because of a request of the referee that we show how to accomplish some of the complicated integrations. We are grateful for the referee's careful reading and for that request, as it now seems clear that it significantly improved the paper.

While the reader may work through the presentation with approval at each step, there is the nagging problem of all those integrations which start with D^* in (7) and progress to (**) in Theorem 5.5. Fortunately, they turn out to be fairly simple, even for the deep problems which occur throughout so much of applied mathematics.

To address this issue we begin with a fractional differential equation and an explicit integration. The interested reader might consult [6] for more complete details on fractional differential equations, although that is not essential. Let $g(t,x)$ be continuous on $[0,\infty)\times\Re$ and consider the scalar fractional differential equation of Caputo type

$$^{c}D^{q}x(t) = -g(t,x(t)), \quad 0<q<1, \quad x(0)\in\Re,$$

which is inverted as the ordinary integral equation

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s,x(s))ds$$

where Γ is the gamma function. A myriad of real-world problems take this form and the value $q=1/2$ is at the forefront. We listed a large set of essentially different classical and modern problems in [6] taking this form with $q=1/2$. That is emphasized here because when $q=1/2$ that intimidating integral D^* in (7) is simply a constant.

We first show how to compute D^* for this kernel, and then show how to use that for the calculations in Corollary 5.8 with a very different kernel. From those details the reader will see that a whole class of problems is solved for smaller kernels when the one calculation is done.

Example 7.1. If

$$D(t,s) = \frac{1}{\Gamma(q)}(t-s)^{q-1}, \quad 0<q<1,$$

then for $0\le s<t$ we have

$$D^*(t,s) = \int_s^t D(t,u)D(u,s)du = \frac{1}{\Gamma(2q)}(t-s)^{2q-1}.$$

In particular, if $q=1/2$ then $D^*(t,s) = 1/\Gamma(2q) = 1$.

Proof. We begin with

$$D^*(t,s) = \frac{1}{\Gamma^2(q)}\int_s^t (t-u)^{q-1}(u-s)^{q-1}du$$

and make the change of variable

$$v := (t-s)^{-1}(u-s)$$

so that

$$u = s+(t-s)v, \quad du = (t-s)dv,$$

while

$$t-u = t-s-(t-s)v = (t-s)(1-v).$$

Thus,

$$D^*(t,s) = \frac{1}{\Gamma^2(q)} \int_0^1 (t-s)^{q-1}(1-v)^{q-1}(t-s)^{q-1}v^{q-1}(t-s)dv$$
$$= \frac{1}{\Gamma^2(q)}(t-s)^{2q-1} \int_0^1 v^{q-1}(1-v)^{q-1}dv.$$

Now the beta function is

$$B(p,q) = \int_0^1 v^{p-1}(1-v)^{q-1}dv = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p>0, q>0.$$

Thus,

$$B(q,q) = \frac{\Gamma^2(q)}{\Gamma(2q)}.$$

Using this in the display we arrive at the desired conclusion. □

We continue on into Section 5 and see that we are also needing D to be an L^1 kernel. Referring back to [6] we see that through transformations we map our fractional differential equation into an equation having an L^1 kernel, so this is no real surprise. We now want to parlay the above example into D^* with an L^1 kernel. There is actually a simple way to do this. The point can be made in the following example and the reader will see that there is some generality in the method.

Example 7.2. The kernel

$$D(t,s) = \frac{1}{\Gamma(q)} \frac{(t-s)^{q-1}}{(t-s+1)^2}, \quad 0<q<1,$$

satisfies the conditions of Corollary 5.8.

Proof. Use the change of variable in the above proof and note that $0 \le v \le 1$ so $1-v$ or v is always as large as $1/2$. Thus for $t>s$ we have

$$D^*(t,s) = \frac{1}{\Gamma^2(q)} \int_s^t \frac{(t-u)^{q-1}(u-s)^{q-1}}{(t-u+1)^2(u-s+1)^2}du$$
$$= \frac{1}{\Gamma^2(q)} \int_0^1 \frac{(t-s)^{q-1}(1-v)^{q-1}(t-s)^{q-1}v^{q-1}(t-s)}{[(t-s)(1-v)+1]^2[(t-s)v+1]^2}dv$$
$$\le \frac{(t-s)^{2q-1}}{\Gamma^2(q)[(1/2)(t-s)+1]^2} \int_0^1 v^{q-1}(1-v)^{q-1}dv$$
$$= \frac{(t-s)^{2q-1}}{[(1/2)(t-s)+1]^2\Gamma(2q)}.$$

Since

$$\int_s^t |D^*(u,s)|du \le \frac{1}{\Gamma(2q)} \int_s^t \frac{(u-s)^{2q-1}}{[(1/2)(u-s)+1]^2}du$$
$$= \frac{1}{\Gamma(2q)} \int_0^{t-s} \frac{w^{2q-1}}{[(1/2)w+1]^2}dw$$

and $2q-1>-1$ so this integral converges at the lower limit. It also converges as the upper limit tends to ∞ since $2-2q+1>1$. Thus, in Corollary 5.8 we have

$$\int_0^t \int_s^t |D^*(u,s)|du|a(s)|ds$$

and this is finite since $a \in L^1[0,\infty)$.

Next,

$$\int_s^t |D(u,s)|du = \frac{1}{\Gamma(q)} \int_s^t \frac{(u-s)^{q-1}}{(u-s+1)^2} du = \frac{1}{\Gamma(q)} \int_0^{t-s} \frac{w^{q-1}}{(w+1)^2} dw.$$

As $q-1>-1$, this converges at the lower limit. As the denominator is of order w^2, $w^{2-q+1}=w^{3-q}$ so the integral converges as $t-s\to\infty$. As $a \in L^1[0,\infty)$, the integral condtion in Corollary 5.8 is satisfied. $\square$

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