SMOOTHED INTEGRAL EQUATIONS

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ABSTRACT. For a linear integral equation $x(t) = a(t) - \int_0^t B(t,s)x(s)ds$ there is a resolvent equation $R(t,s) = B(t,s) - \int_s^t B(t,u)R(u,s)du$ and a variation of parameters formula $x(t) = a(t) - \int_0^t R(t,s)a(s)ds$. It is assumed that B is a perturbed convex function and that $a(t)$ may be badly behaved in several ways. When the first two equations are treated separately by means of a Liapunov functional, restrictive conditions are required separately on $a(t)$ and $B(t,s)$. Here, we treat them as a single equation $f(t) = S(t) - \int_0^t B(t,u)f(u)du$ where S is an integral combination of $a(t)$ and $B(t,s)$. There are two distinct advantages. First, possibly bad behavior of $a(t)$ is smoothed. Next, properties of S needed in the Liapunov functional can be obtained from an array of properties of $a(t)$ and $B(t,s)$ yielding considerable flexibility not seen in standard treatment. The results are used to treat nonlinear perturbation problems. Moreover, the function $y(t) = a(t) - \int_0^t B(t,s)a(s)ds$ is shown to converge pointwise and in $L^2[0,\infty)$ to $x(t)$.

1. Introduction

We consider a perturbed integral equation

$$z(t) = a(t) - \int_0^t B(t,s)[z(s) + G(s, z(s))]ds \tag{1}$$

in which a satisfies a variety of integral conditions, $|G(t,z)| \leq \phi(t)|z|$ where $\phi \in L^p[0,\infty)$, and $B(t,s)$ is a perturbed convex kernel. In a paper to follow this one, we will show that B can be either a growing memory kernel or a fading memory kernel and parallel treatments will yield essentially the same results. We derive conditions under which $z \in L^p$ and $z(t) \to a(t)$. When a, B, and G are continuous then there is a local solution of (1) and if it remains bounded then it can be continued to $0 \leq t < \infty$. (See for example [5; Chapter 3].)

First note we may rewrite (1) as

$$z(t) = [a(t) - \int_0^t B(t,s)z(s)ds] - \int_0^t B(t,s)G(s,z(s))ds, \tag{1'}$$

in which we could refer to the bracketed term as the unperturbed part and the second integral term as the perturbed part. The unperturbed

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equation

$$x(t) = a(t) - \int_0^t B(t,s)x(s)ds \tag{2}$$

has as its solution $x(t) = a(t) - \int_0^t R(t,s)a(s)ds$, where $R(t,s)$ solves the resolvent equation

$$R(t,s) = B(t,s) - \int_s^t B(t,u)R(u,s)du \tag{3}$$

for $0 \le s \le t < \infty$. We may then claim a solution to (1) has been found if we can find a solution to the equation

$$z(t) = x(t) - \int_0^t R(t,s)G(s,z(s))ds. \tag{4}$$

By direct substitution it is a relatively simple exercise to show that if z solves (4) with x solving (2) and R solving (3) then z solves the perturbed equation (1). (See [5; p. 163] or [15; p. 190] for example.)

The long term project is to begin with a convex kernel $C(t,s)$ which we perturb to $B(t,s) = C(t,s) + D(t,s)$ and then determine the conditions on $a(t)$ and on the perturbation $D(t,s)$ so that if x satisfies the standard variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds$$

and if $y(t)$ is defined by

$$y(t) = a(t) - \int_0^t B(t,s)a(s)ds$$

then $x(t)$ converges to $y(t)$ both pointwise and in $L^2[0,\infty)$. That is, we seek conditions under which the totally unknown resolvent $R(t,s)$ can be replaced with the given kernel $B(t,s)$ and the error made with that substitution tends to zero pointwise and is in $L^2[0,\infty)$. To view the history of this project, the reader is referred to [5; p. 118], [7; Theorem 2.4], and [6].

In order to effectively use Liapunov theory we need to first smooth $a(t)$ which we do in an interesting and unexpected way, bypassing the x equation and working directly with the resolvent equation which, obviously, is independent of a. The Liapunov functional is applied to the smoothed problem producing the result that $x(t) \to a(t)$ both in L^2 and pointwise. Then the Liapunov functional is applied to the resolvent equation itself yielding precise knowledge about $\int_0^t |R(t,s)|\phi(s)ds$. Finally, the components are assembled to show that $z(t) \to a(t)$ pointwise and that $z \in L^p$.

2. Smoothing $a(t)$

In this section we present several very elementary propositions. The function a will be replaced by a function S and it is crucial that the reader have in mind the manner in which S will differ from a. We first consider $a : [0, \infty) \to R^n$ to be continuous and $B(t, s)$ to be an $n \times n$ matrix of functions which are continuous for $0 \le s \le t < \infty$. In this case, the resolvent equation (3) has a continuous solution $R(t, s)$, and variation of parameters gives $x(t) = a(t) - \int_0^t R(t, s)a(s)ds$ as the unique solution to (2). We also wish to consider cases where $a(t)$ is badly behaved, although frequently we will still assume a belongs to some L^p space. We could have $\limsup_{t\to\infty} |a(t)| \ge \epsilon > 0$, with $\epsilon = \epsilon(t) \to \infty$ as $t \to \infty$.

Under certain conditions, it is known that when $B(t, s)$ is small in three different measures then it is true that the functions $x(t)$, $a(t)$, $\int_0^t R(t, s)a(s)ds$, and $\int_0^t B(t, s)x(s)ds$ all lie in the same space. When the kernel is large, then a and $\int_0^t R(t, s)a(s)ds$ often lie in the same space, while x lies in an unrelated space. Much of Chapter 2 of [5] is devoted to such a study. One of the projects we have here is to investigate how various types of irregular behavior for $a(t)$ might be shared by the other functions. In such cases, our first task is to tame the behavior of $a(t)$. To the list of equations given in Section One, we add

$$f(t) = \int_0^t R(t, s)a(s)ds \tag{5}$$

and

$$S(t) = \int_0^t B(t, s)a(s)ds. \tag{6}$$

Note with (5) the solution to (2) becomes $x(t) = a(t) - f(t)$, so $f(t) = a(t) - x(t)$ measures the difference between the input $a(t)$ and the unperturbed output $x(t)$.

If we multiply (3) by $a(s)$ and integrate, we obtain

$$\begin{aligned}\int_0^t R(t, s)a(s)ds &= \int_0^t B(t, s)a(s)ds - \int_0^t \int_s^t B(t, u)R(u, s)du\, a(s)ds \\ &= \int_0^t B(t, s)a(s)ds - \int_0^t B(t, u) \int_0^u R(u, s)a(s)ds\, du,\end{aligned}$$

which, upon making the substitutions (5) and (6), gives

$$f(t) = S(t) - \int_0^t B(t, u)f(u)du. \tag{7}$$

The following propositions point out that (7) may be preferable to (2) in two very distinct ways. First, notice that (7) has the same form as (2), with a possibly badly behaved $a(t)$ being replaced by

$S(t)$, a substantially smoother function when B has nice properties. Also, the smoothed integral equation (7) has as its solution $f(t) = S(t) - \int_0^t R(t,s)S(s)ds$, with R solving (3) as before. Thus, the act of smoothing $a(t)$ might be expected to also smooth the difference $f(t) = a(t) - x(t)$. Next, there is flexibility in conditions concerning a and B, in contrast to conditions often imposed on (2) requiring separate conditions on a and B.

In Theorem 4.2 we will see that it is critical to have S bounded and $S \in L^2[0,\infty)$. Proposition 2.5 will give a continuum of different conditions on B and a to ensure $S \in L^2$. Here, we give two different conditions to ensure S bounded. The reasons are as follows. We will employ a Liapunov functional on (7) which is parallel to two functionals used previously on (2) and (3) separately. When used on (2) we absolutely must ask $a \in L^2[0,\infty)$. When the functional is used on (3) we are forced to ask $\int_s^t B^2(u,s)du$ bounded. By forming (7) we gain flexibility in taking combinations of properties of a and B, a situation which is entirely new.

Next, we will see several places where the smoothing of $a(t)$ using $S(t)$ enhances the study of (7) over that of (2). But if we differentiate (2) or (3) and use the aforementioned Liapunov functional then we are forced to ask a and a' in L^2 or $\int_s^t B^2(u,s)du + \int_s^t B_t^2(u,s)du$ bounded. Differentiation of (7) requires no such conditions on a' or B_t.

In Propositions 1 and 3 note that $a(t)$ may be unbounded, but $S(t)$ is bounded. Moreover in Proposition 2.3 even if $\limsup a(t) = \infty$, still $S(t) \to 0$. While (2) and (7) have the same form, $a(t)$ has been radically changed.

Proposition 2.1. *If $a \in L^2[0,\infty)$ and if there is an $M > 0$ with $\int_0^t |B(t,s)|^2 ds \leq M < \infty$ for all $t \geq 0$ then S is bounded. Also, if $a \in L^1[0,\infty)$ and B is bounded, then S is bounded.*

Proof. Note the stated condition on B is an L^2 condition for $B(t,\cdot)$, and from (6) $S(t)$ may be viewed as the L^2 inner product between $a(t)$ and $B(t,\cdot)$. The Schwarz inequality may then be used to bound S in terms of the L^2 norms for a and B. The second alternative is immediate from (6). □

Proposition 2.2. *If $a \in L^1[0,\infty)$ and if there is an $M > 0$ with $\int_s^t |B(u,s)|du \leq M < \infty$ for all $0 \leq s \leq t < \infty$ then $S \in L^1[0,\infty)$.*

Again, the stated condition on B is an L^1 type of condition, but here the integration is in the first variable as contrasted with Proposition 2.1. Also, as L^1 is not a Hilbert space, this proof requires a bit more work.

Proof. Given $t > 0$, we have

$$\begin{aligned}\int_0^t |S(u)|\,du &= \int_0^t \Big|\int_0^u B(u,s)a(s)ds\Big|\,du \\ &\leq \int_0^t\int_0^u |B(u,s)a(s)|\,dsdu = \int_0^t\int_s^t |B(u,s)a(s)|\,duds \\ &\leq \int_0^t\int_s^t |B(u,s)|\,du\,|a(s)|\,ds \\ &\leq M\int_0^t |a(s)|\,ds \leq M||a||_1 < \infty, \text{ as required.}\end{aligned}$$

□

Thus we have so far that if $a \in L^2$ and $B \in L^2$ (in a sense) then S is bounded, although this does not give $S \in L^2$ (however, see Proposition 2.5). Also, if $a \in L^1$ and $B \in L^1$ (in a different sense) then we do have $S \in L^1$, although this does not necessarily imply S is bounded.

We should also mention we are using $|\cdot|$ to denote the vector norm for $a(t) \in R^n$ and also to denote the operator matrix norm for $B(t,s)$. We choose the operator norm for B so that $|B(t,s)a(s)| \leq |B(t,s)||a(s)|$ holds, as was used in the preceding proof.

Proposition 2.3. *If $a \in L^1$, if $|B(t,s)|$ is bounded, and if for every $T > 0$ we have $\lim_{t\to\infty}\int_0^T |B(t,s)|\,ds = 0$, then $S(t) \to 0$ as $t \to \infty$.*

Proof. Note $|S(t)| \leq \int_0^t |B(t,s)a(s)|\,ds$ for all $t > 0$, and so it suffices to show $\lim_{t\to\infty}\int_0^t |B(t,s)a(s)|\,ds = 0$.

Let $M_B = \sup\{|B(t,s)| : 0 \leq s \leq t < \infty\}$ ($M_B < \infty$ by hypothesis), and let $\epsilon > 0$ be given. Since $a \in L^1$, choose $T > 0$ such that $\int_T^\infty |a(s)|\,ds < \frac{\epsilon}{2M_B}$, and let $M_T = \sup\{|a(s)| : 0 \leq s \leq T\}$. (Recall in this section we are assuming a is continuous, and so $M_T < \infty$.) Next, use $\lim_{t\to\infty}\int_0^T |B(t,s)|\,ds = 0$ to find $\tau > T$ so that $t > \tau$ implies $\int_0^T |B(t,s)|\,ds < \frac{\epsilon}{2M_T}$. Then we have, for $t > \tau$,

$$\begin{aligned}\int_0^t |B(t,s)a(s)|\,ds &\leq \int_0^T |B(t,s)||a(s)|\,ds + \int_T^t |B(t,s)||a(s)|\,ds \\ &\leq M_T\int_0^T |B(t,s)|\,ds + M_B\int_T^\infty |a(s)|\,ds \\ &< M_T\cdot\frac{\epsilon}{2M_T} + M_B\cdot\frac{\epsilon}{2M_B} = \epsilon, \text{ as required.}\end{aligned}$$

□

Proposition 2.4. *If $a(t) \to 0$ as $t \to \infty$, if $\int_0^t |B(t,s)|ds \leq M < \infty$ for all $t \geq 0$, and if $\lim_{t\to\infty}\int_0^T |B(t,s)|\,ds = 0\ \forall T > 0$, then $S(t) \to 0$ as $t \to \infty$.*

Proof. Again it suffices to show $\int_0^t |B(t,s)a(s)|\,ds \to 0$ as $t \to \infty$. Since $a(t) \to 0$, given $\epsilon > 0$ there exists $T > 0$ such that $||a||_{[T,\infty)} = \sup_{s\geq T}|a(s)| < \epsilon$. Combining this with the continuity of a also shows $||a||_\infty = \sup_{t\geq 0}|a(t)| < \infty$. Then taking $t > T$ gives

$$\begin{aligned}\int_0^t |B(t,s)||a(s)|\,ds &\leq ||a||_\infty \cdot \int_0^T |B(t,s)|\,ds + ||a||_{[T,\infty)} \cdot \int_T^t |B(t,s)|\,ds\\ &\leq ||a||_\infty \cdot \int_0^T |B(t,s)|\,ds + ||a||_{[T,\infty)} \cdot M.\end{aligned}$$

Thus, given $\epsilon > 0$, first find $T > 0$ such that $||a||_{[T,\infty)} < \frac{\epsilon}{2M}$, and then find $\tau > T$ such that $t > \tau$ implies $\int_0^T |B(t,s)|\,ds < \frac{\epsilon}{2||a||_\infty}$. Then $t > \tau$ will give $\int_0^t |B(t,s)a(s)|\,ds < \epsilon$, as required. □

Proposition 2.5. *Let $a(t)$ and $B(t,s)$ be scalar functions. Suppose that r and d are numbers with $0 \leq r \leq 1, 0 \leq d \leq 1$ and there are positive numbers M, K with*

$$\int_0^u |B^{2r}(u,s)||a^{2d}(s)|ds \leq M,$$

$$\int_0^t \int_s^t |B^{2(1-r)}(u,s)|du|a^{2(1-d)}(s)|ds \leq K.$$

Then $S \in L^2[0,\infty)$.

Proof. We have

$$\begin{aligned}\int_0^t S^2(u)du &= \int_0^t (\int_0^u B(u,s)a(s)ds)^2 du\\ &= \int_0^t \left(\int_0^u B^r(u,s)a^d(s)B^{1-r}(u,s)a^{1-d}(s)ds\right)^2 du\\ &\leq \int_0^t \int_0^u |B^{2r}(u,s)||a^{2d}(s)|ds \int_0^u |B^{2(1-r)}(u,s)||a^{2(1-d)}(s)|dsdu\\ &\leq M\int_0^t \int_0^u |B^{2(1-r)}(u,s)||a^{2(1-d)}(s)|dsdu\\ &= M\int_0^t \int_s^t |B^{2(1-r)}(u,s)|du|a^{2(1-d)}(s)|ds \leq MK,\end{aligned}$$

as required. □

As an example, let $a \geq 0$, $B(t,s) = g(t)s^2$ with $g \in L^2[0,\infty)$, let $r = 0$, and let $\int_0^t a^{2d}(s)s^4 ds$ be bounded, so that the first inequality in Proposition 2.5 is satisfied. Next, let

$$\sup_{u\geq 0} \int_0^u a^{2d}(s)ds \leq M,\ \int_0^t a^{2(1-d)}(s)s^4 ds \leq K$$

for some positive constants M and K. We then have

$$\begin{aligned}\int_0^t \int_s^t B^2(u,s)a^{2(1-d)}(s)dsdu &= \int_0^t \int_s^t g^2(u)dua^{2(1-d)}(s)s^4 ds \\ &\leq \int_0^\infty g^2(u)du \int_0^t a^{2(1-d)}(s)s^4 ds \\ &\leq K \int_0^\infty g^2(u)du.\end{aligned}$$

Less formally, one may note that if $|B(t,s)| \leq b_1(t)b_2(s)$ then to have $S \in L^2[0,\infty)$ we need only ask that $b_1(t)\int_0^t b_2(s)|a(s)|ds \in L^2$. We will continue this after Theorem 4.2 and it will introduce a new way to show that the resolvent satisfies $\sup_{0\leq t<\infty}\int_0^t |R(t,s)|ds < \infty$.

In the convolution case, it seems that L^p spaces are appropriate. But this example suggests that they are too coarse in the non-convolution case. If we were to use the upcoming Liapunov functional on the resolvent equation, it would demand that $\int_s^t B^2(u,s)du$ be bounded so examples of this type could not be considered.

Properties of the smoothing function $S(t) = \int_0^t B(t,s)a(s)ds$ are clearly dictated by properties of a and B. Proposition 2.1 gives conditions sufficient for S to be bounded, and the subsequent propositions give properties of B which are sufficient to show S follows a, meaning the smoothing process does not have to be done at the cost of giving up nice properties a might have.

Proposition 2.6. *If the partial derivative $B_t(t,s)$ is continuous, then $S(t)$ has a continuous first derivative.*

Proof. The continuity of both a and B will imply $S(t) = \int_0^t B(t,s)a(s)ds$ is continuous, and direct computation gives the derivative of S as $S'(t) = B(t,t)a(t) + \int_0^t B_t(t,s)a(s)ds$, and so the continuity of B_t gives both the differentiability of S and the continuity of its first derivative. □

Simple as Proposition 2.6 is, it is critical. From (2) we write

$$w(t) := x(t) - a(t) = -\int_0^t B(t,s)[w(s) + a(s)]ds$$

or

$$w(t) = -S(t) - \int_0^t B(t,s)w(s)ds$$

and then

$$w'(t) = -S'(t) - B(t,t)w(s) - \int_0^t B_t(t,s)w(s)ds,$$

where

$$S'(t) = B(t,t)a(t) + \int_0^t B_t(t,s)a(s)ds.$$

Theorem 2.7. *Suppose that for every bounded and continuous $a(t)$ there is a $J > 0$ with $|S'(t)| \leq J$. Suppose also that there is an $\alpha > 0$ with*

$$B(t,t) - \int_0^t |B_t(t,s)|\, ds \geq \alpha$$

for $t \geq 0$. Then $x(t)$, the solution of (2), is bounded, and

$$\sup_{t\geq 0} \int_0^t |R(t,s)|ds < \infty.$$

Proof. We will show that w is bounded for every bounded and continuous $a(t)$ and, hence, that $x(t)$ is bounded. But this then means $\int_0^t R(t,s)a(s)ds$ is bounded for every bounded and continuous $a(t)$. By Perron's theorem ([16] or [4; p. 116]) this will yield the conclusion. To that end, let $a(t)$ be fixed and let J be found. If, by way of contradiction, $w(t)$ is not bounded then there is a fixed $t_1 > 0$ with $|w(t_1)| > J/\alpha$ and with $|w(s)| \leq |w(t_1)|$ for $0 \leq s \leq t_1$. Define the Razumikhin function $V(t) = |w(t)|$ and notice that

$$V'(t) \leq |S'(t)| - B(t,t)|w(t)| + \int_0^t |B_t(t,s)||w(s)|\, ds.$$

Clearly, at t_1 we have $V'(t_1) \geq 0$. However,

$$V'(t_1) \leq J - \alpha|w(t_1)| < 0,$$

a contradiction. □

3. The Perturbed Volterra Kernel

In 1928 Volterra [17] noted that common fading memory kernels, $C(t,s)$, followed $e^{-(t-s)}$ in that

$$C(t,s) \geq 0,\ C_s(t,s) \geq 0,\ C_{st}(t,s) \leq 0,\ C_t(t,0) \leq 0. \tag{8}$$

He proposed this for problems in biology and it was adopted in many places including nuclear reactor theory of more than one type, in viscoelasticity, and in neural networks, among many other places. See [2,3, 5 Chapter 4, 8–14, 15 Chapter IV, 17–19] for much discussion of convex kernels and applications both for integral and integrodifferential equations. Volterra conjectured that a Liapunov functional could be constructed for use with such kernels. Levin constructed one for integrodifferential equations in 1963 and we constructed one for integral equations in 1992.

But it is only with great trepidation that we could attribute such precision as is embodied in (8) to any physical process. Moreover, the Liapunov functionals completely failed if any of those conditions in (8) failed. The analysis of a truncated integral equation, especially, collapsed if anything in (8) failed.

Thus, we have a clever mathematical theory, applied to physical processes without the robustness that is absolutely essential to the integrity of analysis.

At the same time in [1] we also constructed another Liapunov functional for integral equations which was robust and focused on integration, not differentiation. To the investigator's surprise and delight the two kernels can be added and the resulting integral equation can be analysed by the sum of the two Liapunov functionals. The result is that we have the clever theory of Volterra supplemented with perturbations which give integrity to the process.

Thus, we will suppose that $B(t,s) = C(t,s) + D(t,s)$ where $C(t,s)$ satisfies (8). We will also assume that both the matrix function $D(t,s)$ and the scalar function $\int_{t-s}^{\infty} |D(u+s,s)|\,du$ are continuous,

$$\exists\, \beta > 0 \text{ with } \int_0^\infty |D(u+t,t)|\,du \leq \beta, \tag{9}$$

and

$$\exists\, \alpha > 0 \text{ with } \int_0^t |D(t,s)|\,ds \leq \alpha \text{ and } \alpha + \beta < 2. \tag{10}$$

4. The first Liapunov functional: Scalar case

While we call this the scalar case, it is true that when $C(t,s) = 0$ it works for vectors and it also works for some vector equations discussed by Bo Zhang [19].

Consider (7) in the scalar case as

$$f(t) = S(t) - \int_0^t [C(t,u) + D(t,u)]f(u)du. \tag{11}$$

We will use the Liapunov functional

$$\begin{aligned} V(t) = &\int_0^t \int_{t-s}^\infty |D(u+s,s)|\,du f^2(s)ds + C(t,0)\left(\int_0^t f(s)ds\right)^2 \\ &+ \int_0^t C_s(t,s)\left(\int_s^t f(u)du\right)^2 ds. \end{aligned} \tag{12}$$

This is a combination of those Liapunov functionals constructed in 1992 which we now apply to (7), not (2), with $a(t)$ smoothed by Proposition 2.5 so that $S \in L^2[0,\infty)$ under the stated conditions. But what is so strategic here is that Propositions 2.1 and 2.3 will yield S bounded even when $a(t)$ is not, and that is a fundamental advance. The differentiability of S (Proposition 2.6) will be central with our second Liapunov functional.

We are particularly interested here in cases in which $a(t)$ has persistent spikes as $t \to \infty$. Here is the situation in more detail. While it is true that x, a, and $\int_0^t R(t,s)a(s)ds$ frequently all lie in the same L^p space, we see here that x inherits all the spikes of $a(t)$. We are then led to ask if $R(t,s)$ inherits all the spikes of $B(t,s)$, or if the integral in the resolvent equation absorbs some of those spikes. It is known that $R(t,s)$ is a remarkable function in that, while R depends only on B, there are vector spaces of functions ϕ for which the mapping on such a space defined by

$$(P\phi)(t) = \phi(t) - \int_0^t R(t,s)\phi(s)ds$$

is in L^p for some $p \in [1,\infty)$. Thus, taking $a(t) = \phi(t)$ gives

$$x(t) = \phi(t) - \int_0^t R(t,s)\phi(s)ds \in L^p.$$

Now

$$\frac{d}{dt}\int_0^t R(t,s)\phi(s)ds = R(t,t)\phi(t) + \int_0^t R_t(t,s)\phi(s)ds$$

so the integral is not necessarily smooth if $\phi(t)$ is not smooth. Can we say that the integral absorbs some of the spikes of $\phi(t)$? It does not: $x(t) \to a(t)$ pointwise. In the same way $R(t,s) \to B(t,s)$ for fixed s pointwise in t.

Two things should be noted. The result holds without change if either $D = 0$ or $C = 0$. When $C = 0$ it is actually the general vector case.

Theorem 4.1. *Let (8), (9), and (10) hold. Then there is a $K > 0$ with*

$$\int_0^t f^2(s)ds \le K\int_0^t S^2(s)ds, \tag{13}$$

$$(f(t) - S(t))^2 \le 4C(t,t)V(t) + \int_0^t D^2(t,u)du\int_0^t f^2(u)du, \tag{14}$$

and there are positive constants M and μ with

$$V(t) \le V(0) + M\int_0^t S^2(u)du - \mu\int_0^t f^2(s)ds$$

along the solution of (11).

Proof. In view of $C_{st}(t,s) \le 0$ and $C_t(t,0) \le 0$ we have from (12) that

$$\begin{aligned} V'(t) \le &\int_0^\infty |D(u+t,t)|du f^2(t) - \int_0^t |D(t,s)|f^2(s)ds \\ &+ 2f(t)C(t,0)\int_0^t f(s)ds + 2f(t)\int_0^t C_s(t,s)\int_s^t f(u)duds. \end{aligned}$$

Integration of the last term by parts yields

$$\begin{aligned}
&2f(t)\Big[C(t,s)\int_s^t f(u)du\Big|_{s=0}^{s=t} + \int_0^t C(t,s)f(s)ds\Big] \\
&= 2f(t)\Big[-C(t,0)\int_0^t f(u)du + \int_0^t C(t,s)f(s)ds\Big]
\end{aligned}$$

so

$$\begin{aligned}
V'(t) &\le \beta f^2(t) - \int_0^t |D(t,s)|f^2(s)ds \\
&+ 2f(t)[S(t) - f(t) - \int_0^t D(t,s)f(s)ds] \\
&\le \beta f^2(t) - \int_0^t |D(t,s)|f^2(s)ds + 2f(t)S(t) - 2f^2(t) \\
&+ \int_0^t |D(t,s)|(f^2(t) + f^2(s))ds \\
&\le \beta f^2(t) + \alpha f^2(t) + 2f(t)S(t) - 2f^2(t) \\
&\le \gamma f^2(t) + MS^2(t) - \eta f^2(t)
\end{aligned}$$

where $\gamma < \eta < 2$ and the inequalities are obtained as follows. We have

$$2|f(t)S(t)| \le MS^2(t) + \frac{1}{M}f^2(t)$$

so

$$2f(t)S(t) - 2f^2(t) \le MS^2(t) - (2 - \frac{1}{M})f^2(t).$$

But $\alpha + \beta =: \gamma < 2$ so choose M so large that

$$\gamma < 2 - \frac{1}{M} =: \eta$$

and then choose

$$\eta - \gamma =: \mu > 0.$$

Hence,

$$0 \le V(t) \le V(0) + M\int_0^t S^2(s)ds - \mu\int_0^t f^2(s)ds$$

or

$$\int_0^t f^2(s)ds \le \frac{M}{\mu}\int_0^t S^2(s)ds.$$

This proves (13) and the last part of the theorem. To prove (14) we set

$$H := \int_0^t D^2(t,u)du \int_0^t f^2(u)du$$

so that from (7) and $B = C + D$ we have that

$$
\begin{aligned}
&(1/2)(f(t) - S(t))^2 \\
&\leq \Big(\int_0^t C(t,u)f(u)du\Big)^2 + \Big(\int_0^t D(t,u)f(u)du\Big)^2 \\
&\leq \Big(-C(t,u)\int_u^t f(v)dv\Big|_0^t + \int_0^t C_u(t,u)\int_u^t f(v)dvdu\Big)^2 + H \\
&= \Big(C(t,0)\int_0^t f(v)dv + \int_0^t C_u(t,u)\int_u^t f(v)dvdu\Big)^2 + H \\
&\leq 2\Big(C(t,0)\int_0^t f(v)dv\Big)^2 \\
&+ 2\Big(\int_0^t C_u(t,u)\int_u^t f(v)dvdu\Big)^2 + H \\
&\leq 2C^2(t,0)\Big(\int_0^t f(v)dv\Big)^2 \\
&+ 2\int_0^t C_u(t,u)du\int_0^t C_u(t,u)\Big(\int_u^t f(v)dv\Big)^2 du + H \\
&\leq 2[C(t,0) + \int_0^t C_u(t,u)du][C(t,0)\Big(\int_0^t f(v)dv\Big)^2 \\
&+ \int_0^t C_u(t,u)\Big(\int_u^t f(v)dv\Big)^2 du] + H \\
&= 2C(t,t)V(t) - 2C(t,t)\int_0^t\int_{t-s}^\infty |D(u+s,s)|duf^2(s)ds + H
\end{aligned}
$$

where the last line comes from (12).

Hence we have

$$
\begin{aligned}
&(f(t) - S(t))^2 \leq 4C(t,t)V(t) + 2\int_0^t D^2(t,u)du\int_0^t f^2(u)du \\
&\leq 4C(t,t)[V(0) + M\int_0^t S^2(u)du] + 2\int_0^t D^2(t,u)du\int_0^t f^2(u)du
\end{aligned}
$$

with M chosen above. □

Volterra's interest in convex kernels stemmed from their description of fading memory and he conjectured, correctly, that the kernels could be very large and still generate bounded solutions, in marked contrast to so many results requiring very small kernels. The next result shows that with several types of integrability of both the kernel and $a(t)$, the convex kernel is a very good global approximation to the unknown resolvent, $R(t,s)$, even when the convex kernel has a substantial perturbation.

Theorem 4.2. *Let (8), (9), (10) hold, and let $S \in L^2[0,\infty)$. If for each large T we have $\lim_{t\to\infty}\int_0^T B^2(t,u)du = 0$ and if there is an $L > 0$ with $\int_0^t B^2(t,u)du \le L$, then $f(t) \to S(t)$ pointwise and in $L^2[0,\infty)$. Moreover, if $y(t) = a(t) - \int_0^t B(t,s)a(s)ds$ and if $x(t) = a(t) - \int_0^t R(t,s)a(s)ds$ then $x(t) \to y(t)$ pointwise and in $L^2[0,\infty)$. If $S(t) \to 0$, so does $f(t)$ and by (5) and the variation of parameters formula $x(t) \to a(t)$ as $t \to \infty$ and $x - a \in L^2[0,\infty)$.*

Proof. First, return to the proof of Theorem 4.1 with

$$\begin{aligned}V'(t) &\le (-2+\alpha+\beta)f^2(t) + 2f(t)S(t)\\ &= -(f(t)-S(t))^2 + (\alpha+\beta-1)f^2(t) + S^2(t).\end{aligned}$$

Thus

$$0 \le V(t) \le V(0) - \int_0^t (f(u)-S(u))^2du + (\alpha+\beta-1)\int_0^t f^2(u)du + \int_0^t S^2(u)du.$$

With $\int_0^t S^2(u)du$ bounded, from (13) we have $f \in L^2[0,\infty)$ and then $f(t) - S(t) \in L^2[0,\infty)$.

Next, for a given $\epsilon > 0$ find $T > 0$ with $\int_T^\infty f^2(u)du < \epsilon^2/(4L)$. Set $\int_0^\infty f^2(u)du = M$ and with T fixed take t so large that $\int_0^T B^2(t,u)du < \epsilon^2/(4M)$. We now have

$$\begin{aligned}|f(t)-S(t)| &\le \int_0^t |B(t,u)f(u)|du\\ &= \int_0^T |B(t,u)f(u)|du + \int_T^t |B(t,u)f(u)|du\\ &\le \sqrt{\int_0^T B^2(t,u)du\int_0^T f^2(u)du} + \sqrt{\int_T^t B^2(t,u)du\int_T^t f^2(u)du}\\ &\le \sqrt{(\epsilon^2/(4M))M} + \sqrt{(L\epsilon^2)/(4L)} = \epsilon.\end{aligned}$$

Now,

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds = a(t) - f(t)$$

and

$$y(t) = a(t) - \int_0^t B(t,s)a(s)ds = a(t) - S(t)$$

so

$$x(t) - y(t) = S(t) - f(t) \to 0$$

as $t \to \infty$, while $\int_0^\infty (x(t)-y(t))^2dt < \infty$. Moreover, if $S(t) \to 0$ so does $f(t)$ yielding $x(t) \to a(t)$ as $t \to \infty$, while $x(t)-a(t) = f(t) \in L^2[0,\infty)$. This completes the proof. □

Main remark. We can refine the condition $\int_0^t B^2(t,u)du \le L$ as follows. Use Theorem 4.1 to obtain f bounded; this requires S bounded which, in turn, asks a combination of conditions on B and a. Thus the burden is distributed between a and B. Then $f \in L^p$ for all $p \ge 2$. Find $q > 0$ with $(1/p) + (1/q) = 1$ and use the Hölder inequality in the last set of estimates so that we only need to consider $\int_0^t |B(t,u)|^q du$ instead of $\int_0^t B^2(t,u)du$. As $p \to \infty$, $q \to 1$ so we get a range of conditions on B. Theorem 4.2 tells us that $x(t) \cong y(t) = a(t) - \int_0^t B(t,s)a(s)ds$ is a variation of parameters approximation which becomes more accurate (both L^2 and pointwise) as $t \to \infty$.

If we return to the informal statement following Proposition 2.5 with $a(t)$ arbitrary and continuous, so long as $S \in L^2[0,\infty)$ nothing more is required of $a(t)$ in this theorem and we have $x(t) \to y(t)$ pointwise and in $L^2[0,\infty)$. In particular, if the result holds for every bounded and continuous $a(t)$ then in the variation of parameters formula we see that $\int_0^t R(t,s)a(s)ds$ is bounded for all such a and Perron's theorem will then say that $\sup_{0\le t<\infty} \int_0^t |R(t,s)|ds < \infty$.

5. Discontinuities in $a(t)$

Like Section 4, this section mainly concerns the scalar case. Section 4 was devoted to evidence that perturbed convex kernels will continue to generate behavior similar to that produced by unperturbed convex kernels. Such results tend to show that there is integrity to the process of describing real world systems using convex kernels even when it is not possible to make measurements establishing such intricate properties as seen in (8).

In this section we continue the idea by allowing $a(t)$ to have infinite discontinuities and still show that $x(t)$ converges to $a(t)$ both pointwise and in $L^2[0,\infty)$ when $B(t,s) = C(t,s) + D(t,s)$ and (8)–(10) hold. To fix ideas consider the example

$$a(t) = e^{-t}t^{-1/3} \text{ so that } a \in L^1 \cap L^2[0,\infty)$$

with additional such discontinuities occurring at points $t_n \to \infty$.

In this discussion we will show $x \to a$ by considering $w = x - a$, which is the same as setting $w = -f$, but f is defined in terms of R and we wish to remove references to the resolvent. Thus, we will obtain properties of w without referring to the resolvent, and to clarify this distinction we will also write $q(t)$ in place of $S(t)$.

To establish existence and uniqueness we return to (2) and have

$$\begin{aligned} w(t) := x(t) - a(t) = -\int_0^t B(t,s)x(s)ds = -\int_0^t B(t,s)[w(s) + a(s)]ds \\ = -\int_0^t B(t,s)a(s)ds - \int_0^t B(t,s)w(s)ds \end{aligned}$$

or

$$w(t) = -q(t) - \int_0^t B(t,s)w(s)ds,\ q(t) := \int_0^t B(t,s)a(s)ds. \tag{15}$$

We now see that if q is continuous then there is a unique continuous solution $w(t)$ so $x(t)-a(t)$ is unique and continuous on $[0,\infty)$. Our task is to show that $w \in L^2[0,\infty)$ and tends to zero pointwise. Continuity of q can be established in a number of simple ways with discontinuities of $a(t)$ as above. We will also depend on q being in L^2 and the following lemma gives one of the many ways in which that can also be established.

Lemma 5.1. *If there is an $M > 0$ with $\int_0^\infty |a(t)|dt < M$ and $\int_s^t B^2(u,s)du < M$, then $q \in L^2[0,\infty)$.*

Proof. We have

$$\begin{aligned}\int_0^t \left(\int_0^u B(u,s)a(s)ds\right)^2 du &\le \int_0^t \int_0^u |a(s)|ds \int_0^u B^2(u,s)|a(s)|dsdu \\ &\le M\int_0^t \int_s^t B^2(u,s)du|a(s)|ds \\ &\le M^2 \int_0^t |a(s)|ds \le M^3.\end{aligned}$$

□

In the next theorem we consider (15) and use the combination Liapunov functional to get $w \in L^2[0,\infty)$ because $q \in L^2[0,\infty)$ and that $w(t) \to 0$. This means $x(t) \to a(t)$ both pointwise and in $L^2[0,\infty)$.

Theorem 5.2. *Let q be continuous and in $L^2[0,\infty)$ and let $B(t,s) = C(t,s) + D(t,s)$ satisfy (8)-(10).*

(i) Then $x - a = w \in L^2$.

(ii) Let $a \in L^1[0,\infty)$ and let $a \in L^2$ locally. If B is bounded and if for each $T > 0$ we have $\lim_{t\to\infty} \int_0^T B^2(t,s)ds = 0$, then w is bounded and $q(t) \to 0$ as $t \to \infty$.

(iii) Let w be bounded, let $\int_0^t B^2(t,s)ds$ be bounded, and let $q(t) \to 0$ as $t \to \infty$. If for each $T > 0$ we have $\lim_{t\to\infty} \int_0^T |B(t,s)|ds = 0$, then $w(t) \to 0$ as $t \to \infty$.

Proof. Define

$$\begin{aligned}V(t) &= \int_0^t C_s(t,s)\left(\int_s^t w(u)du\right)^2 ds \\ &\quad + C(t,0)\left(\int_0^t w(u)du\right)^2 + \int_0^t \int_{t-s}^\infty |D(u+s,s)|\,du\,w^2(s)ds\end{aligned}$$

and follow the proof of Theorem 4.1 to obtain a $\gamma > 0$ with

$$\int_0^t w^2(s)ds \le \gamma \int_0^t q^2(s)ds,$$

proving (i).

To prove (ii), for a given $\epsilon > 0$ take T so large that $\|B\| \int_T^\infty |a(s)|ds < \epsilon/2$. Then for $t > T$ we have

$$\begin{aligned}
|q(t)| &\leq \int_0^t |B(t,s)a(s)|ds \\
&= \int_0^T |B(t,s)a(s)|ds + \int_T^t |B(t,s)a(s)|ds \\
&\leq \sqrt{\int_0^T B^2(t,s)ds \int_0^T a^2(s)ds} + \|B\| \int_T^\infty |a(s)|ds \\
&< \epsilon
\end{aligned}$$

for large t so $q(t) \to 0$ as $t \to \infty$.

Next,

$$\begin{aligned}
|w(t)| &\leq |q(t)| + \int_0^t |B(t,s)w(s)|ds \\
&\leq |q(t)| + \sqrt{\int_0^t B^2(t,s)ds \int_0^t w^2(s)ds}
\end{aligned}$$

so $w(t)$ is bounded and (ii) is proved.

To prove (iii) note that from (i) we have $w \in L^2[0,\infty)$. Thus,

$$\begin{aligned}
\int_0^t |B(t,s)w(s)|ds &= \int_0^T |B(t,s)w(s)|ds + \int_T^t |B(t,s)w(s)|ds \\
&\leq \|w\| \int_0^T |B(t,s)|ds + \sqrt{\int_T^t B^2(t,s)ds \int_T^t w^2(s)ds} \\
&\leq \|w\| \int_0^T |B(t,s)|ds + \sqrt{\int_0^t B^2(t,s)ds \int_T^\infty w^2(s)ds}.
\end{aligned}$$

For a given $\epsilon > 0$, fix T so that the last term is smaller than $\epsilon/2$. Then take t so large that the next-to-last term is smaller than $\epsilon/2$. □

Main remark. The result can be generalized in several places by using the Hölder inequality instead of the Schwarz inequality as we did in Proposition 2.5 and such details are left to the reader. But the weak part of the result is in (ii) where we ask for $B(t,s)$ bounded. This is a reminder that L^p spaces are too coarse for this work. That condition can be replaced by the more cumbersome, but much better, condition: For each $\epsilon > 0$ there is a $T > 0$ such that $t \geq T$ implies that

$$\sup_{T \leq s \leq t < \infty} |B(t,s)| \int_T^t |a(s)|ds < \epsilon.$$

Thus, with a fixed B we can choose that class of a to dominate in the indicated manner. With this changed, no boundedness of a or B is needed.

In preparation for part (iii) of the next result, we consider the perturbed equation

$$z(t) = a(t) - \int_0^t B(t,s)[z(s) + G(s,z(s))]ds \tag{16}$$

with

$$|G(t,z)| \le \phi(t)|z|. \tag{17}$$

It is known that if x satisfies (2) then

$$z(t) = x(t) - \int_0^t R(t,s)G(s,z(s))ds. \tag{18}$$

There is a result from Burton-Dwiggins [7] which will play a main role here. It is stated as (i) in the theorem. We then apply (i) and the work of Proposition 2.3 with B replaced by R to obtain (ii). Finally, we put them all together to get (iii).

Theorem 5.3. *If (8)-(10) hold, then there is a $\gamma > 0$ with*

$$\int_s^t R^2(u,s)du \le \gamma \int_s^t B^2(u,s)du.$$

(i) If, in addition, $\int_s^t B^2(u,s)du$ and R_t are bounded, then $R(t,s) \to 0$ as $t \to \infty$ for fixed s. If, in addition, R_s is bounded then, for every $T > 0$, $\int_0^T |R(t,s)|ds \to 0$ as $t \to \infty$.

(ii) If, in addition, we assume that $\phi \in L^1[0,\infty)$, is bounded on bounded sets, with $\phi(t) \ge 0$, and that $C(t,t)$, $\int_s^t D^2(t,u)du$, and $B(t,s)$ are all bounded, then $R(t,s)$ is bounded and $\int_0^t |R(t,s)|\phi(s)ds \to 0$ as $t \to \infty$.

(iii) If, in addition, $x(t)$ is bounded, then any solution, z, of (18) is bounded. Thus $G(t,z) \in L^1[0,\infty)$ so $\int_0^t |R(t,s)G(s,z(s))|ds$ tends to zero as $t \to \infty$ and $z(t) \to x(t)$ as $t \to \infty$.

Proof. The conclusions in (i) are found in [7].

We follow the proof of Theorem 4.1 using

$$\begin{aligned} V(t) &= \int_s^t C_v(t,v)\left(\int_v^t R(u,s)du\right)^2 dv \\ &\quad + C(t,s)\left(\int_s^t R(u,s)du\right)^2 + \int_s^t \int_{t-v}^\infty |D(u+v,v)|du R^2(v,s)dv \end{aligned}$$

to get the first relation. The details are straightforward but lengthy and it would be a distraction to give them here. Thus, we will supply them in the appendix, along with the details of the counterpart of (14).

To prove (ii), we begin by following the proof of (14) and show that $K(R(t,s)-B(t,s))^2$ is a lower bound on that Liapunov functional for some $K>0$, resulting in $R(t,s)$ bounded. (See the appendix.) Next, take $T>0$ and write

$$\begin{aligned}\int_0^t |R(t,s)|\phi(s)ds &= \int_0^T |R(t,s)|\phi(s)ds + \int_T^t |R(t,s)|\phi(s)ds \\ &\le \|\phi\|^{[0,T]} \int_0^T |R(t,s)|ds + \|R\| \int_T^t \phi(s)ds.\end{aligned}$$

To see that this tends to zero, first take T large and then take t large.

To prove (iii), from (18) with $x(t)$ bounded we have

$$\begin{aligned}|z(t)| &\le \|x\| + \int_0^t |R(t,s)G(s,z(s))|ds \\ &\le \|x\| + \int_0^t |R(t,s)|\phi(s)|z(s)|ds.\end{aligned}$$

Since $\int_0^t |R(t,s)|\phi(s)ds \to 0$, there is a $T>0$ such that $t \ge T$ implies that $\int_0^t |R(t,s)|\phi(s)ds < 1/2$. Now, if z is not bounded then there is a sequence $\{t_n\} \uparrow \infty$ such that $|z(s)| \le |z(t_n)|$ if $0 \le s \le t_n$. Hence,

$$\begin{aligned}|z(t_n)| &\le \|x\| + |z(t_n)| \int_0^{t_n} |R(t_n,s)\phi(s)|ds \\ &\le \|x\| + (1/2)|z(t_n)|,\end{aligned}$$

a contradiction for large n. □

Main remark The conclusion in (i) that $R(t,s) \to 0$ as $t \to \infty$ for fixed s does not bound $R(t,s)$. It is (ii) which yields $R(t,s)$ bounded through a lower bound on a Liapunov functional and a bound on B. That is a crucial result. Parts (i) and (ii) are the only places in which we have not been able to avoid asking a bound on B with no other alternatives. It is certainly a weak part of the result since one of our main goals is to ask bounds only on integrals.

6. Applications and Future Research

In the study of behavior of solutions to (1), it is usually assumed that $a(t)$ is continuous, or that its only discontinuities are jumps of finite size. In this paper we have obtained results which hold even when $a(t)$ has infinite singularities, as long as a belongs to some L^p space. We have also demonstrated how the value of p can be altered by shifting some of the burden to integral bounds for the kernel $B(t,s)$. In particular, Proposition 2.5 illustrates a new type of flexibility in deciding what conditions need be imposed on a and B.

As an example of a physical situation in which weakening the condition of continuity of $a(t)$ may be applied, Ergen [8] gives differential equations relating the temperature $T(t)$ in a circulating-fuel nuclear reactor to a function $P(t)$ representing the power of the system. There, a Liapunov functional $H(t)$ (related to the total energy in the system) is used to show that, if there are oscillations in $P(t)$, then those oscillations cannot be undamped. As Ergen pointed out, this form of stability is crucial in the operation of modern nuclear reactors, since it is neither desirable nor feasible to rely on intervention (manual or servo-mechanical) in cases where undamped power oscillations might occur.

Ergen's equation (9) [8; page 708] can be written in the form $P(t) = a(t) + \int_0^t K(t-s)P(s)ds$, where the kernel K dictates different fuel travel times along different paths, and $a(t)$ is proportional to the time rate change of $T(t)$. In this paper, we have shown how the continuity of $a(t)$ need not be assumed, provided $a \in L^p$ or some other condition holds. Thus, in [8] we need not assume the temperature function is differentiable. There may be temperature spikes of undetermined magnitude occurring at many times t_n, but as long as the derivative of T is, say, an L^2 function, then we have demonstrated (with appropriate assumptions on K) that P must also be an L^2 function, which again implies the system cannot have undamped oscillations. Moreover, once we have removed the need for $T(t)$ to be differentiable, further results might be obtained by studying nuclear reactor kinetics on a more microscopic level.

Ergen [8; Sections V and VI] also discussed how the signs of the derivatives of $K(t-s)$ affect the demonstration that $H(t)$ will decrease over one period of oscillation. Levin [11] determined that, if $C(t,s) = -K(t-s)$, it is the signs of the partial derivatives of C which are important, and not that the integral equation is necessarily of convolution type. Levin presented examples of such convex kernels (that is, those which satisfy the derivative sign conditions) which are not of convolution type, noting however that each example was a multiplicative perturbation of a standard convolution kernel.

Levin's examples in [11] obey the desired derivative sign properties, which we have listed in this paper as (8). Yet we maintain these properties cannot be reasonably be expected to hold in any type of physical situation. That is why we have studied, both here and in [7], kernels of the type $B = C + D$, which represent additive perturbations of the types of kernels studied by Levin. We need not assume a given kernel B satisfies all the conditions listed in (8), as long as the difference between B and a Levin-type kernel C is small enough, with the size of the perturbation $D = B - C$ kept small by imposing particular integral bounds on D.

When investigating the behavior of solutions to (2), if we use the variation of parameters solution $x(t) = a(t) - \int_0^t R(t,s)a(s)ds$, then we must begin with the behavior of $a(t)$, which is given, and the resolvent

$R(t, s)$, which is unknown. Using the smoothing process introduced here, we are able to obtain results even when $a(t)$ is very badly behaved, and we have also discovered that this process leads to a first step in removing reference to properties of the resolvent in trying to determine the behavior of $x(t)$.

In our presentation of Theorem 5.3, we noted that part (i) is a result which appears in [7]. There, a Lipschitz condition on the resolvent was used, while here we have used the stronger condition of a bounded derivative, in order to clarify the application of this result. We also note that the conclusion to part (iii) of Theorem 5.3, that $z(t)$ is bounded and $z(t) \to x(t)$ as $t \to \infty$, is the same conclusion found in [7], but with different assumptions. In [7] we assumed $x(t)$ is bounded and $x \in L^2$, while here we have replaced $x \in L^2$ with the assumption that $B(t, s)$ is bounded. As noted in the remark after Theorem 5.3, this is an unsatisfactory result, because we prefer to work instead with assumed integral bounds for B.

One way to improve this result would be instead to invoke assumptions which will already give $x \in L^2$ (for example, some suitable modification of Theorem 4.2). However, it may be that more fruitful research will come from the flexible approach suggested by Proposition 2.5, that is, replace the conditions B bounded and $x \in L^2$ with some other set of conditions involving an integral bound for B and, hopefully, a weaker condition on x.

We stated as our long-term project the study of behavior of solutions to (2), not by using properties of the resolvent $R(t, s)$, but by finding error bounds between the solution $x(t)$ and some approximant $y(t)$. We have taken the first step, defining y by replacing R with B in the variation of parameters version of x. Along the way we also obtained estimates of the quantity $R - B$, which may be viewed as a first-order error bound.

We have obtained preliminary results for a second-order error bound, which will give a second approximant for $x(t)$, leading to improved results. Continuing this process, using the expansion of the resolvent as a series of iterated integrals, we will then be able to obtain approximants for $x(t)$ of any order, with the behavior of $x(t)$ thus being determined completely by $a(t)$ and the kernel $B(t, s)$. This future research will help us in our ultimate goal of being able to study the behavior of solutions to integral equations without first needing to determine properties of their resolvents.

Appendix

We are going to give the details concerning the derivative of

$$V(t)=\int_s^t C_v(t,v)\Big(\int_v^t R(u,s)du\Big)^2 dv$$
$$+\,C(t,s)\Big(\int_s^t R(u,s)du\Big)^2+\int_s^t\int_{t-v}^\infty |D(u+v,v)|duR^2(v,s)dv$$

along a solution of (3) and also the counterpart of (14) for (3) and this Liapunov functional. The counterpart of (14) is given first. In the last steps we will use the first relation of Theorem 5.3 and this will be obtained later in this appendix.

Define $H=\int_s^t D^2(t,u)du\int_s^t R^2(u,s)du$ so that from (3) we have

$$\begin{aligned}
&(1/2)(R(t,s)-B(t,s))^2\\
&\le\Big(\int_s^t C(t,u)R(u,s)du\Big)^2+\Big(\int_s^t D(t,s)R(u,s)du\Big)^2\\
&\le\Big(-C(t,u)\int_u^t R(v,s)dv\Big|_s^t+\int_s^t C_u(t,u)\int_u^t R(v,s)dvdu\Big)^2+H\\
&=\Big(C(t,s)\int_s^t R(v,s)dv+\int_s^t C_u(t,u)\int_u^t R(v,s)dvdu\Big)^2+H\\
&\le 2\Big(C(t,s)\int_s^t R(v,s)dv\Big)^2+2\Big(\int_s^t C_u(t,u)\int_u^t R(v,s)dvdu\Big)^2+H\\
&\le 2C^2(t,s)\Big(\int_s^t R(v,s)dv\Big)^2\\
&+2\int_s^t C_u(t,u)du\int_s^t C_u(t,u)\Big(\int_u^t R(v,s)dv\Big)^2du+H\\
&\le 2\Big[C(t,s)+\int_s^t C_u(t,u)du\Big]\Big[C(t,s)\Big(\int_s^t R(v,s)dv\Big)^2\\
&+\int_s^t C_u(t,u)\Big(\int_u^t R(v,s)dv\Big)^2du\Big]+H\\
&\le 2C(t,t)V(t)-2C(t,t)\int_s^t\int_{t-v}^\infty |D(u+v,v)|duR^2(v,s)dv\\
&+\int_s^t D^2(t,u)du\,\cdot\,\gamma\int_s^t B^2(u,s)du
\end{aligned}$$

or

$$\big(R(t,s)-B(t,s)\big)^2\le 4C(t,t)V(t)+2\gamma\int_s^t D^2(t,u)du\int_s^t B^2(u,s)du,$$

a counterpart of (14). It is now clear that if $V(t), C(t,t), \int_s^t D^2(t,u)du$, $\int_s^t B^2(u,s)du, B(t,s)$ are all bounded then $R(t,s)$ is bounded which is the first thing needed to prove Theorem 5.3(ii).

We will now compute the derivative of V along the solution of (3) to establish the first relation of Theorem 5.3.

Taking into account that $C_{vt}(t,v) \le 0$ and $C_t(t,s) \le 0$ we have

$$\begin{aligned} V'(t) \le & \int_0^\infty |D(u+t,t)|du R^2(t,s) - \int_s^t |D(t,v)|R^2(v,s)dv \\ & + 2R(t,s)C(t,s)\int_s^t R(u,s)du + 2R(t,s)\int_s^t C_v(t,v)\int_v^t R(u,s)dudv. \end{aligned}$$

If we integrate the last term by parts we have

$$\begin{aligned} & 2R(t,s)\Big[C(t,v)\int_v^t R(u,s)du\Big|_s^t + \int_s^t C(t,v)R(v,s)dv\Big] \\ & = 2R(t,s)\Big[-C(t,s)\int_s^t R(u,s)du + \int_s^t C(t,v)R(v,s)dv\Big]. \end{aligned}$$

Canceling terms and taking (3) into account we have

$$\begin{aligned} V' & \le \beta R^2(t,s) - \int_s^t |D(t,v)|R^2(v,s)dv \\ & + 2R(t,s)\Big[C(t,s) + D(t,s) - R(t,s) - \int_s^t D(t,u)R(u,s)du\Big] \\ & \le \beta R^2(t,s) - \int_s^t |D(t,v)|R^2(v,s)dv \\ & + 2R(t,s)[C(t,s) + D(t,s) - R(t,s)] \\ & + \int_s^t |D(t,u)|\big(R^2(u,s) + R^2(t,s)\big)du \\ & \le (\alpha+\beta)R^2(t,s) + 2R(t,s)[C(t,s) + D(t,s) - R(t,s)] \\ & \le (\alpha+\beta)R^2(t,s) + M\big(C^2(t,s) + D^2(t,s)\big) - \lambda R^2(t,s) \end{aligned}$$

where λ can be chosen so that $\alpha+\beta < \lambda < 2$ and then for $\eta = \lambda-(\alpha+\beta)$ we have

$$V'(t) \le -\eta R^2(t,s) + M\big(C^2(t,s) + D^2(t,s)\big)$$

so that an integration will yield the first relation in Theorem 5.3.

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