Perron-type Stability Theorems for Neutral Equations

ptstne1.tex

T A Burton

Northwest Research Institute

732 Caroline St.

Port Angeles, WA 98362

taburton at olypen.com

Abstract. In this paper we present two Perron-type asymptotic stability results for a neutral functional differential equation of the form

$$x'(t) = Sx(t) + Px(t-r) + \frac{d}{dt}Q(t, x_t) + G(t, x_t)$$

when the linear part ($x'(t) = Sx(t) + Px(t-r)$) is asymptotically stable. In particular, Q and G are allowed to be unbounded functions of t and Q need not be differentiable. The results are based on Krasnoselskii's fixed point theorem. It is to be emphasized that, unlike Perron, we obtain only asymptotic stability because of the unboundedness of Q and G.

1991 Mathematics Subject Classification. 47H10, 34K40

Key words and phrases. Fixed points, stability, neutral equations, Perron theorem

1. Introduction The broad purpose of this paper is to present a Perron-type stability theorem for a neutral functional differential equation. Among many other results, Perron [7] showed that uniform asymptotic stability of the zero solution of the vector system

$$x'(t) = Sx(t) + F(t, x(t)) \tag{1.1}$$

is inherited from the uniform asymptotic stability of

$$y' = Sy \tag{1.2}$$

whenever

$$\lim_{|x|\to 0} \frac{|F(t, x)|}{|x|} = 0 \tag{1.3}$$

uniformly for $0 \leq t < \infty$. Here, S is an $n \times n$ constant matrix, all of whose characteristic roots have negative real parts, while $F : [0, \infty) \times R^n \to R^n$ is continuous. Bellman [1] gives many associated criteria, as do Lakshmikantham and Leela [6; p. 115].

We are interested in similar results for a system

$$x'(t) = Sx(t) + Px(t-r) + \frac{d}{dt}Q(t, x_t) + G(t, x_t) \tag{1.4}$$

where r is a positive constant and x_t is an element of the Banach space $(C, \|\cdot\|)$ of continuous functions $\phi : [-\gamma, 0] \to R^n$, $\gamma > 0$, with $x_t(s) = x(t+s)$, $-\gamma \leq s \leq 0$. For the vector case we find that we need to take $P = 0$, but for the scalar case both S and P can be functions of t.

The condition (1.3) might be expressed as

$$|F(t,x)| \leq h(|x|)|x|$$

where h is continuous with $h(0) = 0$. We ask a variant of this on both Q and G , allowing them to grow rapidly in t. We also ask strong continuity properties.

The nice thing about Perron's result is that in so many problems we simply glance at a very formidable equation and see that Perron's result applies. For example, in

$$x'' + (\cos x)x' + \sin x = 0 \tag{E_1}$$

we recognize that the linear part is the asymptotically stable equation

$$x'' + x' + x = 0$$

and that the rest of the terms are of higher order. Hence, (E_1) is uniformly asymptotically stable.

It would be very interesting to have a fairly simple criterion which would tell us that the zero solution of

$$x'' + (\cos x)x' + \sin x = \frac{d}{dt}t^3x^2(t - r(t)) + t^5x^3(t - r(t)) \tag{E_2}$$

is also asymptotically stable even when r is a bounded continuous function which is not differentiable. Our first result (Theorem 2.1) gives us just such a criterion and (E_2) is asymptotically stable. The technique will also work with a term of the form $\frac{d}{dt}tg(t-r(t))$ where g satisfies a Lipschitz condition with sufficiently small Lipschitz constant. This is done in Theorem 3.1, but it requires careful calculations, while Theorem 2.1 relies on general order of growth.

In the scalar first order case we consider equations having a linear part of the form

$$(E_3) \qquad x'(t) = -a(t)x(t) - b(t)x(t-r)$$

in which

$$a(t) + b(t+r) \geq 0, \int_0^\infty [a(t) + b(t+r)]dt = \infty,$$

and

$$\sup_{t\geq 0} \int_{t-1}^{t} [a(s) + b(s+r)]ds$$

is bounded. A simple example of that is

$$x'(t) = -(\sin t)x(t) - x(t-r).$$

We show that the equation is asymptotically stable under perturbations of the type in (E_2).

Neutral equations arise in natural ways such as in describing the vibration of masses on an elastic bar. Often it is necessary to convert a functional differential equation of the form $x'(t) = f(t, x_t)$ into a neutral equation in order to secure a tractible linear term from which to launch a stability investigation. That type of work is seen in the following pages.

The methods used here include the Krasnoselskii fixed point theorem, to be stated later. It is found in Krasnoselskii [5] or Smart [11; p. 31]. Krasnoselskii's theorem is a natural tool for neutral equations being motivated by the fact that, frequently, the inversion of the perturbed differential operator yields the sum of a contraction and compact map. Neutral equations offer a perfect example of that property.

The idea developed is a Perron type stability theorem for neutral equations. A survey of work concerning Perron's theorem can be found in many standard works on ordinary differential equations. The use of fixed point theory in stability may be found in [2-4] and [9], for example. Recently, Serban [10] has used fixed point theory to prove asymptotic stability for a difference equation using the Picard operator theory of Rus [8].

Compactness requirements in Krasnoselskii's theorem depend on good estimates of the rate of decay of solutions. Study of Bernoulli equations leads us to those estimates. The author and Furumochi used similar ideas in [4] where we studied an ordinary differential equation

$$x' = a(t,x) + b(t,x)$$

by means of a modified Krasnoselskii fixed point theorem. That work was essentially different from the present one. There, we obtained mappings by solving the equation

$$x' = a(t,x) + b(t,\psi(t))$$

where ψ is in a specified set. It was closely tied to Liapunov's direct method, which we pointedly avoid here. Among other difficulties, we now see no way to treat the term $\frac{d}{dt}Q(t,x_t)$ in (1.4) by that method.

Concerning our avoidance of Liapunov's direct method, in earlier work [3] we have studied difficulties encountered using Liapunov's direct method with a view to circumventing these difficulties by means of fixed point theory. Here are some of the considerations.

A. When the delay is not well-behaved, then Liapunov's direct method is difficult to use, but these difficulties vanish with fixed point theory.

B. Liapunov's direct method tends to require pointwise relations, but fixed point theory uses averaged conditions.

C. On the other hand, Liapunov theory yields stability behavior for all solutions with a given initial function, but fixed point theory often only specifies that one solution with the given initial function has the prescribed behavior; when solutions are uniquely determined by initial conditions, the fixed point argument does yield stability.

D. One of the continuing difficulties with Liapunov's direct method is the need for the right hand side of the differential equation to be bounded for the state variable bounded. Such difficulties are seldom seen in application of fixed point theory to stability.

E. In neutral theory with Liapunov's direct method it is frequently necessary that the neutral term, $\frac{d}{dt}Q(t,x_t)$, satisfy $|Q(t,x_t)| \le \alpha\|x_t\|, \alpha < 1$, since we often show that $x(t) + Q(t,x_t) \to 0$ and wish to infer that $x(t) \to 0$. That problem does not occur with fixed point theory.

F. There is also a balance. In Liapunov theory one must find a Liapunov function; in fixed point theory one must find an appropriate mapping. Both can be challenging. The interested reader will find several different ways of defining the mappings in [2-4].

G. Fixed point theory is effective in studying stability by restricting the functions to a specified complete metric space so that conditions for stability can be greatly reduced. Liapunov functions are defined in a cylindrical domain of the form $[0, \infty) \times D$ where D is an open neighborhood of 0. The fixed point method works in a much smaller set.

2. A general neutral equation Let $\gamma > 0$ and $(C, \|\cdot\|)$ be the Banach space of continuous functions $\phi : [-\gamma, 0] \to R^n$ with the supremum norm, C_H the H-ball in C. We suppose also that Q and G map $[0, \infty) \times C_H \to R^n$ are at least continuous. Finally, let S be an $n \times n$ real constant matrix, all of whose characteristic roots have negative real parts. We then choose positive constants K and α with

$$|e^{St}| \leq Ke^{-2\alpha t}, t \geq 0.$$

With this K and α in mind, consider the neutral functional differential equation

$$x'(t) = Sx(t) + \frac{d}{dt}Q(t, x_t) + G(t, x_t) \tag{2.1}$$

with continuous initial function $\psi : [-\gamma, 0] \to R^n$, $|e^{St}| \leq Ke^{-2\alpha t}, t \geq 0$.

In order to prove asymptotic stability by fixed point theory, write (2.1) as

$$\frac{d}{dt}[x(t) - Q(t, x_t)] = S[x(t) - Q(t, x_t)] + SQ(t, x_t) + G(t, x_t)$$

so that

$$x(t) = Q(t, x_t) + e^{St}[\psi(0) - Q(0, \psi)] + \int_0^t e^{S(t-s)}[SQ(s, x_s) + G(s, x_s)]ds. \tag{2.2}$$

Our results will be based on Krasnoselskii's theorem which Smart [10; p. 31] states as follows.

THEOREM (Krasnoselskii). Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map M into S such that

(i) $Ax + By \in M (\forall x, y \in M)$,

(ii) A is continuous and AM is contained in a compact set,

(iii) B is a contraction with constant $\alpha < 1$.

Then there is a $y \in M$ with $Ay + By = y$.

Looking at the theorem and at (2.2) we see that we must define a mapping from (2.2) and, hence, must separate it into two operators, one a contraction and one a compact mapping. The choice here is fairly clear since Q is exposed and does not smooth. Our only question concerns whether or not that Q should be joined with the Q under the integral (which does get smoothed); it turns out to make little difference.

Study of simple Bernoulli equations leads us to conjecture that our solution should reside in

$$M = \{\phi : [-\gamma, \infty) \to R^n | \phi_0 = \psi, |\phi(t)| \leq Le^{-\alpha t}\} \tag{2.3}$$

where $L > 0$ is to be determined, ψ is the initial function with $\|\psi\|$ small. Now, from (2.2) we define two operators $A, B : M \to M$ by $\phi \in M$ implies that

$$(B\phi)(t) = Q(t, \phi_t) + e^{St}[\psi(0) - Q(0, \psi)] \tag{2.4}$$

and

$$(A\phi)(t) = \int_0^t e^{S(t-s)}[SQ(s, \phi_s) + G(s, \phi_s)]ds. \tag{2.5}$$

There is great latitude in the conditions which can be placed on Q and G. From the following discussions it is hoped that the reader can see how to modify our conditions to fit a wide variety of problems.

Discussion of Q

If B is to define a contraction, Q will need to be Lipschitz with a very small Lipschitz constant. But we have in mind functions such as $Q(t, \phi) = r(t)\phi^n$ so, for example, if $Q(t, \phi) = r(t)\phi^2$then we have

$$|Q(t, \phi_1) - Q(t, \phi_2)| = |r(t)||\phi_1 + \phi_2||\phi_1 - \phi_2|.$$

The fact that we are working entirely in M means that this will define a contraction with small contraction constant, proveded only that $|r(t)|Le^{-\alpha(t-\gamma)}$ is small; we have the freedom to take L as small as we please, $L > 0$. The following assumption illustrates this.

Assume that there is a continuous increasing function $\beta : [0, \infty) \to [0, \infty)$, $\beta(0) = 0$, a continuous function $q : [0, \infty) \to [0, \infty)$, and a positive constant q^* such that $\phi, \eta \in C_H$ implies that

$$(2.6) \quad |Q(t, \phi) - Q(t, \eta)| \leq \beta(q(t)[\|\phi\| + \|\eta\|])\|\phi - \eta\| \qquad \text{and} \qquad q(t)e^{-\alpha(t-\gamma)} < q^*.$$

Assume also that $Q(t, 0) = 0$ so that

$$(2.7) \qquad |Q(t, \phi)| \leq \beta(q(t)[\|\phi\|])\|\phi\|.$$

Discussion of the function G

We need three things. The growth of G must be controlled so that $A : M \to M$. A must map M into an equicontinuous set. A must be continuous. To those ends, for the function G we suppose there is a continuous function $f : [0, \infty) \to [0, \infty)$ and a continuous function $h : [0, H] \to [0, \infty)$ which is increasing.

Making $A : M \to M$

It will take the next two conditions to ensure that $A : M \to M$. Assume that for $\phi \in C_H, t \geq 0$ then

$$(2.8) \qquad |G(t, \phi)| \leq f(t)h(\|\phi\|)\|\phi\|, h(0) = 0,$$

and

$$(2.9) \qquad \int_0^t e^{-\alpha(t-s)} f(s)h(Le^{-\alpha(s-\gamma)})ds \to 0$$

as $L \to 0$ uniformly for $0 \leq t < \infty$. These two conditions may be considered as a generalization of (1.3).

Continuity of A

EXAMPLE. We will give a sample condition to ensure continuity of A based on a prototype of $G(t, \phi) = f(t)\phi^n$. To see how continuity proceeds, take $G(t, \phi) = f(t)\phi^2$ so that for a $\delta > 0$ and $\phi, \eta \in M$ with $\|\phi - \eta\| < \delta$ we have

$$\begin{aligned} |G(t,\phi) - G(t,\eta)| &= |f(t)[\phi^2 - \eta^2]| \\ &\leq |f(t)|[\|\phi\| + \|\eta\|]\|\phi - \eta\| \\ &\leq 2\delta |f(t)| L e^{-\alpha(t-\gamma)}. \end{aligned}$$

Thus, the part of $|(A\phi)(t) - (A\eta)(t)|$ from Q offers no difficulty, while the part from G satisfies

$$\int_0^t e^{-2\alpha(t-s)} 2\delta |f(s)| L e^{-\alpha(s-\gamma)} ds \leq 2\delta e^{\alpha\gamma} L \int_0^t e^{-2\alpha(t-s)} |f(s)| e^{-\alpha s} ds,$$

yielding the simple requirement that the last integral be bounded, which is an averaging condition on f.

Thus, to ensure that A is continuous we ask that for each $\epsilon > 0$ there is a $\delta > 0$ such that $\phi_1, \phi_2 \in M$ with $\|\phi_1 - \phi_2\| < \delta$ imply that

$$|G(t,\phi_1) - G(t,\phi_2)| \leq \epsilon f(t)(\|\phi_1\| + \|\phi_2\|) \tag{2.10}$$

and

$$\int_0^t e^{-2\alpha(t-s)} f(s) e^{-\alpha(s-\gamma)} ds \tag{2.11}$$

is bounded, another averaging condition on f.

A maps M into an equicontinuous set

Finally, to ensure that the operator A smooths, we may note that $(A\phi)(t)$ has a bounded derivative provided that

$$f(t)h(Le^{-\alpha(t-\gamma)})Le^{-\alpha(t-\gamma)} \text{ is bounded .} \tag{2.12}$$

This is not an averaging condition, but now we have the factor $h(Le^{-\alpha(t-\gamma)})$ which can go to zero very quickly.

THEOREM 2.1. If (2.6)-(2.12) hold then for L and ψ sufficiently small there is a solution $x(t,0,\psi) \in M$ which satisfies (2.2) and $x(t) - Q(t,x_t)$ is continuously differentiable. In particular, $x(t) \to 0$.

Proof. First, we will show that for L and ψ small enough, if $\phi, \eta \in M$ then $A\phi + B\eta \in M$. We have

$$|(B\phi)(t)| \leq \beta(q(t)\|\phi_t\|)\|\phi_t\| + Ke^{-2\alpha t}[|\psi(0)| + \beta(q(0)\|\psi\|)\|\psi\|]$$

$$\leq \beta(Lq(t)e^{-\alpha(t-\gamma)})Le^{-\alpha(t-\gamma)} + Ke^{-2\alpha t}\|\psi\|[1 + \beta(q(0)\|\psi\|)]$$

$$\leq \beta(Lq^*)Le^{-\alpha(t-\gamma)} + Ke^{-2\alpha t}\|\psi\|[1 + \beta(q(0)\|\psi\|)]$$

$$\leq \frac{1}{3}Le^{-\alpha t}$$

if L and $\|\psi\|$ are sufficiently small since $\beta(0) = 0$, β is continuous, and $q(t)e^{-\alpha(t-\gamma)} < q^*$.

Find K^* with $|S| \leq K^*$. Then $\phi \in M$ implies that

$$|(A\phi)(t)| \leq K\int_0^t e^{-2\alpha(t-s)}[K^*\beta(q(s)\|\phi_s\|)\|\phi_s\| + f(s)h(\|\phi_s\|)\|\phi_s\|]ds$$

$$\leq K\int_0^t e^{-2\alpha(t-s)}[K^*\beta(Lq^*)Le^{\gamma\alpha}e^{-\alpha s} + f(s)h(Le^{\alpha\gamma-\alpha s})Le^{\alpha\gamma}e^{-\alpha s}]ds$$

$$= KK^*\beta(Lq^*)Le^{\alpha\gamma}e^{-\alpha t}\int_0^t e^{-\alpha(t-s)}ds + KLe^{\alpha\gamma}e^{-\alpha t}\int_0^t e^{-\alpha(t-s)}f(s)h(Le^{-\alpha(s-\gamma)})ds$$

$$\leq \frac{1}{3}Le^{-\alpha t}$$

if L is sufficiently small since $\beta(0) = h(0) = 0$, both are continuous, and (2.9) holds.

Next, we can show that AM is equicontinuous by noting that $\frac{d}{dt}(A\phi)(t)$ is bounded.

Also, we can use (2.6), (2.8), and (2.10) to prove that A is continuous.

Now M resides in the Banach space of bounded continuous functions with the supremum norm on the interval $[-\gamma, \infty)$. To see that B is a contraction, if $\phi, \eta \in M$ then

$$|(B\phi)(t) - (B\eta)(t)| \leq |Q(t, \phi_t) - Q(t, \eta_t)|$$

$$\leq \beta(q(t)[\|\phi_t\| + \|\eta_t\|])\|\phi_t - \eta_t\|$$

$$\leq \beta(2q(t)Le^{-\alpha(t-\gamma)})\|\phi_t - \eta_t\|$$

$$\leq \beta(2q^*L)\|\phi_t - \eta_t\|$$

$$\leq d\|\phi - \eta\|_{[-\gamma,\infty)}$$

for some $d < 1$ if L is small.

The conditions of Krasnoselskii's theorem are satisfied and there is a $\phi \in M$ with $A\phi + B\phi = \phi$.

EXAMPLE. Consider once more the equation

$$(E_2.) \qquad x'' + (\cos x)x' + \sin x = \frac{d}{dt}t^3x^2(t - r(t)) + t^5x^3(t - r(t))$$

Theorem 2.1 was constructed so that the conditions were all based on the order of magnitude of terms, thereby avoiding tedious computations. This equation does satisfy the conditions of Theorem 2.1. We will write it as a system and separate the terms. First, take $\gamma = 1$ so that we have $0 \le r(t) \le 1$. Next, write

$$\begin{aligned} x' &= y \\ y' &= -\sin x - (\cos x)y + \frac{d}{dt}t^3x^2(t - r(t)) + t^5x^3(t - r(t)) \end{aligned}$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{d}{dt}t^3x^2(t - r(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ x - \sin x + y(1 - \cos x) + t^5x^3(t - r(t)) \end{pmatrix}.$$

It is an elementary exercise to show that these functions satisfy the conditions of Theorem 2.1. We showed how to obtain the continuity requirements in the development of (2.6) and (2.10).

3. A scalar problem. The scalar case goes beyond the vector case in two ways. First, we add a delay term so that the basic equation being perturbed is a delay equation. Next, stability of the equation being perturbed is shown by an averaging process which averages in two ways.

Let r and γ be positive constants, $(C, \|\cdot\|)$ be the Banach space of continuous functions $\phi : [-\gamma, 0] \to R$, C_H the H-ball in C for some $H > 0$, and let $Q, G : [0, \infty) \times C_H \to R$ be continuous. Suppose also that $a, b : [0, \infty) \to R$ are continuous and define a function c by

$$2c(t) := a(t) + b(t + r), t \ge 0,$$

$$c(t) = 0, t < 0.$$

Thus, c is the average of $a(t)$ and $b(t+r)$.

Consider the equation

$$x'(t) = -a(t)x(t) - b(t)x(t-r) + \frac{d}{dt}Q(t, x_t) + G(t, x_t) \tag{3.1}$$

which we can write as

$$\frac{d}{dt}[x - \int_{t-r}^{t} b(u+r)x(u)du - Q(t, x_t)] =$$

$$-[a(t) + b(t+r)][x(t) - \int_{t-r}^{t} b(u+r)x(u)du - Q(t, x_t)]$$

$$-[a(t) + b(t+r)][\int_{t-r}^{t} b(u+r)x(u)du + Q(t, x_t)] + G(t, x_t).$$

For an initial function ψ, by the variation of parameters formula we can write this as

$$x(t) = \int_{t-r}^{t} b(u+r)x(u)du + Q(t, x_t) + e^{-2\int_0^t c(s)ds}[\psi(0) - \int_{-r}^{0} b(u+r)\psi(u)du - Q(0, \psi)]$$

$$-\int_0^t e^{-\int_s^t 2c(u)du} 2c(s)[\int_{s-r}^{s} b(u+r)x(u)du + Q(s, x_s)]ds$$

$$+\int_0^t e^{-\int_s^t 2c(u)du} G(s, x_s)ds.$$

Let

$$\int_0^{\infty} c(u)du = \infty \text{ and } c(t) \geq 0 \tag{3.2}$$

and for some $L > 0$ let

$$M = \{\phi : [-\gamma, \infty) \to R : \phi_0 = \psi, |\phi(t)| \leq Le^{-\int_0^t c(s)ds} \text{ for } t \geq 0\}. \tag{3.3}$$

For $\phi \in M$ define operators A and B by

$$(B\phi)(t) := \int_{t-r}^{t} b(u+r)\phi(u)du + Q(t, \phi_t) + e^{-2\int_0^t c(s)ds}[\psi(0) - \int_{-r}^{0} b(u+r)\psi(u)du - Q(0, \psi)]$$

$$-\int_0^t e^{-\int_s^t 2c(u)du} 2c(s)[\int_{s-r}^s b(u+r)\phi(u)du + Q(s,\phi_s)]ds \tag{3.4}$$

and

$$(A\phi)(t) = \int_0^t e^{-\int_s^t 2c(u)du} G(s,\phi_s)ds. \tag{3.5}$$

To get a bound on the first term in B we suppose there are positive constants α_1 and J with

$$\sup_{t\geq 0}\int_{t-r}^t |b(s+r)|e^{\int_s^t c(u)du}ds \leq \alpha_1 \qquad \text{and} \qquad \sup_{t\geq 0}\int_{t-\gamma}^t c(u)du \leq J. \tag{3.6}$$

To prepare the second term of B for a contraction argument we suppose there is a continuous increasing function $\beta : [0,\infty) \to [0,\infty)$, a continuous function $q : [0,\infty) \to [0,\infty)$, and a continuous increasing function $\alpha_2(L)$ such that for $\phi, \eta \in M$ we have

$$\begin{aligned} &\beta(2q(t)Le^{-\int_0^{t-\gamma} c(s)ds}) \leq \alpha_2(L) \\ &|Q(t,\phi_t) - Q(t,\eta_t)| \leq \beta(q(t)[\|\phi_t\| + \|\eta_t\|])\|\phi_t - \eta_t\|, Q(t,0) = 0. \end{aligned} \tag{3.7}$$

To ensure that $A : M \to M$ we suppose there is a continuous function $f : [0,\infty) \to [0,\infty)$ and a continuous function $h : [0,H] \to [0,\infty)$ which is increasing and for which $\phi \in C_H, t \geq 0$ imply that

$$|G(t,\phi)| \leq f(t)h(\|\phi\|)\|\phi\|, h(0) = 0, \tag{3.8}$$

and that

$$\int_0^t e^{-\int_s^t c(u)du} f(s)h(Le^{-\int_0^s c(u)du})ds \to 0 \tag{3.9}$$

as $L \to 0$ uniformly for $0 \leq t < \infty$.

To prove continuity of A we ask that for each $\epsilon > 0$ there is a $\delta > 0$ such that $\phi_1, \phi_2 \in C_H$ with $\|\phi_1 - \phi_2\| < \delta$ imply that

$$|G(t,\phi_1) - G(t,\phi_2)| \leq \epsilon f(t)(\|\phi_1\| + \|\phi_2\|). \tag{3.10}$$

In addition, let

$$\int_0^t e^{-\int_s^t 2c(u)du} f(s) e^{\int_0^{s-\gamma} c(u)du} ds \tag{3.11}$$

be bounded.

THEOREM 3.1. Suppose that (3.2) and (3.6) to (3.11) hold. Let $L^* > 0$ be fixed with

$$\mu := 2\alpha_1 + 3\alpha_2(L) \sup_{t\geq 0} e^{\int_{t-\gamma}^t c(s)ds} < 1$$

and

$$f(t)h(Le^{-\int_0^{t-\gamma} c(u)du})Le^{-\int_0^{t-\gamma} c(u)du} \qquad \text{be bounded}$$

if $L \leq L^*$. Then for ψ sufficiently small, (3.1) has a solution in M.

The proof will be given in a series of lemmas. For $\phi, \eta \in M$ and for $-\gamma \leq \lambda_1 < \lambda_2 \leq \infty$ we let $\|\phi - \eta\|_{[\lambda_1,\lambda_2]} = \sup_{\lambda_1 \leq t \leq \lambda_2} |\phi(t) - \eta(t)|$. Note that $(M, \|\cdot\|_{[-\gamma,\infty)})$ is a complete metric space.

LEMMA 1. If (3.7) holds and if for $L \leq L^*$ we have

$$\mu = 2 \sup_{t\geq 0} \int_{t-r}^t |b(u+r)|du + 3\alpha_2(L) < 1$$

then B is a contraction on M.

Proof. For $\phi, \eta \in M$ we have

$$|(B\phi)(t) - (B\eta)(t)| \leq \int_{t-r}^t |b(u+r)|du \|\phi_t - \eta_t\|$$

$$+|Q(t,\phi_t) - Q(t,\eta_t)| + \sup_{s\geq 0} \int_{s-r}^s |b(u+r)||\phi(u) - \eta(u)|du + \sup_{0\leq s\leq t} |Q(s,\phi_s) - Q(s,\eta_s)|.$$

Note that

$$|Q(t,\phi_t) - Q(t,\eta_t)| \leq \beta(q(t)[\|\phi_t\| + \|\eta_t\|])\|\phi_t - \eta_t\|$$

$$\leq \beta(2q(t)Le^{-\int_0^{t-\gamma} c(s)ds})\|\phi_t - \eta_t\|$$

$$\leq \alpha_2(L)\|\phi - \eta\|_{[-\gamma,\infty)}.$$

Thus,

$$
\begin{aligned}
&|(B\phi)(t)-(B\eta)(t)| \\
&\le 2\sup_{t\ge 0}\int_{t-r}^{t}|b(u+r)|du\|\phi-\eta\|_{[-\gamma,\infty)}+2\alpha_2(L)\|\phi_t-\eta_t\|_{[-\gamma,\infty)} \\
&\le [2\sup_{t\ge 0}\int_{t-r}^{t}|b(u+r)|du+2\alpha_2(L)]\|\phi-\eta\|_{[-\gamma,\infty)} \\
&\le \mu\|\phi-\eta\|_{[-\gamma,\infty)}.
\end{aligned}
$$

LEMMA 2. If G satisfies (3.8) to (3.10) and if

$$f(t)h(Le^{-\int_0^{t-\gamma}c(u)du})Le^{-\int_0^{t-\gamma}c(u)du}+c(t)e^{-\int_0^t c(s)ds}$$

is bounded for $L\le L^*$, then A maps M into an equicontinuous set.

We note that A has a bounded derivative on M when we see at the end of the proof of Lemma 4 that $|(A\phi)(t)|\le Le^{-\int_0^t c(s)ds}$.

LEMMA 3. If (3.6) and (3.7) hold then for L and ψ sufficiently small, $\phi\in M$ implies that

$$|(B\phi)(t)|\le \mu^* Le^{-\int_0^t c(s)ds} \qquad \text{for} \qquad \mu^*<1.$$

Proof. Taking $\Gamma(\psi)=e^{-2\int_0^t c(s)ds}[|\psi(0)|+\int_{-r}^0|b(u+r)\psi(u)|+|Q(0,\psi)|]$, we have

$$|(B\phi)(t)|\le 2\sup_{t\ge 0}\int_{t-r}^{t}|b(s+r)|Le^{-\int_0^s c(u)du}ds+\alpha_2(L)Le^{-\int_0^{t-\gamma}c(s)ds}$$

$$+\Gamma(\psi)+\int_0^t e^{-2\int_s^t c(u)du}2\alpha_2(L)c(s)\|\phi_s\|ds.$$

The last term is bounded by

$$\int_0^t e^{-2\int_s^t c(u)du}2Lc(s)\alpha_2(L)e^{-\int_0^{s-\gamma}c(u)du}ds$$

$$\le \int_0^t e^{-\int_s^t c(u)du}2Lc(s)\alpha_2(L)e^{\int_{s-\gamma}^s c(u)du}dse^{-\int_0^t c(u)du}.$$

From this we have

$$|(B\phi)(t)|\le 2L\sup_{t\ge 0}\int_{t-r}^{t}|b(u+r)|e^{\int_s^t c(u)du}dse^{-\int_0^t c(u)du}+\alpha_2(L)Le^{-\int_0^{t-\gamma}c(s)ds}$$

$$+\alpha_2(L)Le^{-\int_0^t c(u)du}\int_0^t 2c(s)e^{-\int_s^t c(u)du}e^{\int_{s-\gamma}^s c(u)du}ds+\Gamma(\psi)$$

$$\leq (2\alpha_1+3\alpha_2(L)\sup_{t\geq 0}e^{\int_{t-\gamma}^t c(u)du})Le^{-\int_0^t c(s)ds}+\Gamma(\psi)$$

$$\leq \mu Le^{-\int_0^t c(s)ds}+\Gamma(\psi).$$

Thus, if $L \leq L^*$ and ψ is sufficiently small, then

$$|(B\phi)(t)| \leq \mu^* Le^{-\int_0^t c(s)ds}$$

where $\mu^* < 1$.

LEMMA 4. If (3.8) - (3.10) hold and if $0 < \theta < 1$, then there is an $L_1 > 0$ such that $\phi \in M$ and $L \leq L_1$ implies that $|(A\phi)(t)| \leq \theta Le^{-\int_0^t c(s)ds}$. Thus, for L small enough and under the conditions of Lemma 3, if $\phi, \eta \in M$ then $A\phi + B\eta \in M$.

Proof. We have

$$|(A\phi)(t)| \leq \int_0^t e^{-\int_s^t 2c(u)du}f(s)h(\|\phi_s\|)\|\phi_s\|ds$$

$$\leq \int_0^t e^{-\int_s^t 2c(u)du}f(s)h(Le^{-\int_0^{s-\gamma} c(u)du})Le^{-\int_0^{s-\gamma} c(u)du}ds$$

$$\leq \int_0^t e^{-\int_s^t 2c(u)du}f(s)h(Le^{-\int_0^s c(u)du}e^{\int_{s-\gamma}^s c(u)du})Le^{-\int_0^s c(u)du}e^{\int_{s-\gamma}^s c(u)du}ds$$

$$\leq \int_0^t e^{-\int_s^t c(u)du}f(s)h(Le^J e^{-\int_0^s c(u)du})e^J dsLe^{-\int_0^t c(s)ds}$$

$$\leq \theta Le^{-\int_0^t c(s)ds}$$

if $L \leq L_1 \leq L^*$ is chosen so that

$$\int_0^t e^{-\int_s^t c(u)du}f(s)h(Le^J e^{-\int_0^s c(u)du})e^J ds < \theta$$

by (3.9).

When we take $\mu^* + \theta < 1$ the conditions of Krasnoselskii's theorem are satisfied and there is a $\phi \in M$ with $A\phi + B\phi = \phi$, a solution of (3.1).

References

1. Bellman, R., Stability Theory of Differential Equations, McGraw-Hill, New York, 1953.

2. Burton, T. A., Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem, Nonlinear Studies 9(2002), 181-190.

3. Burton, T. A. and Furumochi, Tetsuo, Fixed points and problems in stability theory, Dynamical Systems and Appl. 10(2001), 89-116.

4. Burton, T. A. and Furumochi, Tetsuo, Krasnoselskii's fixed point theorem and stability, Nonlinear Analysis 49(2002), 445-454.

5. Krasnoselskii, M. A., Some problems of nonlinear analysis, Amer. Math. Soc. Transl. (2) 10(1958), 345-409.

6. Lakshmikantham, V. and Leela, S., Differential and Integral Inequalities, Vol. I, Academic Press, New York, 1969.

7. Perron, O., Die Stabilitatsfrage bei Differentialgleichungen, Math. Z. 32(1930), 703-728.

8. Rus, I. A., Picard Operators and Applications, Babes-Bolyai Univ., Cluj-Napoca, Romania, 1996.

9. Saito, Seiji, Global stability of solutions for quasilinear ordinary differential systems, Math. Japonica 34(1989), 821-929.

10. Serban, Marcel-Adrian, Global asymptotic stability for some difference equations via fixed point technique, Seminar on Fixed Point Theory Cluj-Napoca 2(2002), 87-96.

11. Smart, D. R., Fixed Point Theorems, Cambridge Univ. Press, Cambridge, 1974