# PARALLEL THEORIES OF INTEGRAL EQUATIONS

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Abstract. In this paper we show that we can offer a study of integral equations of the form $x(t) = a(t) - \int_0^t C(t,s)[x(s)+G(s,x(s))]ds$ by placing conditions on either the first or the second coordinate of $C$. One obtains parallel theories yielding $L^\infty$ results by working with the second coordinate, or $L^1$ results by working with the first coordinate. Both lines of study are quite elementary.

## 0. Introduction

A comparison of both classical [8] and modern [3] studies of nonconvolution integral equations using non-Liapunov techniques with studies [2] using Liapunov techniques suggests that there are two very distinct, but parallel, theories of integral equations. Starting with a scalar linear equation

$$y(t) = a(t) - \int_0^t C(t,s)y(s)ds$$

we observe three main techniques. First order Liapunov functionals place conditions on the first coordinate of $C$ and yield results about solutions being in $L^1[0,\infty)$. Fixed point methods and Razumikhin techniques place conditions on the second coordinate of $C$ and yield results about solutions being in $L^\infty[0,\infty)$. The assumption

$$\int_0^t |C(t,s)|ds \le \alpha < 1$$

will bring us through a sequence of results culminating with a perturbation theorem for the nonlinear equation

$$x(t) = a(t) - \int_0^t C(t,s)[x(s) + G(s,x(s))]ds,$$

while the assumption

$$\int_s^t |C(u,s)|du \le \beta < 1$$

will bring us through a parallel sequence of results culminating with a perturbation theorem for the same equation. Moreover, one can then

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construct a result in which those two parallel lines meet with a joint perturbation theorem. Each of those lines is a type of "bare bones" theory which can then be filled in with many related results. The first mentioned line is essentially classical, while the second line is quite new and promises to hold many new and intresting problems.

All of those results concern the integral equations. We then extend the theory by differentiating the linear equation, then integrating it to obtain a new integral equation with an explicit relation between the resolvent of the old equation with that of the new equation. This allows us to continue those parallel lines. It is very interesting to note that the newer line of results is more natural than the classical line.

The work begins with

$$x(t) = a(t) - \int_0^t C(t,s)[x(s) + G(s,x(s))]ds \tag{0.1}$$

in which $a : [0,\infty) \to \Re, C : [0,\infty)\times[0,\infty) \to \Re$, and $G : [0,\infty)\times\Re \to \Re$ are all at least continuous. Under these conditions there is at least one solution of (0.1) on some interval $[0,T)$ and if the solution remains bounded then $T = \infty$. This is explained in some detail in [1; p. 85].

The reader may verify without difficulty that virtually everything goes through without change if we take $x$ and $G(t,x)$ to be column vectors and $C$ to be an $n\times n$ matrix. In marked contrast to vector systems of ordinary differential equations, the fact that many high order differential equations, many partial differential equations, and problems from control theory can be expressed as scalar integral equations often leaves us with more respect for the scalar equation than for its vector counterpart.

We have great flexibility in studying (0.1) because of a variation of parameters formula. First, we separate out the linear part as

$$y(t) = a(t) - \int_0^t C(t,s)y(s)ds, \tag{0.2}$$

leaving us with

$$x(t) = y(t) - \int_0^t R(t,s)G(s,x(s))ds \tag{0.3}$$

(See [4; p. 191] or [2; p. 163].) where $R(t,s)$ is the resolvent which solves

$$\begin{aligned} R(t,s) &= C(t,s) - \int_s^t C(t,u)R(u,s)du \\ &= C(t,s) - \int_s^t R(t,u)C(u,s)du, \qquad (0.4) \end{aligned}$$

as shown by Miller [4; p. 193 ] and yields

$$y(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{0.5}$$

The two choices in (0.4) yield more flexibility which we will use repeatedly.

Early work on (0.1) is found in Strauss [8] and the focus of that study is in showing and using

$$\sup_{t\geq 0} \int_0^t |R(t,s)|ds < \infty \tag{0.6}$$

which is clearly a useful result in (0.5) yielding $y \in L^\infty$ whenever $a \in L^\infty$.

Section 1 will now show a number of central results with (0.6) as the main focus. Then Section 2 will focus on

$$\sup_{0\leq s\leq t<\infty} \int_s^t |R(u,s)|du < \infty \tag{0.7}$$

which is also used in (0.5) to show $y \in L^1[0,\infty)$ whenever $a \in L^1[0,\infty)$.

## 1. The $s$-coordinate of $C(t,s)$

We begin with a well-known (See Strauss [8].) sufficient condition for (0.6).

**Theorem 1.1.** *If there is an $\alpha < 1$ with*

$$\int_0^t |C(t,s)|ds \leq \alpha \tag{1.1}$$

*for $t \geq 0$, then for $t \geq 0$ we also have*

$$\int_0^t |R(t,s)|ds \leq \frac{\alpha}{1-\alpha}. \tag{1.2}$$

A nice collection of basic results can be extracted from this theorem and the proof is simple in the extreme. The classical proof, found in [8] or [2] is a simple integration of the first choice in (0.4) (after taking appropriate absolute values), followed by interchange of order of integration. Another proof is by means of contraction mappings and it also yields the same bound by resorting to standard error bounds introduced in Smart [7; p.3]. That proof is most elegant, yielding in one blow existence, uniqueness, boundedness, and a simple avenue for advancement to a nonlinear form of (0.2). It rests on the following result of Perron.

**NOTATION.** Throughout this note the symbol $(\mathcal{BC}, \|\cdot\|)$ will denote the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm. Also, $\|x\|^{[a,b]} = \sup_{a\leq t\leq b} |x(t)|$.

**Theorem 1.2.** *(Perron [5]) If $R(t,s)$ is an $n\times n$ matrix of continuous functions $r_{ij}:[0,\infty)\times[0,\infty)\to\Re$, $1\le i,j\le n$, and if $\int_0^t R(t,s)\phi(s)ds$ is bounded for every $\phi\in\mathcal{BC}$, then (0.6) holds.*

With the aid of this theorem we readily prove (See [2; p. 54].) the following fundamental theorem of integral equations.

**Theorem 1.3.** *Every solution of (0.2) is bounded for every $a\in\mathcal{BC}$ if and only if (0.6) holds.*

This result will allow us to link two very different problems. With Theorem 1.1 we can now formally state a consequence of (1.1) and (0.5).

**Theorem 1.4.** *If (1.1) (or (1.2)) holds and if $a\in\mathcal{BC}$, then the unique solution, $y$, of (0.2) is bounded.*

The algebraic theory of Ritt [6] shows that, however simple $C(t,s)$ may be, $R(t,s)$ is usually arbitrarily complicated. Yet, there are startling similarities between $R(t,s)$ and $C(t,s)$, as we see in Theorem 1.1. There are two more in quick succession.

**Theorem 1.5.** *Suppose that (1.1) (or (1.2)) holds and there is an $M>0$ with*

$$\sup_{0\le s\le t<\infty}|C(t,s)|\le M. \tag{1.3}$$

*Then*

$$\sup_{0\le s\le t<\infty}|R(t,s)|\le\frac{M}{1-\alpha} \tag{1.4}$$

*where $\alpha$ is defined in (1.1).*

*Proof.* From the second choice in (0.4) we have

$$\begin{aligned}|R(t,s)| &\le |C(t,s)|+\int_s^t|R(t,u)C(u,s)|du\\ &\le M+M\int_s^t|R(t,u)|du\\ &\le M+M\int_0^t|R(t,u)|du\\ &\le M+\frac{M\alpha}{1-\alpha},\end{aligned}$$

as required. □

There is a second proof using the contraction mapping principle, yielding the identical result. The next result has no known counterpart in Section 2. It would be so interesting to find such a counterpart.

**Theorem 1.6.** *(Strauss [8]) Suppose (1.1) holds and that for every $T > 0$ we have*

$$\int_0^T |C(t,s)|ds \to 0 \text{ as } t \to \infty. \tag{1.5}$$

*Then*

$$\int_0^T |R(t,s)|ds \to 0 \text{ as } t \to \infty. \tag{1.6}$$

There are two classical results from convolution theory in analysis which are of great utility. In this section and the next we find exact counterparts for our non-convolution problem. Here is the counterpart to the theorem which states that the convolution of an $L^1$ function with a function tending to zero does, itself, tend to zero.

**Theorem 1.7.** *If (1.1) (or (1.2)) and (1.5) hold, then*

$$\int_0^t R(t,s)a(s)ds \to 0 \text{ as } t \to \infty \tag{1.7}$$

*whenever $a(t) \to 0$ as $t \to \infty$.*

*Proof.* It is, of course, the consequences of (1.1) and (1.5), being (1.2) and (1.6), which we need and we often have those without (1.1) and (1.5). To be definite, let $\int_0^t |R(t,s)|ds \le M$. Let $\epsilon > 0$ be given and find $T > 0$ for a fixed $a(t)$ so that $|a(t)| \le \epsilon/2M$ if $t \ge T$. Then we have

$$\begin{aligned}\int_0^t |R(t,s)a(s)|ds &\le \int_0^T |R(t,s)a(s)|ds + \int_T^t |R(t,s)a(s)|ds \\ &\le \|a\| \int_0^T |R(t,s)|ds + \frac{\epsilon}{2M}\int_0^t |R(t,s)|ds \\ &\le \|a\| \int_0^T |R(t,s)|ds + \frac{\epsilon}{2},\end{aligned}$$

while the first term tends to zero as $t \to \infty$. □

This small collection of results can be greatly enlarged by gleaning Chapter 2 of [2]. However, it is "complete" in the sense that we now have enough to attack a nontrivial nonlinear perturbation. To that end, we suppose there is a continuous function $\phi : [0,\infty) \to [0,\infty)$, a $J \ge 0$, and a $\gamma < 1$ with

$$|G(t,x)| \le \phi(t)|x| \tag{1.8}$$

and for $\alpha$ satisfying (1.1) then

$$\phi(t)\frac{\alpha}{1-\alpha} \le \gamma \text{ if } t \ge J. \tag{1.9}$$

**Theorem 1.8.** *Let (1.1) (or (1.2)), (1.8), and (1.9) hold.*

*(i) If $a \in \mathcal{BC}$ and if $J = 0$, then $y(t)$ and any solution $x(t)$ of (0.1) are bounded.*

*(ii) If $a \in \mathcal{BC}$ and if (1.5) holds, then $x$ and $y$ are bounded.*

*(iii) If $\phi(t) \to 0$ and if (1.5) holds, then $x(t) \to y(t)$ as $t \to \infty$.*

*Proof.* If (1.1) holds and $a \in \mathcal{BC}$, then certainly $y$ is bounded. If $J = 0$ then

$$|x(t)| \le \|y\| + \|\phi\| \int_0^t |R(t,s)||x(s)|ds.$$

If $x(t)$ is unbounded then there is a sequence $\{t_n\} \uparrow \infty$ such that $|x(t)| \le |x(t_n)|$ if $0 \le t \le t_n$. Thus

$$|x(t_n)| \le \|y\| + \|\phi\| \frac{\alpha}{1-\alpha} |x(t_n)| \le \|y\| + \gamma |x(t_n)|,$$

a contradiction for large $t_n$.

If $a(t)$ is bounded and if (1.5) holds, then the above argument changes to $J < T < t_n$ and

$$\begin{aligned}
|x(t_n)| \le \|y\| &+ \int_0^T |R(t_n,s)|\phi(s)|x(s)|ds \\
&+ \int_T^{t_n} |R(t_n,s)|\phi(s)|x(s)|ds \\
&\le \|y\| + \int_0^T |R(t_n,s)|\phi(s)ds\|x\|^{[0,T]} \\
&+ \frac{\alpha}{1-\alpha}\|\phi\|^{[T,\infty)}|x(t_n)| \\
&\le \|y\| + 1 + \gamma|x(t_n)|.
\end{aligned}$$

if $t_n$ is large because of (1.5). We get the same contradiction for large $|x(t_n)|$ proving (ii).

By Theorem 1.7, if $a(t) \to 0$ it is clear that $y(t) \to 0$. For (iii) we apply Theorem 1.7 noting that for $x$ bounded and $\phi(t) \to 0$ we have $G(t, x(t)) \to 0$ so $\int_0^t R(t,s)G(s,x(s))ds \to 0$ and so $x(t) \to y(t)$. □

We have provided a brief sketch of a line of results based on (1.1), virtually alone, sufficient to yield an informative result concerning a nonlinear perturbation. We now turn to a parallel line of results based on $\int_s^t |C(u,s)|du \le \beta < 1$ and end with a result letting the parallel lines meet.

## 2. The $t$-coordinate of $C(t,s)$.

Much of Section 1 is classical in character. This is not. Here is the counterpart of Theorem 1.1.

**Theorem 2.1.** *If there is a $\beta < 1$ with*

$$\int_s^t |C(u,s)|du \le \beta \text{ for } 0 \le s \le t < \infty, \tag{2.1}$$

*then*

$$\int_s^t |R(u,s)|du \le \frac{\beta}{1-\beta} \text{ and } \int_0^\infty |R(v+t,t)|dv \le \frac{\beta}{1-\beta}. \tag{2.2}$$

There are two very different and instructive proofs of this result. Here is the first proof.

*Proof.* Using the unique function $R(t,s)$ we define a Liapunov functional

$$V(t) = \int_s^t \int_{t-u}^\infty |C(u+v,u)|dv|R(u,s)|du \tag{2.3}$$

for fixed $s \in [0,\infty)$. Then we note that we have

$$|R(t,s)| \le |C(t,s)| + \int_s^t |C(t,u)R(u,s)|du$$

and so

$$\begin{aligned} V'(t) &= \int_0^\infty |C(t+v,t)|dv|R(t,s)| - \int_s^t |C(t,u)||R(u,s)|du \\ &\le \int_t^\infty |C(w,t)|dw|R(t,s)| + |C(t,s)| - |R(t,s)| \\ &\le \beta|R(t,s)| + |C(t,s)| - |R(t,s)| \\ &= -(1-\beta)|R(t,s)| + |C(t,s)|. \end{aligned}$$

Thus,

$$\begin{aligned} 0 \le V(t) &\le V(s) - (1-\beta)\int_s^t |R(u,s)|du + \int_s^t |C(u,s)|du \\ &\le -(1-\beta)\int_s^t |R(u,s)|du + \beta \end{aligned}$$

so

$$\int_s^t |R(u,s)|du \le \frac{\beta}{1-\beta},$$

as required.

Moreover,

$$\int_0^\infty |R(v+t,t)|dv = \int_t^\infty |R(w,t)|dw \le \frac{\beta}{1-\beta}.$$

Next,

$$\int_s^t |R(v,s)|dv \le \frac{\beta}{1-\beta}$$

implies

$$\int_s^\infty |R(v,s)|dv \le \frac{\beta}{1-\beta}$$

so

$$\begin{aligned}\int_{t-s}^\infty |R(u+s,s)|du &= \int_t^\infty |R(v,s)|dv\\ &\le \int_s^\infty |R(v,s)|dv \le \frac{\beta}{1-\beta}.\end{aligned}$$

□

**Notice.** In going from the first line to the second, we discarded the entire term

$$\int_s^t |R(v,s)|dv$$

and that can be very careless. In Section 7 we will try to recycle that quantity.

Here is the second proof.

*Proof.* From (0.4) we have

$$\begin{aligned}\int_s^t |R(u,s)|du &\le \int_s^t |C(u,s)|du + \int_s^t \int_s^v |C(v,u)R(u,s)|dudv\\ &= \int_s^t |C(u,s)|du + \int_s^t \int_u^t |C(v,u)R(u,s)|dvdu\end{aligned}$$

$$\le \beta + \beta \int_s^t |R(u,s)|du,$$

yielding the result. □

We now have the counterpart of Theorem 1.4.

**Theorem 2.2.** *If (2.1) (or (2.2)) holds and if $a \in L^1[0,\infty)$, then the solution $y$ of (0.2) is in $L^1[0,\infty)$.*

This theorem, too, has two interesting and very different proofs. Here is the first one.

*Proof.* From (0.5) we have

$$\begin{aligned}\int_0^t |y(s)|ds &\le \int_0^t |a(s)|ds + \int_0^t \int_0^v |R(v,s)a(s)|dsdv\\ &= \int_0^t |a(s)|ds + \int_0^t \int_s^t |R(v,s)||a(s)|dvds\\ &\le \int_0^t |a(s)|ds + \sup_{0\le s\le t<\infty} \int_s^t |R(v,s)|dv \int_0^t |a(s)|ds\\ &\le \left(1+\frac{\beta}{1-\beta}\right)\int_0^t |a(s)|ds = \frac{1}{1-\beta}\int_0^t |a(s)|ds.\end{aligned}$$

□

For the second proof, use the Liapunov functional

$$V(t) = \int_0^t \int_{t-s}^{\infty} |C(u+v,v)|du|y(v)|dv, \tag{2.4}$$

as we have done in a previous proof. Differentiate it along the solution of (0.2).

We now quickly obtain the counterpart of Theorem 1.7. It is the counterpart also of the classical theorem of analysis stating that the convolution of two $L^1$-functions is an $L^1$-function.

**Corollary 2.3.** *If (2.1) (or (2.2)) holds and if $a \in L^1[0,\infty)$, so is $\int_0^t R(t,s)a(s)ds$.*

This is an immediate consequence of Theorem 2.2.

**Theorem 2.4.** *Let (2.1) hold, let $a \in L^1[0,\infty)$, and let*

$$|G(t,x)| \le \phi(t)|x| \tag{2.5}$$

*where $\phi : [0,\infty) \to [0,\infty)$ is continuous and there is a $\gamma < 1$ with*

$$\phi(t)\frac{\beta}{1-\beta} \le \gamma. \tag{2.6}$$

*Then the solution $y$ of (0.2) and any solution $x$ of (0.1) are both in $L^1[0,\infty)$.*

*Proof.* Theorem 2.2 already yields $y \in L^1[0,\infty)$. Define a functional

$$V(t) = \int_0^t \int_{t-s}^{\infty} |R(u+s,s)|du|G(s,x(s))|ds$$

and since

$$-\int_0^t |R(t,s)G(s,x(s))|ds \le |y(t)| - |x(t)|$$

we have

$$\begin{aligned}
V'(t) &= \int_0^{\infty} |R(u+t,t)|du|G(t,x(t))| - \int_0^t |R(t,s)||G(s,x(s))|ds \\
&\le \frac{\beta}{1-\beta}|G(t,x(t))| + |y(t)| - |x(t)| \\
&\le \frac{\beta}{1-\beta}\phi(t)|x(t)| + |y(t)| - |x(t)| \\
&\le -(1-\gamma)|x(t)| + |y(t)|
\end{aligned}$$

or

$$0 \le V(t) \le V(0) - (1-\gamma)\int_0^t |x(s)|ds + \int_0^t |y(s)|ds,$$

yielding the result. □

We have shown two parallel lines of results stemming from (1.1) and (2.1). Both lines showed a variety of simple results culminating in a nonlinear perturbation. We now let those parallel lines meet, resulting in a particularly nice pair of results.

**Theorem 2.5.** *Let (2.1) hold, let $a \in L^1[0,\infty)$, and let*

$$|G(t,x)| \le \phi(t)|x| \text{ where } \phi \in L^1[0,\infty). \tag{2.7}$$

*If $x$ is any bounded solution of (0.1), then $x \in L^1[0,\infty)$.*

*Proof.* We know from Theorem 2.2 that $y \in L^1[0,\infty)$ and from Theorem 2.3 that

$$\int_0^t R(t,s)\psi(s)ds \in L^1[0,\infty) \text{ whenever } \psi \in L^1[0,\infty).$$

Let $|x(t)| \le U$ so that $|G(t,x(t))| \le \phi(t)|x(t)| \le \phi(t)U$ and is in $L^1$. Thus,

$$\int_0^t R(t,s)G(s,x(s))ds \in L^1[0,\infty),$$

completing the proof. □

We now have a corollary to Theorems 1.8(i) and 2.5.

**Corollary 2.6.** *Let (1.1), (2.5), (1.9), (2.1), and (2.7) hold. If $a \in L^1[0,\infty)$ and bounded, then so is $x$.*

Corollary 2.3 asks for $a \in L^1$ and (2.1) to yield $\int_0^t R(t,s)a(s)ds \in L^1$. We now set out to obtain conditions for $\int_0^t |R(t,s)a(s)|ds \in L^1$. We continue with the idea of transferring properties of $C(t,s)$ to $R(t,s)$. More and more, we see the two very different functions merge.

**Theorem 2.7.** *If*

*(i) there is an $M > 0$ with $\int_s^t |C(u,s)|du \le M$,*

*(ii) for each $T > 0$, then $\int_0^T |C(t,s)|ds \in L^1$, and*

*(iii) $a \in L^1$,*

*then*

$$\int_0^t |C(t,s)a(s)|ds \in L^1[0,\infty).$$

*Proof.* We have

$$\int_0^t |C(t,s)||a(s)|ds = \int_0^T |C(t,s)||a(s)|ds + \int_T^t |C(t,s)||a(s)|ds.$$

As $a$, hence $|a|$, is continuous, it is bounded on $[0,T]$, so by hypothesis the first integral on the right-hand-side is in $L^1$. Now

$$\int_0^u |C(u,s)||a(s)|ds = \int_0^T |C(u,s)||a(s)|ds + \int_T^u |C(u,s)||a(s)|ds.$$

Integrating the last integral we obtain

$$\int_T^t \int_T^u |C(u,s)||a(s)|dsdu = \int_T^t \int_s^t |C(u,s)|du|a(s)|ds$$
$$\leq M \int_T^t |a(s)|ds \leq M \int_T^\infty |a(s)|ds.$$

so that $\int_T^t |C(t,s)||a(s)|ds \in L^1$. □

**Theorem 2.8.** *If*

*(i) there is a* $\beta < 1$ *with* $\int_s^t |C(u,s)|du \leq \beta$,

*(ii) for each* $T > 0$, $\int_0^T |C(t,s)|ds \in L^1$, *and*

*(iii)* $a \in L^1$,

*then*

$$\int_0^t |R(t,s)a(s)|ds \in L^1.$$

*Proof.* By Theorem 2.7 we have $\int_0^t |C(t,s)||a(s)|ds \in L^1$. From

$$|R(t,s)| \leq |C(t,s)| + \int_s^t |R(t,u)||C(u,s)|du,$$

we have

$$|R(t,s)||a(s)| \leq |C(t,s)||a(s)| + \int_s^t |R(t,u)||C(u,s)|du|a(s)|,$$

or

$$\int_0^t |R(t,s)||a(s)|ds \leq \int_0^t |C(t,s)||a(s)|ds + \int_0^t \int_s^t |R(t,u)||C(u,s)|du|a(s)|ds$$
$$= \int_0^t |C(t,s)||a(s)|ds + \int_0^t \int_0^u |R(t,u)||C(u,s)||a(s)|dsdu.$$

Now the first term on the right is $L^1$ and

$$\int_0^t \left( \int_0^v \int_0^u |R(v,u)||C(u,s)||a(s)|dsdu \right) dv$$
$$= \int_0^t \int_u^t \int_0^u |R(v,u)||C(u,s)||a(s)|dsdvdu$$
$$= \int_0^t \int_u^t |R(v,u)| \int_0^u |C(u,s)||a(s)|dsdvdu$$
$$= \int_0^t \int_0^u |C(u,s)||a(s)|ds \left( \int_u^t |R(v,u)|dv \right) du$$
$$\leq \frac{\beta}{1-\beta} \int_0^t \int_0^u |C(u,s)||a(s)|dsdu.$$

so

$$\int_0^t \left( \int_0^u |C(u,s)||a(s)|ds \right) du < \infty.$$

□

## 3. Preparation of (0.1) and (0.2)

Both (1.1) and (2.1) frequently fail, but so often $C_t(t,s)$ is a very nice function. The kernels

$$(1/2)[1+t-s]^{-1/2} \text{ and } 1+[1+t-s]^{-3/2}$$

are typical examples of kernels which become integrable after differentiation and also satisfy (1.5). While (0.2) may become more tractable upon differentiation, (0.1) may become less so. The idea then is to differentiate (0.2), derive properties of $R$ and $y$, and apply those properties to the study of (0.3). Referring back to Theorems 1.1 and 2.1, we now wish to deduce (1.2) without (1.1) and (2.2) without (2.1). That is our challenge in Sections 4 and 5.

It will severely complicate matters if we differentiate $a(t)$. However, under our stated conditions in Section 0, it is true that (0.2) has a unique continuous solution (see [9]) and, hence, each side of

$$(y(t)-a(t)) = -\int_0^t C(t,s)y(s)ds$$

has a continuous derivative whenever

$$C_t(t,s) \text{ is continuous.} \tag{3.1}$$

Here, $C_t(t,s) = \frac{\partial C(t,s)}{\partial t}$. Thus, we write

$$\begin{aligned}(y(t)-a(t))' &= -C(t,t)y(t) - \int_0^t C_t(t,s)y(s)ds \\ &= -C(t,t)[y(t)-a(t)] - \int_0^t C_t(t,s)[y(s)-a(s)]ds \\ &\quad - C(t,t)a(t) - \int_0^t C_t(t,s)a(s)ds \end{aligned}\tag{3.2}$$

which we now write as

$$w'(t) = A(t) - C(t,t)w(t) - \int_0^t C_t(t,s)w(s)ds \tag{3.3}$$

where

$$A(t) = -C(t,t)a(t) - \int_0^t C_t(t,s)a(s)ds, \; w = y-a. \tag{3.4}$$

Notice that $y(0)=a(0)$ so

$$w(0)=0. \tag{3.5}$$

We will now write (3.3) as an integral equation

$$\begin{aligned} w(t) &= \int_0^t e^{-\int_u^t C(s,s)ds}A(u)du - \int_0^t e^{-\int_u^t C(s,s)ds}\int_0^u C_t(u,s)w(s)dsdu \\ &= \int_0^t e^{-\int_u^t C(s,s)ds}A(u)du - \int_0^t \int_s^t e^{-\int_u^t C(s,s)ds}C_t(u,s)duw(s)ds \end{aligned}$$

or

$$w(t) = \int_0^t e^{-\int_u^t C(s,s)ds} A(u)du - \int_0^t L(t,s)w(s)ds \tag{3.6}$$

where

$$L(t,s) = \int_s^t e^{-\int_u^t C(s,s)ds} C_t(u,s)du \tag{3.7}$$

and finally

$$w(t) = d(t) - \int_0^t L(t,s)w(s)ds \tag{3.8}$$

where

$$d(t) = \int_0^t e^{-\int_u^t C(s,s)ds} A(u)du. \tag{3.9}$$

Now, for (3.8) there is the new resolvent equation

$$Q(t,s) = L(t,s) - \int_s^t Q(t,u)L(u,s)du \tag{3.10}$$

with variation of parameters formula

$$w(t) = d(t) - \int_0^t Q(t,s)d(s)ds. \tag{3.11}$$

From (0.5) we have

$$y(t) - a(t) = -\int_0^t R(t,s)a(s)ds$$

so from this and (3.11) with $w = y - a$ from (3.4), we then have our most fundamental relation

$$-\int_0^t R(t,s)a(s)ds = d(t) - \int_0^t Q(t,s)d(s)ds \tag{3.12}$$

with $d(t)$ defined in (3.9) and $A(t)$ in (3.4).

Equation (3.12) will allow us to infer properties of $\int_0^t |R(t,s)|ds$ from properties of $\int_0^t |Q(t,s)|ds$ which come directly from $L(t,s)$ as in Theorem 1.5. It turns out that the parallel line to determine (2.2) is far more direct. Starting with (0.4) we again assume (3.1) and write

$$R_t(t,s) = C_t(t,s) - C(t,t)R(t,s) - \int_s^t C_t(t,u)R(u,s)du$$

where $s \geq 0$ is fixed and $t \geq s$. Treating this as a first order ode we have

$$\begin{aligned} R(t,s) = R(s,s)e^{-\int_s^t C(q,q)dq} &+ \int_s^t e^{-\int_v^t C(q,q)dq} C_t(v,s)dv \\ &- \int_s^t e^{-\int_v^t C(q,q)dq} \int_s^v C_t(v,u)R(u,s)dudv. \end{aligned}$$

If we interchange the order of integration in the last term we have

$$\begin{aligned}R(t,s) &= C(s,s)e^{-\int_s^t C(q,q)dq} + \int_s^t e^{-\int_v^t C(q,q)dq} C_t(v,s)dv \\ &\quad - \int_s^t \int_u^t e^{-\int_v^t C(q,q)dq} C_t(v,u)dvR(u,s)du. \end{aligned} \tag{3.13}$$

Now we want to verify that the kernel in the last integral is $L(t,u)$. In $L(t,s)$, change $u$ to $v$, yielding

$$L(t,s) = \int_s^t e^{-\int_v^t C(q,q)dq} C_t(v,s)dv.$$

To complete the verification, change $s$ to $u$. Then rewrite (3.13) as

$$R(t,s) = B(t,s) - \int_s^t L(t,u)R(u,s)du \tag{3.14}$$

where

$$B(t,s) = C(s,s)e^{-\int_s^t C(q,q)dq} + \int_s^t e^{-\int_v^t C(q,q)dq} C_t(v,s)dv. \tag{3.15}$$

From (3.7) and (3.15) we will be able to extend every result in Section 2 to the case in which

$$\int_s^t |L(u,s)|du \le \beta^* < 1.$$

## 4. The $s$-coordinate of $C_t(t,s)$.

We will use (3.12) to deduce (1.2), thereby recreating most of the theorems of Section 1 with $L(t,s)$ from (3.7) replacing $C(t,s)$. First we obtain the counterpart of Theorem 1.1.

**Theorem 4.1.** *If there is an $\alpha^* < 1$ with*

$$\int_0^t |L(t,s)|ds \le \alpha^* \tag{4.1}$$

*then*

$$\int_0^t |Q(t,s)|ds \le \frac{\alpha^*}{1-\alpha^*}. \tag{4.2}$$

*Let $A$ be defined by (3.4) and $d$ be defined by (3.9). If, in addition, $d \in \mathcal{BC}$ whenever $a \in \mathcal{BC}$, then (1.2) holds for some $\alpha < 1$.*

*Proof.* Conclusion (4.2) is no different than the work in Theorem 1.1 of showing (1.2) from (1.1). Note that the right-hand-side of (3.12) is in $\mathcal{BC}$ since $d \in \mathcal{BC}$ and since $\int_0^t Q(t,s)d(s)ds \in \mathcal{BC}$ because of (4.2). Hence, the left-hand-side is in $\mathcal{BC}$ for each $a \in \mathcal{BC}$. By Perron's theorem

$$\sup_{t\ge 0} \int_0^t |R(t,s)|ds = \gamma < \infty.$$

We readily determine the required $\alpha$. $\square$

We now seek a counterpart of Theorem 1.3. Namely, we want to show that if the conditions of Theorem 4.1 hold and if $a \in \mathcal{BC}$ then $y \in \mathcal{BC}$. That is immediate because Theorem 1.3 only asks (1.2) which we have from Theorem 4.1.

**Theorem 4.2.** *Let (4.1) hold and let $d \in \mathcal{BC}$ whenever $a \in \mathcal{BC}$; hence, (1.2) holds. If $a \in \mathcal{BC}$, so is $y$, the unique solution of (0.2).*

Next, we obtain the counterpart of Thorem 1.5. This is just a result of Strauss [8], exactly as Theorem 1.5.

**Theorem 4.3.** *Suppose that (4.1) holds and that for every $T > 0$ we have*

$$\int_0^T |L(t,s)|ds \to 0 \text{ as } t \to \infty. \tag{4.3}$$

*Then*

$$\int_0^T |Q(t,s)|ds \to 0 \text{ as } t \to \infty. \tag{4.4}$$

We now reason through the counterpart of Theorem 1.7.

**Theorem 4.4.** *Let (4.1) and (4.3) hold so that (4.4) holds. Suppose that $d(t) \to 0$ whenever $a(t) \to 0$. Then by Theorem 1.6 it will follow that $\int_0^t Q(t,s)d(s)ds \to 0$ as $t \to \infty$ and, hence, from (3.12) we have that*

$$\int_0^t R(t,s)a(s)ds \to 0 \text{ as } t \to \infty.$$

**Theorem 4.5.** *Let (4.1) hold and for $B$ defined in (3.15) we suppose there is an $M > 0$ with $|B(t,s)| \le M$. Then*

$$|R(t,s)| \le \frac{M}{1-\alpha^*}$$

*Proof.* Now $R$ is defined in (3.14) and for fixed $s$ we define a mapping as follows. Let $(X, \|\cdot\|)$ denote the Banach space of bounded continuous functions $\phi : [s,\infty) \to \Re$ with the supremum norm. Define $P : X \to X$ by $\phi \in X$ implies

$$(P\phi)(t) = B(t,s) - \int_s^t L(t,u)\phi(u)du.$$

This is a contraction by (4.1) with unique solution $\phi(t)$ and it is clearly $R(t,s)$ for $t \ge s$. It is known (See Smart [8; p. 3].) that if we take $\phi_0 = 0$, then

$$\|R(t,s) - \phi_0\| \le \frac{1}{1-\alpha^*}\|\phi_0 - P\phi_0\| = \frac{1}{1-\alpha^*}\|B(t,s)\| \le \frac{M}{1-\alpha^*}$$

and that holds for every $s$. □

We can get a close approximation to Theorem 1.6.

**Theorem 4.6.** *Let (4.1) and (4.3) hold and let $|R(t,s)| \le J$. Suppose that $d(t) \to 0$ whenever $a(t) \to 0$ so that by Theorem 1.7 we have $\int_0^t Q(t,s)d(s)ds \to 0$ as $t \to \infty$. Then for each $T > 0$ we have*

$$\int_0^T R(t,s)ds \to 0 \text{ as } t \to \infty. \tag{4.5}$$

*Proof.* Let $T > 0$ and $\epsilon > 0$ be given. Define $a(t) = 1$ on $[0,T]$, $a(t) = 0$ on $[T + \frac{\epsilon}{J}, \infty]$, and linear and continuous in between. Then for $t > T + \frac{\epsilon}{J}$we have from (3.12) that

$$\begin{aligned}\left|\int_0^t R(t,s)a(s)ds\right| &= \left|\int_0^T R(t,s)ds + \int_T^{T+\epsilon/J} R(t,s)a(s)ds\right| \\ &\le |d(t)| + \left|\int_0^t Q(t,s)d(s)ds\right| < \epsilon\end{aligned}$$

if $t \ge T^*$ by Theorem 4.4 for some $T^* > 0$. But

$$\left|\int_T^{T+\epsilon/J} R(t,s)ds\right| \le \frac{\epsilon J}{2J} = \epsilon/2$$

and so

$$\left|\int_0^T R(t,s)ds\right| < \epsilon$$

if $t \ge T^*$. $\square$

**Theorem 4.7.** *Let (4.1) hold and suppose that $d \in \mathcal{BC}$ whenever $a \in \mathcal{BC}$; hence, (1.2) holds by Theorem 4.2 for some $\alpha < 1$. Let (1.8) hold and let $\phi(t) \to 0$ as $t \to \infty$. Suppose also that $|C(t,s)| \le M$ for some $M > 0$. Then $y(t)$ and $x(t)$ are bounded. If, in addition, $a(t) \to 0$ , so does $y(t)$ and $x(t)$.*

*Proof.* From Theorem 1.5 we have $|R(t,s)| \le \frac{M}{1-\alpha}$. We have already shown that $y$ is bounded. Since $\int_0^t |R(t,s)|ds \le \frac{\alpha}{1-\alpha}$ and $\phi(s) \to 0$, we can find $J > 0$ so that $\int_J^t |R(t,s)\phi(s)|ds \le 1/2$ if $t \ge J$. Thus, for $J < t$ we have

$$\begin{aligned}&\int_0^t |R(t,s)G(s,x(s))|ds \\ &\qquad \le \int_0^J |R(t,s)G(s,x(s))|ds + \int_J^t |R(t,s)|\phi(s)|x(s)|ds \\ &\qquad \le \|x\|^{[0,J]} \int_0^J |R(t,s)|ds\|\phi\|^{[0,J]} + \int_J^t |R(t,s)|\phi(s)ds\|x\|^{[0,t]} \\ &\qquad \le \|x\|^{[0,J]} \int_0^t |R(t,s)|ds\|\phi\|^{[0,J]} + (1/2)\|x\|^{[0,t]}.\end{aligned}$$

If $x$ is not bounded, then there is a sequence $\{t_n\} \uparrow \infty$ such that $|x(t)| \le |x(t_n)|$ if $0 \le t \le t_n$. Thus

$$|x(t_n)| \le \|y\| + \|x\|^{[0,J]} \frac{\alpha}{1-\alpha} \|\phi\|^{[0,J]} + (1/2)|x(t_n)|$$

yielding $x$ bounded.

Finally, if $a(t) \to 0$, so does $y$. If $\phi(t) \to 0$ as $t \to \infty$, so does $\int_0^t R(t,s)\phi(s)ds$. Thus, if $x$ is bounded then $G(t,x(t)) \to 0$ as $t \to \infty$ and so $\int_0^t R(t,s)G(s,x(s))ds \to 0$ as $t \to \infty$. Thus, $x(t) \to y(t) \to 0$. □

## 5. The $t$-coordinate of $C_t(t,s)$

Because (3.14) is so close to (0.4) virtually everything in Section 2 is trivially advanced to (3.14).

**Theorem 5.1.** *If there is a $\beta^* < 1$ with*

$$\int_s^t |L(u,s)|du \le \beta^* \tag{5.1}$$

*and if there is a $\mu > 0$ so that for $B$ defined by (3.15) we have*

$$\int_s^t |B(u,s)|du \le \mu \tag{5.2}$$

*then*

$$\int_s^t |R(u,s)|du \le \frac{\mu}{1-\beta^*}. \tag{5.3}$$

*Proof.* From (3.14) we have

$$\begin{aligned}\int_s^t |R(u,s)|du &\le \int_s^t |B(u,s)|du + \int_s^t \int_s^v |L(v,u)||R(u,s)|dudv \\ &\le \mu + \int_s^t \int_u^t |L(v,u)||R(u,s)|dvdu \\ &\le \mu + \int_s^t \beta^*|R(u,s)|du\end{aligned}$$

or

$$(1-\beta^*)\int_s^t |R(u,s)|du \le \mu,$$

as required. □

We readily combine the counterparts of Theorem 2.2 and Corollary 2.3 as follows.

**Theorem 5.2.** *If (5.1) and (5.2) hold and if $a \in L^1[0,\infty)$, then so is $y$, the solution of (0.2), as well as $\int_0^t R(t,s)a(s)ds$.*

**Theorem 5.3.** *If (5.1) and (5.2) hold and if both $B(t,s)$ and $L(t,s)$ are bounded, so is $R(t,s)$.*

We simply note that for $M$ the bound on $B$ and $L$ we have

$$|R(t,s)| \leq M + M\int_s^t |R(u,s)|du \leq M + \frac{M\mu}{1-\beta*}.$$

It is so important to know that $R(t,s)$ is bounded. We have seen it used several times. But if we have (5.3) and if $R$ is bounded then we also have that

$$\int_s^t |R(u,s)|^k du$$

is bounded where $k = 1, 2, ..., \infty$. The following result, found in [2; p. 161] yields both at the same time. Moreover, it is such a contrast to results in a later section. Here we obtain (2.2) by asking smallness conditions on the integral of $C(t,s)$, while we later (Theorem 6.2) obtain (2.2) by asking derivative conditions on $C(t,s)$.

**Theorem 5.4.** *Suppose there is an* $\xi > 0$ *with*

$$C(t,t) - \int_0^\infty |C_t(v+t,t)|dv \geq \xi.$$

*Then*

$$|R(t,s)| + \xi\int_s^t |R(u,s)|du \leq |C(s,s)| + \int_s^t |C_t(u,s)|du$$

*so that if, in addition, there is an* $M > 0$ *with*

$$|C(s,s)| + \int_s^t |C_t(u,s)|du \leq M$$

*then*

$$|R(t,s)| + \xi\int_s^t |R(u,s)|du \leq M.$$

*Proof.* If we differentiate the first choice in (0.4) we have

$$R_t(t,s) = C_t(t,s) - C(t,t)R(t,s) - \int_s^t C_t(t,u)R(u,s)du.$$

Keep $s$ fixed and define the Liapunov functional

$$V(t) = |R(t,s)| + \int_s^t \int_{t-u}^\infty |C_t(v+u,u)|dv|R(u,s)|du$$

so that the derivative of $V$ along a solution satisfies

$$\begin{aligned} V'(t) \leq\ & |C_t(u,s)| - C(t,t)|R(t,s)| + \int_s^t |C_t(t,u)R(u,s)|du \\ & + \int_0^\infty |C_t(v+t,t)|dv|R(t,s)| - \int_s^t |C_t(t,u)||R(u,s)|du \\ & \leq |C_t(t,s)| - \xi|R(t,s)|. \end{aligned}$$

so that

$$|R(t,s)| \le V(t) \le V(s) + \int_s^t |C(u,s)|du - \xi \int_s^t |R(u,s)|du$$

or

$$\begin{aligned}|R(t,s)| + \xi \int_s^t |R(u,s)|du &\le |R(s,s)| + \int_s^t |C(u,s)|du \\ &= |C(s,s)| + \int_s^t |C(u,s)|du.\end{aligned}$$

□

Follow the proof of Theorem 2.4 to verify the following result.

**Theorem 5.5.** *Let (5.1) and (5.2) hold and suppose that*

$$|G(t,x)| \le \phi(t)|x| \tag{5.4}$$

*where $\phi$ is continuous and*

$$\frac{\phi(t)\mu}{1-\beta^*} \le \gamma < 1. \tag{5.5}$$

*If, in addition, $a \in L^1[0,\infty)$, so are $x$ and $y$, solutions of (0.1) and (0.2).*

**Theorem 5.6.** *Let (5.1) and (5.2) hold, let $a \in L^1[0,\infty)$, and let (2.7) hold. If $x$ is any bounded solution of (0.1), then $x \in L^1[0,\infty)$.*

*Proof.* By Theorem 5.2, $y \in L^1[0,\infty)$, so $\int_0^t |y(v)|dv \le M$ and $|x(t)| \le M$ for some $M$ and all $t \ge 0$.

Now, from (0.3)

$$x(t) = y(t) - \int_0^t R(t,s)G(s,x(s))ds, \ 0 \le s \le t < \infty,$$

so

$$\begin{aligned}|x(t)| &\le |y(t)| + \int_0^t |R(t,s)||G(s,x(s))|ds \\ &\le |y(t)| + \int_0^t |R(t,s)|\phi(s)|x(s)|ds\end{aligned}$$

and

$$\begin{aligned}\int_0^t |x(v)|dv &\le \int_0^t |y(v)|dv + \int_0^t \int_0^v |R(v,s)|\phi(s)|x(s)|dsdv \\ &\le M + M\int_0^t \int_0^v |R(v,s)|\phi(s)dsdv \\ &\le M + M\int_0^t \left(\int_s^t |R(v,s)|dv\right)\phi(s)ds.\end{aligned}$$

Thus, from (5.3) we have

$$\int_0^t |x(v)|dv \le M + M\frac{\mu}{1-\beta^*}\int_0^t \phi(s)ds$$

and the result follows since $\phi \in L^1[0,\infty)$ in (2.7). $\square$

**Theorem 5.7.** *Let (4.1) hold and suppose $d \in \mathcal{BC}$ whenever $a \in \mathcal{BC}$ where $d$ is defined by (3.9); hence, from Theorem 4.1, (1.2) holds for some $\alpha < 1$. Further, suppose (1.9) holds for this $\alpha$ and $J = 0$, and let (2.7) hold. If $a \in \mathcal{BC}$ and $a \in L^1[0,\infty)$, then so is any solution $x$ of (0.1).*

*Proof.* From (3.12)

$$-\int_0^t R(t,s)a(s)ds = d(t) - \int_0^t Q(t,s)d(s)ds.$$

It follows from Theorem 4.1 that

$$\begin{aligned}\left|\int_0^t R(t,s)a(s)ds\right| &\le |d(t)| + \int_0^t |Q(t,s)||d(s)|ds\\ &\le M + M\int_0^t |Q(t,s)|ds\\ &\le M + M\frac{\alpha^*}{1-\alpha^*}\end{aligned}$$

for some $M$ and all $t \ge 0$. But $y(t) = a(t) - \int_0^t R(t,s)a(s)ds$, so $y \in \mathcal{BC}$, say $|y(t)| \le K$ for $t \ge 0$. Now,

$$x(t) = y(t) - \int_0^t R(t,s)G(s,x(s))ds$$

from which we obtain

$$\begin{aligned}|x(t)| &\le |y(t)| + \int_0^t |R(t,s)|\phi(s)|x(s)|ds\\ &\le K + \|x\|^{[0,t]}\int_0^t |R(t,s)|\phi(s)ds\\ &\le K + \|x\|^{[0,t]}\gamma\frac{1-\alpha}{\alpha}\int_0^t |R(t,s)|ds\\ &\le K + \gamma\|x\|^{[0,t]}.\end{aligned}$$

If $x$ is unbounded, then there exists $\{t_n\} \uparrow \infty$ such that $|x(t_n)| \to \infty$ and $|x(t_n)| = \|x\|^{[0,t_n]}$. Thus, for any such $t_n$ we have

$$|x(t_n)| \le K + \gamma\|x\|^{[0,t_n]} = K + \gamma|x(t_n)|$$

or

$$|x(t_n)| \le \frac{K}{1-\gamma},$$

a contradiction. It follows that $x \in \mathcal{BC}$ and, from theorem 5.6, $x \in L^1[0,\infty)$, thereby completing the proof. $\square$

## 6. Convex kernels; the $t$-coordinate

There is a variety of problems going back to Volterra in 1928 which include problems in biology, nuclear reactors, viscoelasticity, and neural networks. They involve a kernel $C(t,s)$ patterned after $e^{-(t-s)}$ which satisfies

$$C(t,s) > 0,\ C_s(t,s) \geq 0,\ C_t(t,s) \leq 0,\ C_{st}(t,s) \leq 0. \tag{6.1}$$

History and results are found throughout [2].

When those conditions hold then there is a perfect Liapunov functional for

$$R(t,s) = C(t,s) - \int_s^t C(t,u)R(u,s)du \tag{6.2}$$

as is shown in [2; p. 130], together with many other interesting properties. We define

(6.3)
$$V(t) = \int_s^t C_v(t,v)\left(\int_v^t R(u,s)du\right)^2 dv + C(t,s)\left(\int_s^t R(u,s)du\right)^2$$

and find that

$$V'(t) \leq C^2(t,s) - R^2(t,s),\ 0 \leq s \leq t \tag{6.4}$$

so that

$$\int_s^t R^2(u,s)du \leq \int_s^t C^2(u,s)du. \tag{6.5}$$

This has three serious defects. First, to use (6.5) we need to ask that there is an $M > 0$ with

$$\int_s^t C^2(u,s)du \leq M \text{ for } 0 \leq s \leq t < \infty. \tag{6.6}$$

But our very "strongest" kernels are those for which there is a $\lambda > 0$ with

$$C(t,s) \geq \lambda \text{ for } 0 \leq s \leq t < \infty. \tag{6.7}$$

Next, even if (6.6) holds, to use $R$ effectively in many perturbation problems we need an $H > 0$ with

$$\int_s^t |R(u,s)|du \leq H \text{ for } 0 \leq s \leq t < \infty, \tag{6.8}$$

instead of $R^2$ as in (6.5). Finally, the motivating problems all concern physical situations in which it is virtually impossible to determine anything so "fine" as $C_{st} \leq 0$, while it is entirely plausible that we can determine $C(t,s) > \lambda$ and $C_t(t,s) \leq 0$ is also plausible in that we will have nonlinear perturbation results of the type studied throughout this paper.

Consider again (3.13) and the conditions of Theorem 5.1. We need $\beta^* < 1$ and $\mu > 0$ with

$$\int_s^t |L(u,s)|du \le \beta^* \tag{5.1}$$

and

$$\int_s^t |B(u,s)|du \le \mu. \tag{5.2}$$

**Theorem 6.1.** *If $C_t(t,s) \le 0$, if $C(s,s)$ is bounded, and if (6.7) holds so does (5.2).*

*Proof.* There is a $\gamma > 0$ with $C(s,s)/C(u,u) \le \gamma$. If $C_t(t,s) \le 0$ and $C(t,s) \ge 0$ then to have (5.2) we first note that

$$\begin{aligned}\int_s^t \frac{C(s,s)}{C(u,u)} C(u,u) e^{-\int_s^u C(q,q)dq} du &\le -\gamma e^{-\int_s^u C(q,q)dq}\Big|_s^t \\ &= \gamma\left(1 - e^{-\int_s^t C(q,q)dq}\right) \le \gamma.\end{aligned}$$

Next, if (6.7) holds and $C_t(t,s) \le 0$ then

$$\begin{aligned}&\left|\int_s^t \int_s^u e^{-\int_v^t C(q,q)dq} C_t(v,s) dv du\right| \\ &= -\int_s^t \int_v^t e^{-\int_v^u C(q,q)dq} du C_t(v,s) dv \\ &\le -\int_s^t \int_v^t e^{-\lambda(u-v)} du C_t(v,s) dv \\ &= \frac{1}{\lambda}\int_s^t e^{-\lambda(u-v)}\Big|_v^t C_t(v,s) dv \\ &= \frac{1}{\lambda}\int_s^t \left[e^{-\lambda(t-v)} - 1\right] C_t(v,s) dv \\ &\le \frac{-1}{\lambda}\int_s^t C_t(v,s) dv \\ &= \frac{-1}{\lambda} C(v,s)\Big|_s^t = \frac{1}{\lambda}[C(s,s) - C(t,s)]\end{aligned}$$

and this is bounded so (5.2) does hold. □

**Theorem 6.2.** *Suppose that $C_t(t,s) \le 0$, that $C(s,s)$ is bounded, and that (6.7) holds. If, in addition, there is a $\beta^* < 1$ with*

$$\frac{1}{\lambda}(C(s,s) - C(t,s)) \le \beta^* \tag{6.9}$$

*then*

$$\int_s^t |R(u,s)|du \leq \frac{\mu}{1-\beta^*} \tag{6.10}$$

*where $\mu$ is defined in (5.2) and $\lambda$ is defined in (6.7).*

*Proof.* We have

$$\begin{aligned}
\int_s^t |L(w,s)|dw &= \int_s^t \int_s^w e^{-\int_u^w C(q,q)dq} |C_t(u,s)|dudw \\
&\leq -\int_s^t \int_s^w e^{-\lambda(w-u)} C_t(u,s)dudw \\
&= -\int_s^t \int_u^t e^{-\lambda(w-u)} dw C_t(u,s)du \\
&= \frac{1}{\lambda} \int_s^t e^{-\lambda(w-u)}\Big|_u^t C_t(u,s)du \\
&= \frac{-1}{\lambda} \int_s^t \left[1 - e^{-\lambda(t-u)}\right] C_t(u,s)du \\
&\leq \frac{-1}{\lambda} \int_s^t C_t(u,s)du = \frac{-1}{\lambda} C(u,s)\Big|_s^t \\
&= \frac{C(s,s)-C(t,s)}{\lambda} \leq \beta^*,
\end{aligned}$$

and the conclusion follows from (5.3). □

We remarked after Theorem 5.3 that the same kind of conclusion follows either from integral conditions or derivative conditions and this is the result mentioned there.

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