ON THE REDUCTION OF KRASNOSELSKII'S THEOREM TO SCHAUDER'S THEOREM

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Abstract. Krasnoselskii noted that many problems in analysis can be formulated as a mapping which is the sum of a contraction and compact map. He proved a theorem covering such cases which is the union of the contraction mapping principle and Schauder's second fixed point theorem. In putting the two results together he found it necessary to add a condition which has been difficult to fulfill, although a great many problems have been solved using his result and there have been many generalizations and simplifications of his result. In this paper we point out that when the mapping is defined by an integral plus a contraction term, the integral can generate an equicontinuous map which is independent of the smoothness of the functions. Because of that, it is possible to set up that mapping, not as a sum of contraction and compact map, but as a continuous map on a compact convex subset of a normed space. An application of Schauder's first fixed point theorem will then yield a fixed point without any reference to that difficult condition of Krasnoselskii. Finite and infinite intervals are handled separately. For the class of problems considered, application is parallel to the much simpler Brouwer fixed point theorem.

1. Introduction

Throughout applied mathematics we see real world problems modeled by various differential equations which are often inverted as integral equations defining natural mappings of certain sets in a Banach space into themselves. In order to get a fixed point solving the differential equation we are frequently faced with severe compactness problems, particularly when we need a solution on an entire interval $[0, \infty)$. Indeed, there is a myriad of real world problems modeled by fractional differential equations which naturally invert as integral equations with singular kernels. These integral equations define a natural mapping which invites either Schauder's or Krasnoselskii's fixed point theorem. Frequently the mapping is continuous and we can locate a convex set mapped into itself. Our task is only beginning as we consider compactness questions and the mixing of contraction and compact maps.

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Here we arrive at the objective of this project. We show that if the kernel and its coefficient function satisfy reasonable conditions, then there is a natural equicontinuity condition on the part of the mapping generated by the integral. We then restrict our mapping to a convex set in the Banach space for which that equicontinuity holds even for the contraction part of the mapping. This allows us to use Schauder's first fixed point theorem to get a fixed point.

Here is the advantage. In Krasnoselskii's theorem there is a complicated condition (item (i) in the theorem below) which ties the contraction mapping to the compact mapping. By the above process we avoid that complication and get the fixed point directly from Schauder's theorem. We now look at the details.

Krasnoselskii studied an old paper of Schauder on elliptic partial differential equations and deduced a working principle which we formalize as follows: The inversion of a perturbed differential operator yields the sum of a contraction and compact map. Accordingly, he offered the following result to facilitate treatment of that sum.

Theorem 1.1 (Krasnoselskii)**.** *Let $(\mathcal{S}, \|\cdot\|)$ be a Banach space, M a closed, convex, nonempty subset of $\mathcal{S}$. Suppose that A, $B : M \to \mathcal{S}$ such that*

(i) $$x, y \in M \Rightarrow Ax + By \in M,$$

A is continuous and

(ii) $$AM \text{ resides in a compact set,}$$

(iii) $$B \text{ is a contraction}$$

with constant $\alpha < 1$.
Then $\exists\ y \in M$ with $Ay + By = y$.

It is clear from (ii) and (iii) that it is intended to be a combination of Schauder's second fixed point theorem and the contraction mapping principle. As such it would seem to be exactly what is needed in so many problems in differential and integral equations. But the marriage of the two principles takes place in (i) and that has been very challenging in so many standard problems from applied mathematics. We addressed item (i) in [3]. A very nice summary of selected results on Krasnoselskii's theorem up through 2007 is found in [16]. Other recent results are found in [1], [10], [11], and [5]. It is very convenient to find Krasnoselskii's result and proof in [17], as well as two forms of Schauder's fixed point theorem used here.

To focus on the need for such a result we consider a neutral functional differential equation

Example. Consider the scalar equation

$$x'(t) = -x(t) + \alpha \frac{d}{dt} x(t-h) + u(t, x(t))$$

with a continuous initial function $\psi : [-h, 0] \to \Re$ in which, for simplicity in this presentation, we ask that $\psi(0) = \alpha\psi(-h)$. By grouping terms and integrating we obtain

$$x(t) = \alpha x(t-h) + \int_0^t e^{-(t-s)}[-\alpha x(s-h) + u(s, x(s))]ds.$$

A full treatment using Krasnoselskii's fixed point theorem is found in [2, pp.180-184].

The first term, $\alpha x(t-h)$, does not smooth but the integral smooths in a most remarkable way. When x is restricted to any given bounded set in BC with a bound of a fixed number K, then the integral maps that set into an equicontinuous set where the equicontinuity is completely independent of the behavior of x. This allows us to place an equicontinuity condition on the mapping set so that the integral equation maps that set into itself. The fact that the contraction term does not smooth causes no trouble at all. We then apply Schauder's first fixed point theorem and obtain a bounded and continuous solution on any interval $[0, T]$. The victory is that in applying the result condition (i) of Krasnoselskii's theorem is completely avoided. If the mapping set is essentially a ball then the work holds for $0 \leq t < \infty$ in a weighted space.

An integral equation with a mild singularity has a natural induced equicontinuity which can be of prime importance in fixed point theory. We will consider two essentially different forms:

$$x(t) = V(t, x(t)) + \int_0^t R(t-s)u(t, s, x(s))ds \tag{1a}$$

and

$$x(t) = f(t, x(\cdot)) + \int_0^t R(t-s)u(t, s, x(\cdot))ds \tag{1b}$$

where V and f are of a nature to generate a contraction while R will generate a compact map. For example, we may find a closed convex nonempty set M in the Banach space $(BC, \|\cdot\|)$ of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the supremum norm with the property that $\phi \in M$ and for P defined by either

$$(P\phi)(t) = V(t, \phi(t)) + \int_0^t R(t-s)u(t, s, \phi(s))ds \tag{2a}$$

or

$$(P\phi)(t) = f(t, \phi(\cdot)) + \int_0^t R(t-s)u(t, s, \phi(\cdot))ds \tag{2b}$$

with a given initial function $\psi : [-h, 0] \to \Re$, we have $P : M \to M$.

Immediately we think of Krasnoselskii's fixed point theorem with all its benefits, together with real challenges. But there is a more direct way if R has mild singularities of the following form. Assume that for the fixed set M there is a positive constant, q, with $0 < q < 1$ so that for $\phi \in M$ then

$$0 \le R(t-s) \le (t-s)^{q-1}. \tag{3}$$

There are many sources for such problems. We can then show (see [6] and Theorem 6.1) that, independently of the particular $\phi \in M$, the integral

$$(L\phi)(t) := \int_0^t R(t-s)u(t,s,\phi(s))ds \tag{4}$$

satisfies

$$|L\phi(t) - L\phi(s)| \le H(t)|t-s|^q \tag{5}$$

where $H(t)$ is an increasing function. That is, LM is an equicontinuous set. In fact, let $\bar{t} \ge 0$ be fixed. For a given $\epsilon > 0$, we find $\delta > 0$ corresponding to the ϵ and $\bar{t}$ in the definition of equicontinuity by $H(\bar{t})\delta^q < \epsilon$ or

$$\delta < (\epsilon/H(\bar{t}))^{1/q}. \tag{6}$$

Next, suppose there is a $\beta < 1$ so that $x, y \in \Re$ implies that

$$|V(t,x) - V(t,y)| \le \beta|x-y|. \tag{7a}$$

Or, suppose there is an $\alpha < 1$ and $h > 0$ so that for $t \ge h$,

$$|f(t,\phi(\cdot)) - f(t,\psi(\cdot))| \le \alpha|\phi_t - \psi_t|^{[-h,0]} \tag{7b}$$

for $\phi, \psi \in M$, where $|\phi_t - \psi_t|^{[-h,0]} = \sup_{-h\le\theta\le 0} |\phi(t+\theta) - \psi(t+\theta)|$.

The process is now clear. We ask that we add to M the property that all functions satisfy the equicontinuity condition (6) in a certain way so that $\phi \in M$ implies $P\phi \in M$.

2. The integral equation without a delay

We are going to continue to use the kernel in the stated form so that the reader can see clearly the exactness of the equicontinuity on which the entire process depends. However, with care one can do the same for a more general equation. For example, Garcia-Falset [10, p. 1746, Item 4] considers the integral equation

$$x(t) = g(t,x) + \int_0^t F(t-s,s,u(s))ds$$

and obtains a bounded mapping set M in which there is a contraction condition on g and a relation

$$\|F(t,s,x) - F(h,s,x)\| \le SG(|t-h|)$$

where G is a continuous function with $G(0) = 0$ and S a constant depending on a bound on the functions in M. The reader is then left to carry out computations parallel to those which we provide below as a template for such work.

Let $T > 0$ and $(BC, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0,T] \to \Re$ with the supremum norm. We consider an integral equation of the form (1a)

$$x(t) = V(t, x(t)) + \int_0^t R(t-s)u(t,s,x(s))ds.$$

The following assumptions will be used.

(i) $R : (0,\infty) \to [0,\infty)$ is continuous, decreasing, and $R(t-s) \leq (t-s)^{q-1}, \quad 0 < q < 1$.

(ii) $u : [0,T] \times [0,T] \times \Re \to \Re$ is continuous.

(iii) There is a closed, bounded, convex, nonempty set $M \subset BC$ with the following properties:

(a) There are positive J, S such that $\phi \in M$ and $0 \leq s \leq t \leq T$ implies that $|u(t,s,\phi(s))| \leq S$ and

$$|u(t,\tau,\phi(\tau)) - u(s,\tau,\phi(\tau))| \leq J|t-s|^q.$$

(b) $V : [0,T] \times \Re \to \Re$ is continuous and there is a positive β with $\beta < 1$ such that $\phi \in M$ and $t, s \in [0,T]$ implies that

$$|V(t,\phi(t)) - V(t,\phi(s))| \leq \beta|\phi(t) - \phi(s)|.$$

We will proceed with a view of constructing a nonempty compact convex subset M^* of M which is mapped into itself by P defined in (2a) in such a way that Schauder's first fixed point theorem can be applied to the restriction mapping $P : M^* \to M^*$. The resulting fixed point is then trivially a fixed point of the original mapping $P : M \to M$.

Theorem 2.1. *Let (i) - (iii) hold. Suppose that the mapping P defined by $\phi \in M$ implies that*

$$(P\phi)(t) = V(t,\phi(t)) + \int_0^t R(t-s)u(t,s,\phi(s))ds, \;\; 0 \leq t \leq T$$

maps $M \to M$. Then P has a fixed point.

Proof. Since M is nonempty, we choose a fixed $\phi^* \in M$ and define

$$M^* = \{\phi \in M : |\phi(t) - \phi(s)| \leq \omega(t,s), \; \forall\, t,s \in [0,T]\}$$

where

$$\omega(t,s) = \frac{1}{1-\beta}\,[\, |\phi^*(t) - \phi^*(s)| + H(t)|t-s|^q + V^*(t,s)]$$

with $V^*(t,s) = \sup_{|x|\leq D} |V(t,x) - V(s,x)|$, $H(t) = J\int_0^t R(s)ds + (2S/q)$ and D the bound of M. We now see that M^* is nonempty since $\phi^* \in M^*$. We shall show that M^* is a compact convex subset of M.

First, we note that M is convex so if $\phi, \eta \in M$ and $0 \leq k \leq 1$ then $k\phi + (1-k)\eta \in M$. Thus, if $\phi, \eta \in M^*$ then

$$|k[\phi(t_1) - \phi(t_2)] + (1-k)[\eta(t_1) - \eta(t_2)]| \leq \omega(t_1, t_2)$$

so M^* is also convex. In fact, M^* is a compact subset of M. To see this, let $\{\phi_n\}$ denote a sequence in M^*. As the sequence is uniformly bounded and equicontinuous on $[0,T]$, by the Ascoli-Arzela theorem there is a subsequence $\{\phi_{n_k}\}$ with a limit ϕ residing in the closed set M. If $t_1, t_2 \in [0,T]$ then

$$|\phi(t_1)-\phi(t_2)| \le |\phi(t_1)-\phi_{n_k}(t_1)|+|\phi_{n_k}(t_1)-\phi_{n_k}(t_2)|+|\phi_{n_k}(t_2)-\phi(t_2)|.$$

The first and last terms on the right-hand-side tend to zero as $k\to\infty$, while the middle term is less than $\omega(t_1,t_2)$ so that for large k the left-hand term is less than $\omega(t_1,t_2)$. Thus, $\phi\in M^*$, and so M^* is a compact subset of M.

Next, we show that $\phi\in M^*$ implies that $P\phi\in M^*$. Certainly, $P\phi\in M$. By Theorem 6.1, we see (5) holds on $[0,T]$. Now, for $t,s\in[0,T]$, by (iii) and Theorem 6.1 we obtain

$$\begin{aligned}
&|(P\phi)(t)-(P\phi)(s)| \\
&\le |V(t,\phi(t))-V(s,\phi(s))|+|L\phi(t)-L\phi(s)| \\
&\le |V(t,\phi(t))-V(t,\phi(s))|+|V(t,\phi(s))-V(s,\phi(s))| \\
&\quad +|L\phi(t)-L\phi(s)| \\
&\le \beta|\phi(t)-\phi(s)|+V^*(t,s)+H(t)|t-s|^q \\
&\le \beta\omega(t,s)+(1-\beta)\omega(t,s)=\omega(t,s)
\end{aligned}$$

so $P: M^*\to M^*$.

It is rather routine to show that P is continuous. Let $\epsilon>0$ be given. First, for $\epsilon/2$, find $\mu>0$ so that

$$\mu\int_0^T R(s)ds<\epsilon/2.$$

Now M is bounded by a number D and $u(t,s,x)$ is uniformly continuous for $0\le s\le t\le T$ and $|x|\le D$ so for the $\mu>0$ there is the $\delta_1>0$ of uniform continuity so that $\|\phi-\eta\|<\delta_1$ implies that $|u(t,s,\phi(s))-u(t,s,\eta(s)|<\mu$. Also, since $V(t,x)$ is uniformly continuous for $0\le t\le T$ and $|x|\le D$, there exists $\delta_2>0$ such that $\|\phi-\eta\|<\delta_2$ implies that

$$|V(t,\phi(t))-V(t,\eta(t))|<\epsilon/2 \ \text{ for all } \ t\in[0,T].$$

Take $\delta=\min[\delta_1,\delta_2]$. Then $\|\phi-\eta\|<\delta$ implies that

$$\|(P\phi)-(P\eta)\|\le\|\widetilde{V}(\phi)-\widetilde{V}(\eta)\|+\mu\int_0^T R(s)ds\le(\epsilon/2)+(\epsilon/2),$$

where $\widetilde{V}(\phi)(t)=V(t,\phi(t))$, as required.

Thus, P is a continuous map of a compact convex nonempty set M^* into itself so, by Schauder's first fixed point theorem, P has a fixed point in $M^*\subset M$. □

2.1. **The case for $T = \infty$.** For the case of $0 \le t \le T$ just covered we allowed M to be any closed bounded convex nonempty set in BC. When we pass to $[0, \infty)$ there is a large change since M^* may no longer be a compact subset of M in BC even if it satisfies the equicontinuity condition. In this case M must be essentially a ball in order for M to be closed and M^* compact in a weighted space being considered here.

Theorem 2.2. *Let (i) - (iii) hold with $T = \infty$. Suppose there are continuous functions $v, w : [0, \infty) \to \Re$ with $v(t) < w(t)$ for $t \ge 0$. Let*

$$M = \{\phi \in BC \mid v(t) \le \phi(t) \le w(t)\}.$$

If the mapping P of Theorem 2.1 maps M into M, then P has a fixed point.

Proof. Let $g : [0, \infty) \to [1, \infty)$ be continuous with $g \in\uparrow \infty$. Then $(W, |\cdot|_g)$ is the Banach space of continuous functions $\phi : [0, \infty) \to \Re$ for which

$$|\phi|_g =: \sup_{0 \le t < \infty} \frac{|\phi(t)|}{g(t)} < \infty.$$

Since g is an arbitrary continuous strictly increasing function with $g(t) \to \infty$ as $t \to \infty$, we may choose g so that

$$\lim_{t \to \infty} \frac{v(t)}{g(t)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{w(t)}{g(t)} = 0. \tag{8}$$

We see that v and w may be unbounded, but the functions in M restricted to any compact interval $[0, T]$ are uniformly bounded.

We define M^* as in Theorem 2.1 with

$$\omega(t, s) = \frac{1}{1-\beta} [\, |\phi^*(t) - \phi^*(s)| + H(t)|t - s|^q + V^*(t, s)]$$

for $t, s \ge 0$.

Note that M^* is a closed convex nonempty subset of $(W, |\cdot|_g)$ and the work in the proof of Theorem 2.1 shows that $P : M^* \to M^*$. Let $\{\phi_n\}$ be a sequence in M^* and use Ascoli's theorem and the diagonalization process to show that there is a subsequence $\{\phi_{n_k}\}$ converging to some $\phi \in BC$ uniformly on compact subsets of $[0, \infty)$ and $\phi \in M$. Moreover, $|\phi_{n_k} - \phi|_g \to 0$ as $k \to \infty$ since (8) holds.

The proof of Theorem 2.1 shows that ϕ satisfies the equicontinuity property so $\phi \in M^*$. Therefore M^* is a compact subset of $(W, |\cdot|_g)$. An argument similar to that in [5] shows that P is continuous in the g-norm on M. Applying Schauder's first fixed point theorem to $P : M^* \to M^*$ in $(W, |\cdot|_g)$, we obtain that there exists a point $\phi \in M^*$ with $P\phi = \phi$. This completes the proof. □

Remark 1. If $\beta = 1$ in (iii)-(b), then the argument in the proof of theorems above fails. However, we may still be able to establish the existence of a fixed point for P under additional assumptions on V. The process goes as follows. We first prove the existence of an ε-fixed point of P; that is, for each $\varepsilon > 0$, there exists $x_\varepsilon \in M$ such that

$$\|Px_\varepsilon - x_\varepsilon\| < \varepsilon.$$

Next, we apply the approximation method to obtain a fixed point of P. This will be demonstrated in Theorem 2.3. Let I denote the identity map and $(I-P)(M)$ denote the range of $I-P$ on M.

Theorem 2.3. *Let (i) - (iii) hold with $\beta = 1$ and $T = \infty$, and let M be defined in Theorem 2.2. Suppose that*

(iv) *the set $(I-P)(M)$ is closed in BC.*

If the mapping P of Theorem 2.1 maps M into M, then P has a fixed point.

Proof. Since M is nonempty, we choose a fixed $\tilde{\phi} \in M$ and define, for any positive integer n, a mapping P_n by $\phi \in M$ implies

$$\begin{aligned}(P_n\phi)(t) &= \big(1 - \frac{1}{n}\big)(P\phi)(t) + \frac{1}{n}\tilde{\phi}(t)\\ &= \Big[\big(1-\frac{1}{n}\big)V(t,\phi(t)) + \frac{1}{n}\tilde{\phi}(t)\Big] + \big(1-\frac{1}{n}\big)\int_0^t R(t-s)u(t,s,\phi(s))ds.\end{aligned}$$

Since $P\phi \in M$ and M is convex, we have $P_n\phi \in M$ and thus, $P_n(M) \subset M$. Letting $V_n(t,x) = (1-1/n)V(t,x) + (1/n)\tilde{\phi}(t)$ for $n = 2, 3, \cdots$, we see for $\phi \in M$ that

$$\begin{aligned}|V_n(t,\phi(t)) - V_n(t,\phi(s))| &= (1 - \frac{1}{n})|V(t,\phi(t)) - V(t,\phi(s))|\\ &\le (1 - \frac{1}{n})|\phi(t) - \phi(s)|\\ &=: \beta|\phi(t) - \phi(s)|\end{aligned}$$

for $t, s \ge 0$. Thus, conditions (i)-(iii) are satisfied with $V(t,x)$, $u(t,s,x)$ replaced by $V_n(t,x)$, $(1-1/n)u(t,s,x)$, respectively. By Theorem 2.2, there is a point $\phi_n \in M$ such that $P_n\phi_n = \phi_n$; that is,

$$\phi_n(t) = V_n(t,\phi_n(t)) + \big(1 - \frac{1}{n}\big)\int_0^t R(t-s)u(t,s,\phi_n(s))ds.$$

We also see from

$$(P\phi_n)(t) - \phi_n(t) = \frac{1}{n}\big[V(t,\phi_n(t)) + \int_0^t R(t-s)u(t,s,\phi_n(s))ds\big] - \frac{1}{n}\tilde{\phi}(t)$$

that P has an ε-fixed point in M for each $\varepsilon > 0$ since

$$\|P\phi_n - \phi_n\| \le \frac{\mu}{n}$$

where $\mu = \sup\{\|P\varphi\| : \varphi \in M\} + \|\tilde{\phi}\|$. Let $G = I - P$. Then

$$\|G(\phi_n)\| = \|P\phi_n - \phi_n\| \to 0 \quad \text{as} \;\; n \to \infty$$

and so $0 \in \overline{G(M)} = G(M)$ since $G(M)$ is closed in BC. Thus, $\exists \phi \in M$ such that $P\phi - \phi = 0$. This completes the proof. □

We now define a mapping $\widetilde{V} : M \to BC$ by $\widetilde{V}(\phi)(t) = V(t, \phi(t))$ for all $t \geq 0$ and $\phi \in M$.

Corollary 1. Let (i) - (iii) hold with $\beta = 1$ and $T = \infty$, and let

$$M = \{ \, \phi \in BC : \|\phi\| \leq K \, \}$$

for a constant $K > 0$. Suppose that

(v) $\quad \sup_{t \geq 0} |V(t, x) - V(t, y)| < |x - y|$

for all $x, y \in [-K, K]$ with $x \neq y$. If the mapping P of Theorem 2.1 maps M into M, then P has a fixed point.

Proof. We only need to show that (iv) holds. To this end, let $\psi_n \in (I - P)(M)$ with $\|\psi_n - \psi\| \to 0$ as $n \to \infty$ for some $\psi \in BC$. We shall show that $\psi \in (I - P)(M)$. Let $\phi_n \in M$ with $\psi_n = (I - P)\phi_n$ for $n = 1, 2, \cdots$. We write $(I - P)\phi_n(t) = \psi_n(t)$ as

$$\phi_n(t) - V(t, \phi_n(t)) = y_n(t) + \psi_n(t) \tag{9}$$

where

$$y_n(t) = \int_0^t R(t-s) u(t, s, \phi_n(s)) ds.$$

Since V is continuous on $[0, \infty) \times [-K, K]$ and $P(M) \subset M$, we see that the sequence $\{y_n\}$ is uniformly bounded and equicontinuous on any compact subset of $[0, \infty)$ by (iii)-(a) and Theorem 6.1. Thus, by the Ascoli-Arzela theorem, there is a subsequence $\{y_{n_k}\}$ converging to some $y \in BC$ uniformly on any closed bounded interval $[0, T]$. Since (v) holds, we have by Theorem 6.2 that $(I - \widetilde{V})^{-1}$ is continuous on $(I - \widetilde{V})(M)$. It now follows from (9) that

$$\phi_{n_k}(t) = (I - \widetilde{V})^{-1} \big[y_{n_k} + \psi_{n_k} \big](t). \tag{10}$$

This implies that $\{\phi_{n_k}\}$ converges to a function $\phi \in M$ uniformly on $[0, T]$. Since $(I - \widetilde{V})^{-1}$ is continuous on $(I - \widetilde{V})(M)$, by letting $k \to \infty$ in (10) we obtain

$$\phi(t) = (I - \widetilde{V})^{-1} [y + \psi](t) \;\; \text{for} \;\; t \in [0, T] \tag{11}$$

Taking the limit in

$$y_{n_k}(t) = \int_0^t R(t-s) u(t, s, \phi_{n_k}(s)) ds$$

we also obtain

$$y(t) = \int_0^t R(t-s)u(t,s,\phi(s))ds \tag{12}$$

Combining (11) and (12), we see that $\psi(t) = (I-P)\phi(t)$ for all $t \geq 0$. Thus, $\psi \in (I-P)(M)$, and the proof is complete. □

Example 1. Consider the fractional differential equation of Caputo type

$$^cD^q(x-\kappa(x)) = -a(t)x^3(t) + G(t,x(t)), \;\; x(0) = x_0, \;\; 0<q<1, \tag{13}$$

with $a : [0,\infty) \to \Re$, $\kappa : \Re \to \Re$, $G : [0,\infty) \times \Re \to \Re$ continuous. See [6] for background and definitions. Suppose that

($\tilde{\text{i}}$) $a(t)$ is bounded on $[0,\infty)$,

($\widetilde{\text{ii}}$) $|G(t,x)| \leq b(t)|x|^3$ for $|x| \leq 1$ and $t \geq 0$,

($\widetilde{\text{iii}}$) $a(t) - b(t) \geq \delta$ for all $t \geq 0$ and a constant $\delta > 0$.

($\widetilde{\text{iv}}$) $\exists\, \gamma > 0$ such that $\kappa : [-\gamma,\gamma] \to \Re$ is nondecreasing, odd with $x - \kappa(x) - x^3$ increasing on $[0,\gamma]$,

Then the zero solution of (13) is stable.

Proof. Choose a constant $\eta > 0$ with $\sup_{t\geq 0} a(t) < \eta$ and define

$$C(t) = \frac{\eta}{\Gamma(q)}t^{q-1}$$

Then the resolvent R satisfies

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds$$

This resolvent R is completely monotone on $(0,\infty)$. Moreover,

$$0 \leq R(t) \leq C(t), \; tR(t) \to 0 \text{ as } t \to \infty, \text{ and } \int_0^\infty R(s)ds = 1.$$

If we write (13) as

$$\begin{aligned} ^cD^q(x-\kappa(x)) &= -a(t)x^3(t) + G(t,x(t)) \\ &= -\eta[x-\kappa(x)] + \eta[x-\kappa(x)-x^3] \\ &\quad + [\eta - a(t)]x^3 + G(t,x(t)) \end{aligned}$$

then the solution $x(t)$ satisfies

$$\begin{aligned}
x(t) &= z(t) + \kappa(x) + \int_0^t R(t-s)[x(s) - \kappa(x) - x^3(s)]ds \\
&\quad + \int_0^t R(t-s)\left(1 - \frac{a(s)}{\eta}\right)x^3(s)ds + \int_0^t R(t-s)\frac{1}{\eta}G(s, x(s))ds \\
&=: \kappa(x) + z(t) + \int_0^t R(t-s)u(s, x(s))ds \\
&=: (Px)(t)
\end{aligned}$$

where $z(t) = (x_0 - \kappa(x_0))(1 - \int_0^t R(s)ds)$.

Let $0 < \varepsilon < \gamma$. We may assume that $\gamma < \sqrt{3}/3, \eta \geq 1$, and $0 < \delta < 1$ so that $\delta\varepsilon^3/\eta < \varepsilon$. Now let $|x_0| < \delta\varepsilon^3/\eta$ and define

$$M = \{\phi \in BC : \|\phi\| \leq \varepsilon\}.$$

For the mapping defined above, we can show that $P : M \to M$. To see this, we observe that $r - \kappa(r) - r^3$ is odd and increasing on $[0, \varepsilon]$ by $\widetilde{\text{(iv)}}$. Apply $\widetilde{\text{(ii)}}$ and $\widetilde{\text{(iii)}}$ to obtain

$$\begin{aligned}
|(Px)(t)| &\leq |z(t)| + \kappa(\varepsilon) + (\varepsilon - \kappa(\varepsilon) - \varepsilon^3) \\
&\quad + \varepsilon^3 \int_0^t R(t-s)\left(1 - \frac{a(s)}{\eta} + \frac{|b(s)|}{\eta}\right)ds \\
&\leq |x_0| + (\varepsilon - \varepsilon^3) + \varepsilon^3(1 - \delta/\eta) < \varepsilon
\end{aligned}$$

if $|x_0| < \delta\varepsilon^3/\eta$.

Since $x - \kappa(x) - x^3$ is increasing on $[0, \gamma]$, we see that $x - \kappa(x)$ is strictly increasing on $[-\gamma, \gamma]$ with

$$|\kappa(x) - \kappa(y)| < |x - y|$$

for all $x, y \in [-\gamma, \gamma]$ with $x \neq y$. We now readily verify that all conditions of Corollary 1 are satisfied with $V(t, x) = \kappa(x) + z(t)$, and so, P has a fixed point $x \in M$ which is the solution of (13). Since $\varepsilon > 0$ is arbitrary, this proves that the solution $x = 0$ of (13) is stable. $\square$

Critique

Notice that conditions (i)-(iii) are simply defining the sets and functions. Theorem 2.1 asks only that the investigator show that $P : M \to M$ and P is continuous. Krasnoselskii's theorem has been replaced by Schauder's theorem for certain Banach spaces.

3. The integral equation with a delay

Let $T > 0$ and $(BC, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0,T] \to \Re$ with the supremum norm. We consider an integral equation with a given continuous initial function $\psi : [-h, 0] \to \Re$ of the form

$$(14)\quad x(t) = f(x(t-h)) + \int_0^t R(t-s)u(s, x(s), x(s-h))ds + F(t),\ t \geq 0$$

with $x(t) = \psi(t)$ for $-h \leq t \leq 0$.

The following assumptions will be used.

(i) $R : (0,\infty) \to [0,\infty)$ is continuous, decreasing, and $R(t-s) \leq (t-s)^{q-1}$, $0 < q < 1$.

(ii) $u : [0,T] \times \Re^2 \to \Re$ and $F : [0,T] \to \Re$ are continuous.

(iii) There is a closed, bounded, convex, nonempty set $M \subset BC$ with the following properties:

(a) $\phi \in M$ implies that $\phi(0) = \psi(0)$.

(b) There is a positive S such that $\phi \in M$ and $0 \leq t \leq T$ implies that $|u(t, \phi(t), \phi(t-h))| \leq S$.

(c) $f : \Re \to \Re$ is continuous and there is an $\alpha < 1$ such that $\phi \in M$ and $t, s \in [0,T]$ implies

$$|f(\phi(t)) - f(\phi(s))| \leq \alpha|\phi(t) - \phi(s)|.$$

It is understood that $\phi(\tau) = \psi(\tau), \forall \tau \in [-h, 0]$, for all $\phi \in M$. We also extend the domain of any function $\eta : [a,b] \to \Re$ to $\Re$ by assigning $\eta(\tau) = \eta(a)$ for $\tau \leq a$ and $\eta(\tau) = \eta(b)$ for $\tau \geq b$. For a real-valued function $\varphi : \Re \to \Re$, we set

$$|\varphi_t - \varphi_s|^{[a,b]} = \sup_{a\leq\tau\leq b} |\varphi(t+\tau) - \varphi(s+\tau)|$$

for all $t, s \in \Re$. Finally, we point out that functions f, F in (14) may depend on ψ so that $\psi(0) = f(\psi(-h)) + F(0)$.

Theorem 3.1. *Let (i) - (iii) hold. Suppose that the mapping P defined by $\phi \in M$ implies that*

$$(P\phi)(t) = f(\phi(t-h)) + \int_0^t R(t-s)u(s, \phi(s), \phi(s-h))ds + F(t),\ 0 \leq t \leq T$$

maps $M \to M$. Then P has a fixed point.

Proof. Let $\phi^* \in M$ be fixed and define

$$M^* = \{\, \phi \in M : |\phi(t) - \phi(s)| \leq \theta(t,s),\ \forall\, t, s \in \Omega \,\}$$

where

$$\Omega = \{\, (t,s) :\ t, s \in [0,h] \ \text{ or } \ t, s \in [h,T] \,\}$$

and

$$\theta(t,s) = \frac{1}{1-\alpha}\left[\,|\phi_t^* - \phi_s^*|^{(-\infty,0\,]} + \frac{2S}{q}|t-s|^q + |F_t - F_s|^{(-\infty,0\,]}\right] + \sup_{\tau\le 0}|f(\psi(t+\tau)) - f(\psi(s+\tau))|\,.$$

for all $t,s \in [0,T]$.

It is clear that $\theta(t,s) \to 0$ as $|t-s| \to 0$. This implies that M^* is uniformly bounded and equicontinuous on $[0,T]$. Since $\phi^* \in M^*$, we see that M^* is a compact convex nonempty subset of M.

We now claim that $\theta(t-h, s-h) \le \theta(t,s)$ for all $t,s \in [h,T]$. In fact, we have

$$\begin{aligned}
&\theta(t-h,s-h)\\
&= \frac{1}{1-\alpha}\left[\,|\phi_{t-h}^* - \phi_{s-h}^*|^{(-\infty,0\,]} + \frac{2S}{q}|t-s|^q + |F_{t-h} - F_{s-h}|^{(-\infty,0\,]}\right]\\
&\qquad + \sup_{\tau\le 0}|f(\psi(t-h+\tau)) - f(\psi(s-h+\tau))|\\
&= \frac{1}{1-\alpha}\left[\,|\phi_t^* - \phi_s^*|^{(-\infty,-h\,]} + \frac{2S}{q}|t-s|^q + |F_t - F_s|^{(-\infty,-h\,]}\right]\\
&\qquad + \sup_{\sigma\le -h}|f(\psi(t+\sigma)) - f(\psi(s+\sigma))|\\
&\le \frac{1}{1-\alpha}\left[\,|\phi_t^* - \phi_s^*|^{(-\infty,0\,]} + \frac{2S}{q}|t-s|^q + |F_t - F_s|^{(\infty,0\,]}\right]\\
&\qquad + \sup_{\tau\le 0}|f(\psi(t+\tau)) - f(\psi(s+\tau))|\\
&= \theta(t,s).
\end{aligned}$$

Next, we show that $\phi \in M^*$ implies that $P\phi \in M^*$. Certainly, $P\phi \in M$. We may assume $h < T$ and still denote the integral term in (14) by $L\phi(t)$ so that (5) holds with $H(t) = 2S/q$. Now, for $t,s \in [h,T]$, we have

$$\begin{aligned}
&|(P\phi)(t) - (P\phi)(s)|\\
&\le |f(\phi(t-h)) - f(\phi(s-h))| + |L\phi(t) - L\phi(s)| + |F(t) - F(s)|\\
&\le \alpha|\phi(t-h) - \phi(s-h)| + \frac{2S}{q}|t-s|^q + |F(t) - F(s)|\\
&\le \alpha\theta(t-h,s-h) + \frac{2S}{q}|t-s|^q + |F_t - F_s|^{(-\infty,0\,]}\\
&\le \alpha\theta(t,s) + (1-\alpha)\theta(t,s) = \theta(t,s).
\end{aligned}$$

For $t, s \in [0, h]$, we observe that

$$|f(\phi(t-h)) - f(\phi(s-h))| = |f(\psi(t-h)) - f(\psi(s-h))|$$

and so

$$\begin{aligned}
&|(P\phi)(t) - (P\phi)(s)| \\
&\le |f(\psi(t-h)) - f(\psi(s-h))| + |L\phi(t) - L\phi(s)| + |F(t) - F(s)| \\
&\le |f(\psi(t-h)) - f(\psi(s-h))| + \frac{2S}{q}|t-s|^q + |F(t) - F(s)| \\
&\le \theta(t, s).
\end{aligned}$$

This implies that $P : M^* \to M^*$. The rest of the proof is exactly as in the proof of Theorem 2.1. □

The extension to the interval $[0, \infty)$ is exactly as before.

4. A fractional integral equation

There is an interesting and important paper [9] dealing with the scalar integral equation

$$x(t) = g(t, x(t)) + \frac{f(t, x(t))}{\Gamma(q)} \int_0^t \frac{v(t, s, x(s))}{(t-s)^{1-q}} ds, \quad 0 < q < 1, \tag{15}$$

which is studied by means of measures of noncompactness and a fixed point theorem of Darbo. It is important because it models a number of real-world problems with a view to showing that at least one solution exists and that any other solution converges to it at infinity. All of the conditions are introduced in Lipschitz form and from these are obtained rapid decay of the functions involved. It is interesting from our point of view because there are no sign conditions on any of the functions. Everything is controlled by the Lipschitz conditions and the decay of functions.

We wish to show that we can avoid the measures of noncompactness and one of the Lipschitz conditions (but not the implied decay), and use Theorem 2.2 to obtain some elementary solutions. Indeed, the point of our results is that they are elementary and, in fact, fairly close to Brouwer's theorem.

The idea is to state the conditions in [9], denoted by (h_i), and then state the consequences which we will use and denote these as (c_i).

(h_1) $g : \Re^+ \times \Re \to \Re$ is continuous, $g(t, 0)$ is bounded, and $g^* = \sup_{t \ge 0} |g(t, 0)|$. Also, there is a continuous function $\ell(t)$ with

$$|g(t, x) - g(t, y)| \le \ell(t)|x - y|$$

for $x, y \in \Re$ and $t \ge 0$.

(c_1) We will use (h_1), but will not ask for g^* (see (c_5)).

(h_2) $f : \Re^+ \times \Re \to \Re$ is continuous and there is a continuous function $m : \Re^+ \to \Re^+$ such that

$$|f(t,x) - f(t,y)| \le m(t)|x-y|$$

for $x, y \in \Re$ and $t \ge 0$.
(c_2) We will use (h_2), but will not ask for $m(t)n(t)t^q \to 0$ as $t \to \infty$.

(h_3) $v : \Re^+ \times \Re^+ \times \Re \to \Re$ is continuous. There exists a continuous functions $n : \Re^+ \to \Re^+$ and a continuous nondecreasing function $\Phi : \Re^+ \to \Re^+$ with $\Phi(0) = 0$ so that for all $t, s \in \Re^+$ with $t \ge s$ we have

$$|v(t,s,x) - v(t,s,y)| \le n(t)\Phi(|x-y|).$$

Also, there is a function $v^*(t) = \max\{|v(t,s,0)| : 0 \le s \le t\}$.
(c_3) We will use

$$|v(t,s,x)| \le v^*(t) + n(t)\Phi(|x|).$$

Now, everything is gathered.

(h_4) The functions $\phi, \psi, \xi, \eta : \Re^+ \to \Re^+$ defined by
$\phi(t) = m(t)n(t)t^q$
$\psi(t) = m(t)v^*(t)t^q$
$\xi(t) = n(t)|f(t,0)|t^q$
$\eta(t) = v^*(t)|f(t,0)|t^q$
are all bounded on $\Re^+$ and $\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \xi(t) = 0$.
(c_4) We will use the notation of (h_4), but will not ask boundedness or limit conditions of these functions.

(h_5) There exists a positive solution r_0 of the inequality

$$(\ell^* r + g^*)\Gamma(q+1) + [\phi^* r\Phi(r) + \psi^* r + \xi^*\Phi(r) + \eta^*] \le r\Gamma(q+1)$$

and $\ell^*\Gamma(q+1) + \phi^*\Phi(r_0) + \psi^* < \Gamma(q+1)$, where $\ell^* = \sup\{\ell(t) : t \ge 0\}$, $\phi^* = \sup\{\phi(t) : t \ge 0\}$, $\psi^* = \sup\{\psi(t) : t \ge 0\}$, $\xi^* = \sup\{\xi(t) : t \ge 0\}$ and $\eta^* = \sup\{\eta(t) : t \ge 0\}$.
(c_5) We will improve (h_5) by asking that there exists a continuous function $r : [0,\infty) \to (0,\infty)$ and a $\beta < 1$ such that for all $t \ge 0$

$$\ell(t) + [\psi(t) + \phi(t)\Phi(r^*(t))]\frac{1}{\Gamma(q+1)} \le \beta$$

and

$$\gamma(t) + \ell(t)r + [\psi(t)r + \phi(t)r\Phi(r^*) + \xi(t)\Phi(r^*)]\frac{1}{\Gamma(q+1)} \le r$$

where $r = r(t)$, $r^*(t) = \sup_{0\le s\le t} r(s)$, $\gamma(t) = |g(t,0)| + \eta(t)/\Gamma(q+1)$.

Under the assumptions (h_1) - (h_5) it is shown in [9, p. 77] that there is at least one solution of the integral equation; moreover, if there are other solutions, then they converge to the given solution. This is proved using the full conditions $(h_1) - (h_5)$ and methods of measures of noncompactness.

Our purpose here is to show that with our theorems there is an elementary proof using only $(c_1) - (c_5)$ to show that there is a solution x of (15) with $|x(t)| \le r(t)$ without the Lipschitz condition on v or the limit conditions in (h_4).

Theorem 4.1. *Let (c_1), (c_2), (c_3), (c_4) and (c_5) hold. Then the integral equation (15) has a solution x with $|x(t)| \le r(t)$ for all $t \ge 0$.*

Proof. As in [9, p. 79], the stated conditions are sufficient to show that if

$$M = \{\phi \in BC : |\phi(t)| \le r(t)\}$$

then the natural mapping defined by the integral equation maps M into itself. The proof of Theorem 2.1 and Theorem 2.2 will establish continuity of the map. The appendix will show the required equicontinuity and other technical details. □

5. A neutral delay equation

5.1. **Neutral equations.** A large body of literature can be found concerning applications of neutral differential equations by simply putting "epidemics and neutral differential equations" into a search engine. The basic and heuristic idea of a neutral equation is that the rate of change, $x'(t)$, is influenced not only by a "position" of x in space and time, together with forces acting on x as we would deduce from Newton's Second Law of Motion, but it is also influenced by the recent rate of change of x. Epidemics and other problems in mathematical biology are widely studied by neutral differential equations. See, for example, Gopalsamy [7], Gopalsamy and Zhang [8], Kuang [13], [14], [15]. Investigators have given heuristic arguments to support their use in describing biological phenomena and much of this is formalized in the final chapter of each of the books by Gopalsamy [7] and Kuang [13].

There are many other problems treated by neutral differential equations and our main contribution here is a quick sketch of the way to express a problem of the form

$$x'(t) = \frac{d}{dt}f(x(t-h)) - u(t, x(t), x(t-h)), \quad x(t) = \psi(t), \quad -h \le t \le 0,$$

as a problem readily attacked by the extension of Theorem 3.1 to $[0, \infty)$. Here, h is a positive constant, u is continuous and bounded for x bounded, and f satisfies a contraction condition. The function ψ is a given continuous initial function and we want a solution for $0 \le t < \infty$. If we were to simply integrate that equation to obtain an integral equation defining a mapping, then almost everything in our theory would fail. Instead we employ a form of a "linearization trick".

Let J be a positive constant to be determined, subtract and add $Jx(t)$ to obtain

$$x'(t) = -Jx(t) + Jx(t) + \frac{d}{dt}f(x(t-h)) - u(t, x(t), x(t-h)).$$

Take all the terms except $x'(t) = -Jx(t)$ as an inhomogeneous term and use the variation of parameters formula to write

$$x(t) = \psi(0)e^{-Jt} + \int_0^t e^{-J(t-s)}[Jx(s) + \frac{d}{ds}f(x(s-h)) - u(s, x(s), x(s-h))]ds$$

Integration by parts of the derivative term in the integral yields

$$\begin{aligned} x(t) &= \psi(0)e^{-Jt} + f(x(t-h)) - e^{-Jt}f(\psi(-h)) \\ &\quad + \int_0^t e^{-J(t-s)}[Jx(s) - Jf(x(s-h)) - u(s, x(s), x(s-h)]ds. \end{aligned}$$

The positive constant, J, can be chosen at will to facilitate the construction of a self-mapping set M of the type required for our theory. This equation defines a mapping which is well-suited to Theorem 3.1 and its extension to $[0, \infty)$. We do not need to satisfy the difficult condition (i) of Krasnoselskii's theorem. Details for a self-mapping set can be found in [4]. The details are quite lengthy and will not be repeated here.

6. Appendix

Theorem 6.1. *Let $u : [0, \infty) \times [0, \infty) \times \Re \to \Re$ be continuous, and let $R : (0, \infty) \to [0, \infty)$ be continuous, decreasing, and $R(t-s) \le D(t-s)^{q-1}$ with $0 < q < 1$ and $D > 0$. Then there is a continuous increasing function H so that if $t, s \ge 0$, if $x \in BC$ with $|u(t, s, x(s))| \le K$ and*

$$|u(t, \tau, x(\tau)) - u(s, \tau, x(\tau))| \le J|t-s|^q \tag{16}$$

then

$$\begin{aligned} L :&= \left| \int_0^t R(t-\tau)u(t, \tau, x(\tau))d\tau - \int_0^s R(s-\tau)u(s, \tau, x(\tau))d\tau \right| \\ &\le H(t)|t-s|^q \end{aligned}$$

where $H(t) = 2KD/q + J\int_0^t R(\tau)d\tau$.

Proof. Note that since $R(t)$ is decreasing and there is a constant D with $0 \leq R(t) \leq Dt^{q-1}$ we have, for $0 \leq s \leq t$, that

$$
\begin{aligned}
L &\leq \int_0^s |R(t-\tau) - R(s-\tau)||u(t,\tau,x(\tau))|d\tau \\
&\quad + \int_s^t |R(t-\tau)||u(t,\tau,x(\tau))|d\tau \\
&\quad + \int_0^s |R(s-\tau)||u(t,\tau,x(\tau)) - u(s,\tau,x(\tau))|ds \\
&\leq \int_0^s K[R(s-\tau) - R(t-\tau)]d\tau + K\int_s^t R(t-\tau)d\tau \\
&\quad + \int_0^s |R(s-\tau)|d\tau \; J|t-s|^q \\
&= K\int_0^s R(s-\tau)d\tau - K\int_0^s R(t-\tau)d\tau + K\int_s^t R(t-\tau)d\tau \\
&\quad + \int_0^s R(\tau)d\tau \; J|t-s|^q \\
&= K\int_0^s R(\tau)d\tau - K\int_{t-s}^t R(\tau)d\tau + K\int_s^t R(t-\tau)d\tau \\
&\quad + \int_0^s R(\tau)d\tau \; J|t-s|^q \\
&= K\int_0^s R(\tau)d\tau - K\int_0^t R(\tau)d\tau + 2K\int_s^t R(t-\tau)d\tau \\
&\quad + \int_0^s R(\tau)d\tau \; J|t-s|^q
\end{aligned}
$$

the sum of the first two terms is negative

$$
\begin{aligned}
&\leq 2DK\int_s^t (t-\tau)^{q-1}d\tau + \int_0^s R(\tau)d\tau \; J|t-s|^q \\
&= -2(KD/q)(t-\tau)^q\Big|_s^t + \int_0^s R(\tau)d\tau \; J|t-s|^q \\
&= [2KD/q + J\int_0^s R(\tau)d\tau]|t-s|^q \\
&\leq [2KD/q + J\int_0^t R(\tau)d\tau]|t-s|^q \\
&=: H(t)|t-s|^q.
\end{aligned}
$$

The same equality holds if $0 \le t \le s$. This completes the proof. $\square$

Theorem 6.2. *Let (i) - (iii) hold with $\beta = 1$ and $T = \infty$, and let*
$$M = \{\ \phi \in BC : \|\phi\| \le K\ \}$$
for a constant $K > 0$. Suppose that

(v) $\sup_{t\ge 0} |V(t,x) - V(t,y)| < |x-y|$

for all $x, y \in [-K, K]$ with $x \ne y$. Then $(I - \widetilde{V})^{-1}$ is continuous on $(I-\widetilde{V})(M)$, where $(\widetilde{V}\phi)(t) = V(t, \phi(t))$ for $\phi \in M$.

Proof. Since (v) holds, we see that $(I-\widetilde{V})$ is one to one, and hence the inverse $(I-\widetilde{V})^{-1}$ exists. We now show that $(I-\widetilde{V})^{-1}$ is continuous on $(I-\widetilde{V})(M)$. To this end, let $\{y_n\}$ be a sequence in $(I-\widetilde{V})(M)$ with $\|y_n - y^*\| \to 0$ as $n \to \infty$ for a function $y^* \in (I-\widetilde{V})(M)$. We need to show that
$$(I-\widetilde{V})^{-1} y_n \to (I-\widetilde{V})^{-1} y^* \ \text{ as } \ n \to \infty.$$
Set $x_n = (I-\widetilde{V})^{-1} y_n$ and $x^* = (I-\widetilde{V})^{-1} y^*$. Then $(I-\widetilde{V})x_n = y_n$ and $(I-\widetilde{V})x^* = y^*$. Suppose that $x_n \not\to x^*$. Then there exists an $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x^*\| \ge \varepsilon_0$ for all $k = 1, 2, \cdots$. Now choose $t_k \in [0, \infty)$ with
$$|x_{n_k}(t_k) - x^*(t_k)| \ge \varepsilon_0/2.$$
Observe that the function $\sup_{t\ge 0} |V(t,x) - V(t,y)|$ is continuous on the compact set $\Omega = [-K, K] \times [-K, K]$. By (v), we have
$$\sup_{|x-y|\ge \varepsilon_0/2} \left\{ \frac{\sup_{t\ge 0} |V(t,x) - V(t,y)|}{|x-y|} :\ x, y \in [-K, K] \right\} = \delta < 1.$$
and therefore
$$|V(t_k, x_{n_k}(t_k)) - V(t_k, x^*(t_k))| \le \delta |x_{n_k}(t_k) - x^*(t_k)|.$$
We now have
$$\begin{aligned}
|y_{n_k}(t_k) - y^*(t_k)| &= |(x_n - \widetilde{V}x_n)(t_k) - (x^* - \widetilde{V}x^*)(t_k)| \\
&\ge |x_{n_k}(t_k) - x^*(t_k)| - |(\widetilde{V}x_{n_k})(t_k) - (\widetilde{V}x^*)(t_k)| \\
&\ge |x_{n_k}(t_k) - x^*(t_k)| - \delta |x_{n_k}(t_k) - x^*(t_k)| \\
&= (1-\delta)|x_{n_k}(t_k) - x^*(t_k)| \ge (1-\delta)\varepsilon_0/2 > 0.
\end{aligned}$$
This yields
$$(1-\delta)\varepsilon_0/2 \le |y_{n_k}(t_k) - y^*(t_k)| \le \|y_{n_k} - y^*\| \to 0 \ \text{ as } \ n \to \infty,$$
a contradiction. So we obtain $x_n \to x^*$ as $n \to \infty$, and thus $(I-\widetilde{V})^{-1}$ is continuous on $(I-\widetilde{V})(M)$. This completes the proof. $\square$

Proof of Theorem 4.1. Let BC be the Banach space of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the supremum norm $\|\cdot\|$. To simplify notations, we set

$$g^*(t,s) = \sup_{|y|\le r(s)} |g(t,y) - g(s,y)|,$$

$$f^*(t,s) = \sup_{|y|\le r(s)} |f(t,y) - f(s,y)|.$$

and

$$I(t, x(\cdot)) = \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s,x(s))}{(t-s)^{1-q}} ds.$$

We define

$$M = \{x \in BC : |x(t)| \le r(t),\ \forall t \ge 0\}$$

where $r(t)$ is given in (c_5).

Now let P be the natural mapping defined by the integral equation (15). We will show that P maps M into itself. To see this, letting $x \in M$, we have

$$\begin{aligned}
|(Px)(t)| &= |g(t,x(t)) + f(t,x(t))I(t,x(\cdot))| \\
&\le |g(t,x(t)) - g(t,0)| + |g(t,0)| \\
&\quad + |f(t,x(t)) - f(t,0)||I(t,x(\cdot))| + |f(t,0)||I(t,x(\cdot))| \\
&\le \ell(t)|x(t)| + |g(t,0)| \\
&\quad + m(t)|x(t)||I(t,x(\cdot))| + |f(t,0)||I(t,x(\cdot))| \\
&\quad \text{(use (16) below)} \\
&\le \ell(t)r(t) + |g(t,0)| \\
&\quad + m(t)r(t)[v^*(t) + n(t)\Phi(r^*(t))]t^q \frac{1}{\Gamma(q+1)} \\
&\quad + |f(t,0)|[v^*(t) + n(t)\Phi(r^*(t))]t^q \frac{1}{\Gamma(q+1)} \\
&= \gamma(t) + \ell(t)r(t) \\
&\quad + [\psi(t)r(t) + \phi(t)r(t)\Phi(r^*(t)) + \xi(t)\Phi(r^*(t))] \frac{1}{\Gamma(q+1)} \\
&\le r(t)
\end{aligned}$$

by applying $(c_1) - (c_3)$ and (c_5).

Next, we define M^* by

$$M^* = \{x \in M : |x(t) - x(s)| \le \nu(t,s),\ \forall t, s \ge 0\}$$

where $\nu(t,s)$ is a continuous function to be defined below with $\nu(t,s) \to 0$ as $|t-s| \to 0$. We want to show that $|(Px)(t) - (Px)(s)| \le \nu(t,s)$ for $\phi \in M$. To this end, we proceed to estimate the terms on the right-hand side of (15) for $x \in M$. By (c_3), we have

$$
\begin{aligned}
(16) \qquad |I(t,x(\cdot))| &\le \frac{1}{\Gamma(q)} \int_0^t \frac{|v(t,s,x(s))|}{(t-s)^{1-q}} ds \\
&\le \frac{1}{\Gamma(q)} \int_0^t \frac{v^*(t) + n(t)\Phi(|x(s)|)}{(t-s)^{1-q}} ds \\
&\le [v^*(t) + n(t)\Phi(r^*(t))] \frac{1}{\Gamma(q)} \int_0^t \frac{1}{(t-s)^{1-q}} ds \\
&= [v^*(t) + n(t)\Phi(r^*(t))] t^q \frac{1}{\Gamma(q+1)} =: J^*(t).
\end{aligned}
$$

and

$$
\begin{aligned}
(17) \qquad |f(t,x(t))| &\le |f(t,x(t)) - f(t,0)| + |f(t,0)| \\
&\le m(t)|x(t)| + |f(t,0)| \\
&\le m(t)r(t) + |f(t,0)| =: \hat{f}(t).
\end{aligned}
$$

From (c_1), we have

$$
\begin{aligned}
(18) \qquad & |g(t,x(t)) - g(s,x(s))| \\
&\le |g(t,x(t)) - g(t,x(s))| + |g(t,x(s)) - g(s,x(s))| \\
&\le \ell(t)|x(t) - x(s)| + g^*(t,s).
\end{aligned}
$$

Apply (c_2) and the estimate in (16) to obtain

$$
\begin{aligned}
(19) \qquad & |f(t,x(t)) - f(s,x(s))||I(t,x(\cdot))| \\
&\le |f(t,x(t)) - f(t,x(s))||I(t,x(\cdot))| \\
&\quad + |f(t,x(s)) - f(s,x(s))||I(t,x(\cdot))| \\
&\le m(t)|x(t) - x(s)|J^*(t) + f^*(t,s)J^*(t) \\
&= [\psi(t) + \phi(t)\Phi(r^*(t))] \frac{1}{\Gamma(q+1)} |x(t) - x(s)| + f^*(t,s)J^*(t).
\end{aligned}
$$

An argument similar to that in the proof of Theorem 6.1 yields, for $0 \le s \le t$, that

$$
\begin{aligned}
& |I(t,x(\cdot)) - I(s,x(\cdot))| \\
&\le [v^*(t) + n(t)\Phi(r^*(t))] \frac{2}{\Gamma(q+1)} |t-s|^q + \frac{1}{\Gamma(q+1)} t^q v^*(t,s) =: I^*(t,s)
\end{aligned}
$$

where $v^*(t,s) = \sup_{|y| \le r(s)} |v(t,\tau,y) - v(s,\tau,y)|$ for $0 \le \tau \le s \le t$.

Taking into account (17), we get

$$|I(t,x(\cdot))-I(s,x(\cdot))||f(s,x(s))| \le I^*(t,s)\hat{f}(s). \tag{20}$$

Combine (18)-(20) to obtain

$$\begin{aligned}
&|(Px)(t)-(Px)(s)| \\
&\le [\ell(t)\Gamma(q+1)+\psi(t)+\phi(t)\Phi(r^*(t))]\frac{1}{\Gamma(q+1)}|x(t)-x(s)| \\
&\quad + g^*(t,s)+f^*(t,s)J^*(t)+I^*(t,s)\hat{f}(s) \\
&\quad (\text{ assigning the last three terms as } (1-\beta)\nu(t,s)\ ) \\
&\le \beta|x(t)-x(s)|+(1-\beta)\nu(t,s) \\
&\le \beta\nu(t,s)+(1-\beta)\nu(t,s)=\nu(t,s)
\end{aligned}$$

where β is given in (c_5). Thus, $P : M^* \to M^*$. The rest of the proof follows that of Theorem 2.2 so P has a fixed point in M which is a solution of (15). The proof is complete.

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