L^p-SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study fractional differential equations of Caputo type ${}^cD^q x(t) = u(t, x(t))$, $0 < q < 1$, of both linear and nonlinear type. That equation is inverted as an integral equation with kernel $C(t-s) := (1/\Gamma(q))(t-s)^{q-1}$. We then transform the integral equation into one with kernel $R(t-s)$ so that $0 < R(t) \le C(t)$ and $\int_0^\infty R(s)ds = 1$. A variety of techniques are introduced by which we are able to show that solutions are in $L^p[0,\infty)$ for appropriate $p \ge 1$.

1. Introduction

The object of this paper is to present basic techniques for showing that the solutions of common fractional differential equations are in $L^p[0,\infty)$ for some positive integer p. Here is a loose description of how we will proceed. Invert the fractional differential equation of Caputo type

$${}^cD^q x(t) = u(t, x(t)), \quad 0 < q < 1, \quad x(0) \in \Re$$

as the standard integral equation

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s, x(s))ds$$

where Γ is the gamma function. Because of the large and singular kernel it can be very difficult to study. Moreover, the $x(0)$ is a constant source of difficulty. There are at least two reasonable ways to avoid those problems. We use both.

Stability theory assumes that we have an object in equilibrium, $x(0) = 0$, and we disturb the equilibrium with $x(0) \neq 0$, causing no further disturbance. We then study the subsequent position of the object.

An equally important study assumes that $x(0) = 0$, but we constantly perturb the object by an external force, say $f(t)$, and study the resulting position of the object. Three very common forms of $f(t)$ are:

(i) $f(t) \to 0$ as $t \to \infty$.
(ii) $f \in L^p[0,\infty)$.
(iii) f is periodic.

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Our choice here is (ii) and we study both $x(0) = 0$ and $x(0) \neq 0$. Moreover, the reader may consult Podlubny [7, p. 224] to see that the assumption of $x(0) = 0$ is of general use and not simply contrived to meet our requirements. For that case we study

$$^cD^q x(t) = u(t, x(t)) + f(t), \quad 0 < q < 1, \quad f \in L^p[0, \infty), \quad x(0) = 0.$$

The inversion is then

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[u(s, x(s)) + f(s)]ds$$

and we are thwarted because $f \in L^1[0, \infty)$ yields $\int_0^t (t-s)^{q-1} f(s)ds \to 0$ as $t \to \infty$, but generally not in $L^p[0, \infty)$.

Fortunately, the kernel is a completely monotone function and that means that the resolvent kernel, $R(t)$, is positive, $\int_0^\infty R(t)dt = 1$, and completely monotone. But to really utilize this wonderful resolvent we need one assumption: $u(t, x)$ must contain a linear term "$-x$", Much of the point here is that we can successfully render this true by a variety of techniques, not the least of which is the elementary device of writing $u(t, x) = -x + x + u(t, x)$.

With this assumption in hand we decompose our equation into a pair:

$$z(t) = x(0) - \int_0^t R(t-s)x(0)ds$$

and

$$x(t) = z(t) + \int_0^t R(t-s)[x(s) + u(s, x(s)) + f(s)]ds.$$

Here, $z(t) \to 0$ as $t \to \infty$; or $z(t) = 0$ if $x(0) = 0$. Moreover, $\int_0^t R(t-s)f(s)ds \in L^1[0, \infty)$ if $f \in L^1[0, \infty)$. This gives us a real chance of proving $x \in L^p[0, \infty)$, a chance that was essentially zero in the original form.

In the next pages we offer a set of examples showing how this can be done. We will introduce a positive constant, J, which preserves the complete monotonicity of the kernel and makes a number of things possible. Not the least of these is that mappings which were only Lipschitz become contractions.

2. Basic theory

Here are the details of the decomposition described in Section 1. The inversion of the Caputo equation into an integral equation when $u(t, x)$ is continuous is verified in ([4, p. 54], [3, pp. 78, 86, 103]). This equation has a unique solution as seen in [1] or [8]). We begin with

$$^cD^q x(t) = u(t, x), \quad 0 < q < 1, \quad x(0) \in \Re \tag{1}$$

where $u : [0,\infty) \times \Re \to \Re$ is continuous. At this point a positive constant J is introduced and we invert (1) as

$$x(t) = x(0) + \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{u(s,x(s))}{J} ds.$$

In our subsequent work, the function " $-x$" is introduced in a well-motivated way. But here we simply add and subtract to obtain

$$x(t) = x(0) + \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[-x(s) + x(s) + \frac{u(s,x(s))}{J} \right] ds \tag{2}$$

Denote the kernel by

$$C(t) = \frac{J}{\Gamma(q)} t^{q-1} \tag{3}$$

so that for any $T > 0$ we have the critical property that

$$\int_0^T |C(u)| du < \infty.$$

Following Miller [6, pp. 193-22] we note that $C(t)$ is completely monotone on $(0,\infty)$ in the sense that $(-1)^k C^{(k)}(t) \geq 0$ for $k = 0,1,2,...$ and $t \in (0,\infty)$. Moreover $C(t)$ satisfies the conditions of Miller's Theorem 6.2 on p. 212. That theorem states that if the resolvent equation for the completely monotone kernel C is

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds \tag{4}$$

then that resolvent kernel, R, satisfies

(5) $0 \leq R(t) \leq C(t)$ for all $t > 0$ so that as $t \to \infty$ then $R(t) \to 0$

and that

$$C \notin L^1[0,\infty) \quad \Longrightarrow \int_0^\infty R(s)ds = 1. \tag{6}$$

Continuing on to [6, pp. 221-224 (Theorem 7.2)] we see that R is also completely monotone.

Next, under the conditions here, it is shown in Miller [6, pp. 191-207] that (2) can be decomposed into

$$z(t) = x(0) - \int_0^t C(t-s)z(s)ds \tag{7}$$

with

$$z(t) = x(0) - \int_0^t R(t-s)x(0)ds = x(0)[1 - \int_0^t R(s)ds]$$

and, having found $z(t)$, then the solution $x(t)$ of (2) solves

$$x(t) = z(t) + \int_0^t R(t-s) \left[x(s) + \frac{u(s,x(s))}{J} \right] ds. \tag{8}$$

Notice that $z(t) \to 0$ as $t \to \infty$.

The kernel in (2) is not integrable on $[0, \infty)$, but in (8) it is replaced, not only by an integrable kernel, but the value of the integral is one and the new kernel is also completely monotone.

3. $f \in L^1[0, \infty)$: Elementary arguments

Our first theorem, together with the preparation for it, can be viewed as an introduction to and a simple and transparent example of several of our subsequent results. It shows the basic selection of the constant J which is one of the most essential parts of the study. The linear equation

$$^cD^q x = f(t) - a(t)x(t), \quad 0 < q < 1, \quad x(0) = 0, \tag{9}$$

is inverted as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[a(s)x(s) - f(s)]ds \tag{10}$$

where $a, f : [0, \infty) \to \Re$ are continuous and there are positive numbers ϵ and M with

$$0 < \epsilon \leq a(t) \leq M. \tag{11}$$

We will exchange the kernel in (10) for $R(t-s)$, but first we will reduce $a(t)$ to a function bounded by $\alpha < 1$. Define $J = \epsilon + (1/2)(M - \epsilon)$. Then there is an α with

$$J > 0, \quad 0 < \alpha < 1, \quad |a(t) - J| < \alpha J. \tag{12}$$

Note that we may choose $\alpha = (M - \epsilon)/(M + \epsilon)$. In fact, if we write $J = (M + \epsilon)/2$, then by (11) we have

$$\epsilon - J \leq a(t) - J \leq M - J.$$

This implies that

$$-\frac{1}{2}(M - \epsilon) \leq a(t) - J \leq \frac{1}{2}(M - \epsilon)$$

and so

$$|a(t) - J| \leq \frac{1}{2}(M - \epsilon) = \frac{M - \epsilon}{M + \epsilon} J =: \alpha J.$$

Since $x(0) = 0$, we write (10) as

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[Jx(s) + (a(s) - J)x(s) - f(s)]ds \\ &= -\frac{1}{\Gamma(q)} \int_0^t J(t-s)^{q-1}[x(s) + \frac{(a(s) - J)}{J} x(s) - \frac{f(s)}{J}]ds \\ &=: -\frac{1}{\Gamma(q)} \int_0^t J(t-s)^{q-1}[x(s) + \beta(s)x(s) - \frac{f(s)}{J}]ds \end{aligned}$$

where $|\beta(t)| \leq \alpha < 1$.

The kernel is still completely monotone so the resolvent has exactly the same properties as the R we discussed before, so we retain the name of R. Next, decompose this equation into $z(t) = 0$ and

$$x(t) = F(t) - \int_0^t R(t-s)[\beta(s)x(s)]ds \tag{13}$$

with

$$F(t) = \frac{1}{J}\int_0^t R(t-s)f(s)ds, \quad \beta(s) = \frac{a(s)-J}{J}.$$

Notice that $f \in L^1$ implies $F \in L^1$.

Condition (11) will be critical in Section 4 for general p, but we will see in two steps that it can be changed to $-\infty \le a(t) \le \infty$.

Theorem 3.1. *If (11) holds and if $f \in L^1[0,\infty)$, then the solution of (9) is in $L^1[0,\infty)$.*

Proof. We have

$$\begin{aligned}\int_0^t |x(s)|ds &\le \int_0^t |F(s)|ds + \int_0^t \int_0^u R(u-s)\alpha|x(s)|dsdu \\ &= \int_0^t |F(s)|ds + \int_0^t \int_s^t R(u-s)du\alpha|x(s)|ds \\ &\le \int_0^t |F(s)|ds + \int_0^t \alpha|x(s)|ds\end{aligned}$$

so that

$$(1-\alpha)\int_0^t |x(s)|ds \le \int_0^t |F(s)|ds,$$

as required. □

In many problems we find that $\beta(t)$ in the above proof can be arbitrarily large for small values of t, but eventually it is dominated by $\alpha < 1$. This can happen in two simple ways. First, it may be that $a(t)$ is asymptotically periodic in the sense that $a(t) = b(t) + c(t)$ where $b(t)$ is a positive periodic function, while $c(t) \to 0$ as $t \to \infty$. In a more complicated way, it may happen that $b(t)$ is again a positive periodic function, but $c \in L^1[0,\infty)$. In this case, $\int_0^t R(t-s)c(s)ds \to 0$ as $t \to \infty$ and some algebraic work must be done to bring us to the desired conclusion. The following lemma allows us to avoid those difficulties.

Lemma 3.2. *If $G(t)$ is continuous on $[0,\infty)$, then for $\int_0^{t/2} R(v)dv > 1/2$ it follows that*

$$(1/2)\int_0^{t/2} |G(s)|ds \le \int_0^t \int_0^u R(u-s)|G(s)|dsdu.$$

Proof. We have

$$\begin{aligned}(1/2)\int_0^{t/2}|G(s)|ds &\le \int_0^{t/2}\int_0^{t/2}R(v)dv|G(s)|ds\\ &\le \int_0^{t/2}\int_0^{t-s}R(v)dv|G(s)|ds\\ &= \int_0^{t/2}\int_s^{t}R(u-s)du|G(s)|ds\\ &\le \int_0^{t}\int_s^{t}R(u-s)du|G(s)|ds\\ &= \int_0^{t}\int_0^{u}R(u-s)|G(s)|dsdu.\end{aligned}$$

□

In the next result, if $K = 0$ then it is essentially Theorem 3.1 and the lemma is not used.

Theorem 3.3. *Suppose that there is a number $K > 0$ and (11) holds for $t \ge K$ and that $f \in L^1[0,\infty)$. Then the solution of (9) is in $L^1[0,\infty)$.*

Proof. Define $J, \alpha, \beta(t)$ as in (12) and (13). Let $\beta^* : [0,\infty) \to \Re$ be a continuous function with $0 \le \beta^*(t) \le 1-\alpha$ if $0 \le t \le K$ and $\beta^*(t) = 1-\alpha$ if $t > K$. We have from (13) that

$$\begin{aligned}&\int_0^t |x(s)|ds \le \int_0^t |F(s)|ds + \int_0^t\int_0^u R(u-s)|\beta(s)||x(s)|dsdu\\ &\le \int_0^t |F(s)|ds + \int_0^K\int_0^u R(u-s)|\beta(s)||x(s)|dsdu\\ &+ \int_K^t\int_0^u R(u-s)|x(s)-(1-\alpha)x(s)|dsdu\\ &=: \mu(t) + \int_K^t\int_0^u R(u-s)[|x(s)|-(1-\alpha)|x(s)|]dsdu\\ &\le \mu(t) + \int_0^t\int_0^u R(u-s)|x(s)|dsdu - \int_0^t\int_0^u R(u-s)\beta^*(s)|x(s)|dsdu.\end{aligned}$$

Interchange the order of integration in the next-to-last integral and then cancel it against the term on the left-hand-side of the display. This yields

$$\int_0^t\int_0^u R(u-s)\beta^*(s)|x(s)|dsdu \le \mu(t).$$

By Lemma 3.2 if t is so large that $\int_0^t R(u)du > 1/2$ and $t > 2K$ then

$$(1/2)\int_K^{t/2}(1-\alpha)|x(s)|ds \le (1/2)\int_0^{t/2}\beta^*(s)|x(s)|ds \le \mu(t),$$

as required. □

Using the lemma we can now let $\epsilon = 0$ in (11). Here, M will take the place of J.

Theorem 3.4. *Let*

$$^cD^q x = -a(t)h(t,x) + f(t),\ x(0) = 0,$$

with $f \in L^1[0,\infty)$, $0 \le a(t) \le M$, *and* $|h(t,x)| \le |x|$ *for all* $(t,x), t \ge 0$. *If, in addition,* $xh(t,x) \ge 0$, *then any solution satisfies*

$$\int_0^{t/2} |(a(s)/M)h(s,x(s))|ds \le 2\int_0^t |\widetilde{F}(s)|ds$$

where t *is large enough that* $\int_0^t R(s)ds > 1/2$ *and*

$$\widetilde{F}(t) = \frac{1}{M}\int_0^t R(t-s)f(s)ds.$$

Proof. Invert the equation as

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[a(s)h(s,x(s)) - f(s)]ds \\ &= -\frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[(a(s)/M)h(s,x(s)) - (f(s)/M)]ds. \end{aligned}$$

Then we add and subtract $x(s)$ in the integrand and decompose in the usual way. As $x(0) = 0$ we have $z(t) = 0$ and our equation is

$$x(t) = \int_0^t R(t-s)[x(s) - \frac{a(s)}{M}h(s,x(s)) + \frac{f(s)}{M}]ds.$$

Here we again retain the name of $R(t)$. Notice that

$$|x(s) - \frac{a(s)}{M}h(s,x(s))| = |x(s)| - |\frac{a(s)}{M}h(s,x(s))|.$$

Now, take absolute values of both sides so that we have (for $\widetilde{F}(t)$ defined above)

$$|x(t)| \le |\widetilde{F}(t)| + \int_0^t R(t-s)[|x(s) - |(a(s)/M)h(s,x(s))|]ds.$$

Integrate both sides from 0 to t. Interchange the order of integration in the first integral and cancel it with the left-hand-side. Apply the result in the previous proof to obtain

$$\int_0^{t/2} |(a(s)/M)h(s,x(s))|ds \le 2\int_0^t |\widetilde{F}(s)|ds,$$

as required. □

We can relate the next theorem to Theorem 3.1 by noting that $A = \epsilon$ and $B = M$; $\frac{\partial G(t,x)}{\partial x} = a(t)$.

Theorem 3.5. *Suppose there are positive constants A and B so that for*

$$^cD^q x = u(t,x) =: -G(t,x) + f(t), \quad x(0) \in \Re$$

we have $f \in L^1[0,\infty)$, $G(t,x) \to 0$ as $x \to 0$, f, G continuous,

$$A \leq \frac{\partial G(t,x)}{\partial x} \leq B, \quad 0 \leq t < \infty.$$

Then there is a unique solution tending to zero. If, in addition, $x(0) = 0$ and for $|x|$ sufficiently small there is a $\beta < 1$ with

$$\left| x - \frac{G(t,x)}{J} \right| \leq \beta |x|$$

for some $J > 0$, then that unique solution is in $L^1[0,\infty)$.

Proof. The equation is inverted as

$$\begin{aligned} x(t) &= x(0) + \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[-\frac{G(s,x(s))}{J} + \frac{f(s)}{J} \right] ds \\ &= \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[-x(s) + x(s) - \frac{G(s,x(s))}{J} + \frac{f(s)}{J} \right] ds \end{aligned}$$

so that $z(t) = x(0)[1 - \int_0^t R(s)ds]$ and

$$x(t) = z(t) + F(t) + \int_0^t R(t-s) \left[x(s) - \frac{G(s,x(s))}{J} \right] ds$$

with

$$F(t) = (1/J) \int_0^t R(t-s) f(s) ds.$$

It follows that $F \in L^1[0,\infty)$ and $F(t) \to 0$ as $t \to \infty$. Now

$$\frac{\partial}{\partial x} \left[x - \frac{G(t,x)}{J} \right] = 1 - \frac{\partial G(t,x))}{J \partial x}$$

and

$$0 < A \leq \frac{\partial G(t,x))}{\partial x} \leq B$$

so take J so large that

$$0 < 1 - (B/J) =: \alpha_1 < 1$$

and

$$0 < 1 - (A/J) =: \alpha_2 < 1.$$

In the second part of the theorem we may want J even larger.

Let $(X_0, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$ such that $\phi(t) \to 0$ as $t \to \infty$. For fixed $x(0)$, define $P : X_0 \to X_0$ by $\phi \in X_0$ implies that

$$(P\phi)(t) = z(t) + F(t) + \int_0^t R(t-s) \left[x(s) - \frac{G(s,x(s))}{J} \right] ds$$

and note that $(P\phi)(t) \to 0$ as $t \to \infty$. Moreover, if $\phi, \eta \in X_0$ then

$$|(P\phi)(t)-(P\eta)(t)| \le \int_0^t R(t-s)\left|\phi(s)-\frac{G(s,\phi(s))}{J}-\eta(s)+\frac{G(s,\eta(s))}{J}\right|ds.$$

This is a contraction with constant $\alpha = \max[\alpha_1, \alpha_2] = \alpha_2$ so P has a unique fixed point $\phi \in X_0$. That proves the first part of the theorem.

For the second part, $z(t) = 0$. Consider the aforementioned unique solution ϕ and find $T > 0$ so that $s \ge T$ implies that

$$\left|\phi(s) - \frac{G(s,\phi(s))}{J}\right| \le \beta|\phi(s)|.$$

Then interchanging the order of integration below, we obtain

$$\begin{aligned}\int_0^t |\phi(s)|ds &\le \int_0^t |F(s)|ds + \int_0^t \int_s^t R(u-s)du|\phi(s) - G(s,\phi(s)|ds \\ &\le \int_0^t |F(s)|ds + \int_0^T |\phi(s) - G(s,\phi(s))|ds + \int_T^t \beta|\phi(s)|ds.\end{aligned}$$

Hence,

$$(1-\beta)\int_T^t |\phi(s)|ds \le \int_0^t |F(s)|ds + \int_0^T |\phi(s) - G(s,\phi(s))|ds,$$

completing the proof. □

There is a simple result which the reader should have in mind when viewing the nonlinear problems.

Proposition 3.6. *Let $G(t) \ge 0$. Then*

$$L(t) := \int_0^t R(t-s)G(s)ds \in L^1[0,\infty) \iff G \in L^1[0,\infty).$$

Proof. If $G \in L^1$, clearly $L(t) \in L^1$. If $L(t) \in L^1$, then for large t

$$\int_0^t \int_0^u R(u-s)G(s)dsdu \ge (1/2)\int_0^{t/2} G(s)ds$$

so $G \in L^1$. □

The proposition reverses the classical theorem that the convolution of two L^1 functions is an L^1 function.

4. $f \in L^p[0,\infty)$

This section is divided into two parts depending on the initial condition.

Case I: $x(0) = 0$.

Return to (9)

$${}^cD^q x = f(t) - a(t)x(t), \quad 0 < q < 1, \quad x(0) = 0$$

and (13)

$$x(t) = F(t) - \int_0^t R(t-s)[\beta(s)x(s)]ds$$

where

$$F(t) := \frac{1}{J}\int_0^t R(t-s)f(s)ds$$

and $R(t)$ satisfies (4)

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds$$

with C defined in (3) as

$$C(t) = \frac{1}{\Gamma(q)}Jt^{q-1} \ \text{ for } \ t > 0.$$

We need the following lemma.

Lemma 4.1. *If $f \in L^p$ then $F \in L^p$ for any $p \geq 1$ with $\|F\|_p \leq \|f\|_p/J$ and $F(t) \to 0$ as $t \to \infty$.*

Proof. By Hölder's inequality, we have

$$\int_0^t R(t-s)|f(s)|ds \leq \left(\int_0^t R(t-s)ds\right)^{\frac{p-1}{p}} \left(\int_0^t R(t-s)|f(s)|^p ds\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^t R(t-s)|f(s)|^p ds\right)^{\frac{1}{p}}$$

since $\int_0^t R(s)ds \leq 1$. This implies that

$$\int_0^t |F(u)|^p du \leq J^{-p}\int_0^t \int_0^u R(u-s)|f(s)|^p dsdu$$

$$\leq J^{-p}\int_0^t \int_s^t R(u-s)du|f(s)|^p ds$$

$$= J^{-p}\int_0^t \int_0^{t-s} R(u)du|f(s)|^p ds$$

$$\leq J^{-p}\int_0^\infty |f(s)|^p ds < \infty$$

and thus, $F \in L^p$ for any $p \geq 1$ and $\|F\|_p \leq \|f\|_p/J$. The fact that $F(t) \to 0$ as $t \to \infty$ follows from the inequality

$$|F(t)|^p \leq J^{-p}\int_0^t R(t-s)|f(s)|^p ds,$$

the convolution of an L^1 function with a function tending to zero. □

Corollary 4.2. *For any* $p \geq 1$,

$$\int_0^t R(t-s)|f(s)|ds \leq \left(\int_0^t R(t-s)|f(s)|^p ds\right)^{1/p}.$$

Theorem 4.3. *If (11) holds and if* $f \in L^p[0,\infty)$ *for* $p \geq 1$, *then the solution of (9) is in* $L^p[0,\infty)$ *with*

$$\|x\|_p \leq \|f\|_p/[J(1-\alpha)].$$

Proof. We integrate (13) on $[0,t]$ and use the triangle inequality of the L^p-norm (or Minkowski inequality) to obtain

$$\begin{aligned}\left(\int_0^t |x(u)|^p du\right)^{1/p} &\leq \left(\int_0^t |F(u)|^p du\right)^{1/p} \\ &\quad + \left[\int_0^t \left(\int_0^u R(u-s)|\beta(s)x(s)|ds\right)^p du\right]^{1/p} \\ &\leq \left(\int_0^t |F(u)|^p du\right)^{1/p} \\ &\quad + \alpha\left(\int_0^t \int_0^u R(u-s)||x(s)|^p ds du\right)^{1/p}.\end{aligned}$$

Interchange the order of integration and use (6), the property for $R(t)$, leaving us with

$$(1-\alpha)\left(\int_0^t |x(s)|^p ds\right)^{1/p} \leq \left(\int_0^t |F(s)|^p ds\right)^{1/p}$$

or

$$\|x\|_p \leq \|f\|_p/[J(1-\alpha)].$$

This completes the proof. □

Case II: $x(0) \neq 0$.

Let $\widetilde{R}(t) = 1 - \int_0^t R(s)ds$ and $z(t) = x(0)\widetilde{R}(t)$. For $x(0) \neq 0$, we write the counterpart of (9) as

$$(9^*) \qquad {}^cD^q x = f(t) - a(t)x(t), \quad 0 < q < 1, \quad x(0) = x_0.$$

With $\beta(s)$ defined in (13), this can be inverted as

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t J(t-s)^{q-1}[x(s) + \beta(s)x(s) - \frac{f(s)}{J}]ds.$$

Then $x(t)$ solves

$$x(t) = z(t) + F(t) - \int_0^t R(t-s)[\beta(s)x(s)]ds. \tag{14}$$

Lemma 4.4. *Let $R(t)$ be defined in (4). If $\widetilde{R}(t) = 1 - \int_0^t R(u)du$, then*

$$\widetilde{R}(t) = 1 - \int_0^t C(t-s)\widetilde{R}(s)ds. \tag{15}$$

Proof. We integrate (4) from 0 to t to obtain

$$\begin{aligned}\int_0^t R(u)du &= \int_0^t C(u)du - \int_0^t \int_0^u C(u-s)R(s)dsdu \\ &= \int_0^t C(u)du - \int_0^t \int_0^{t-s} C(u)duR(s)ds.\end{aligned}$$

This is equivalent to

$$1 - \int_0^t R(u)du = 1 - \int_0^t C(u)du + \int_0^t \int_0^{t-s} C(u)duR(s)ds.$$

Taking into account that $\widetilde{R}'(t) = -R(t)$, we arrive at

$$\widetilde{R}(t) = 1 - \int_0^t C(u)du - \int_0^t \int_0^{t-s} C(u)du\widetilde{R}'(s)ds$$

(integrate the last term by parts)

$$\begin{aligned}&= 1 - \int_0^t C(u)du - \left[\int_0^{t-s} C(u)du\widetilde{R}(s)\Big|_{s=0}^{s=t} + \int_0^t C(t-s)\widetilde{R}(s)ds\right] \\ &= 1 - \int_0^t C(u)du - \left[-\int_0^t C(u)du + \int_0^t C(t-s)\widetilde{R}(s)ds\right]\end{aligned}$$

as required. □

Lemma 4.5. *Let $R(t)$ be defined in (4) and $\widetilde{R}$ be defined in (15). Then $\widetilde{R}(t) \in L^k[0,\infty)$ for $k \geq 1$ if and only if $k > 1/q$.*

Proof. We take Laplace transform of (15) to obtain

$$\mathcal{L}(\widetilde{R}) = \mathcal{L}(1) - \mathcal{L}(C)\mathcal{L}(\widetilde{R})$$

and so

$$\mathcal{L}(\widetilde{R})(s) = \frac{1}{s[1+\mathcal{L}(C)(s)]}.$$

We recall that $\mathcal{L}(t^\nu)(s) = \Gamma(\nu+1)s^{-\nu-1}$ for $\nu > -1$ and $\Re e(s) > 0$ (see [7, p. 28]). Thus,

$$\mathcal{L}(C)(s) = \frac{J}{\Gamma(q)}\mathcal{L}(t^{q-1})(s) = \frac{J}{\Gamma(q)}\Gamma(q)s^{-q} = Js^{-q}.$$

Substitute this equation into the right-hand side of $\mathcal{L}(\widetilde{R})(s)$ above to obtain

$$\mathcal{L}(\widetilde{R})(s) = \frac{1}{s(1+Js^{-q})} = \frac{s^{q-1}}{s^q+J}$$

and hence,

$$\widetilde{R}(t) = \mathcal{L}^{-1}\left[\frac{s^{q-1}}{s^q+J}\right] = E_{q,1}(-Jt^q)$$

(see [7, p. 21]), where $E_{q,1}$ is a member of the two parameter family of Mittag-Leffler functions (generalized exponential functions) defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \ \alpha, \ \beta > 0$$

with the property, in particular, that

$$|E_{q,1}(-Jt^q)| \le \frac{\mu}{1+Jt^q}$$

for some constant μ and all $t \ge 0$ (see [7, p. 35]). Let $k > 1/q$ be fixed. Then

$$|\widetilde{R}(t)|^k \le (\mu/J)^k/t^{kq} \ \text{ for } \ t \ge 1$$

and thus, $\widetilde{R}(t) \in L^k[0,\infty)$ for all $k > 1/q$.

Conversely, suppose that $1 \le k \le 1/q$. We show that $\widetilde{R}(t) \notin L^k[0,\infty)$. In fact, it follows from an asymptotic expansion formula for $E_{\alpha,\beta}(z)$ (see [7, p. 33-34]) that

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O(|z|^{-1-p}) \quad \text{as} \ |z| \to \infty$$

where $0 < \alpha < 2$, $\beta > 0$, $p \ge 1$ is an arbitrary integer, and $\mu \le |\arg(z)| \le \pi$ with $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Now for $p = 1$, we have

$$E_{q,1}(-Jt^q) = -\frac{1}{\Gamma(1-q)}\left(-Jt^q\right)^{-1} + O(t^{-2q}) \quad \text{as} \ t \to \infty.$$

Thus, there exists $\eta > 0$ and $T > 0$ such that

$$\widetilde{R}(t) = E_{q,1}(-Jt^q) \ge \eta t^{-q} \quad \text{for} \ t \ge T$$

and hence,

$$|\widetilde{R}(t)|^k \ge \eta^k t^{-kq} \quad \text{for} \ t \ge T.$$

Since $0 < kq \le 1$, we see that $\widetilde{R} \notin L^k[0,\infty)$. This completes the proof. □

Theorem 4.6. *Suppose that (11) holds and $f \in L^p[0,\infty)$ for $p \ge 1$. Then the solution $x(t)$ of (9*) with $x(0) \neq 0$ is in $L^p[0,\infty)$ if and only if $p > 1/q$. Moreover, if $p > 1/q$, then*

$$\|x\|_p \le L\left[|x(0)|\|\widetilde{R}\|_p + \|f\|_p\right] \tag{16}$$

for some constant $L > 0$.

Proof. If $p > 1/q$, we have $\widetilde{R} \in L^p[0,\infty)$ by Lemma 4.5. It then follows from (14) that

$$\|x\|_p \le \|z\|_p + \|F\|_p + \alpha\|x\|_p$$
$$\le |x(0)|\|\widetilde{R}\|_p + \|f\|_p/J + \alpha\|x\|_p.$$

This implies that $x \in L^p[0,\infty)$ and (16) holds.

Conversely, suppose that $1 \le p \le 1/q$. Then $\widetilde{R} \notin L^p[0,\infty)$ by Lemma 4.5. We write (14) as

$$\widetilde{R}(t)x(0) = x(t) - F(t) + \int_0^t R(t-s)[\beta(s)x(s)]ds.$$

If $x \in L^p[0,\infty)$, then

$$\|\widetilde{R}\|_p \le [\|x\|_p + \|f\|_p/J + \alpha\|x\|_p]/|x(0)| < \infty,$$

which yields $\widetilde{R} \in L^p[0,\infty)$, a contradiction. □

Remark: If $f \equiv 0$, then by (16), the zero solution of (9^*) is L^p-asymptotically stable. We also observe from (16) that solutions of (9^*) are uniformly bounded and uniformly ultimately bounded in $L^p[0,\infty)$ for $p > 1/q$ at $t = 0$.

We consider the nonlinear equation

$$^cD^q x = -a(t)x^3(t) + f(t), \quad 0 < q < 1 \tag{17}$$

where $a, f : [0,\infty) \to \Re$ are continuous and there are positive numbers ϵ and M such that (11) holds. We then find $J > 0, \alpha < 1$ with $|J - a(t)| \le \alpha J$. We invert (17) as

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[a(s)x^3(s) - f(s)]ds$$

which we write

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[Jx(t) - Jx(t) + a(s)x^3(s) - f(s)]ds.$$

We now decompose it into

$$z(t) = x(0) - \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}z(s)ds$$

with solution $z(t) = x(0)\widetilde{R}(t)$ and

$$x(t) = z(t) + \int_0^t R(t-s)[x(s) - x^3(s) + \frac{(J-a(s))}{J}x^3(s)]ds + F(t) \tag{18}$$

where $R(t)$ is defined in (4) and

$$F(t) := \frac{1}{J}\int_0^t R(t-s)f(s)ds.$$

Note that $F \in L^p$ if $f \in L^p$ for $p \geq 1$ and $F(t) \to 0$ as $t \to 0$.

Theorem 4.7. *Suppose (11) holds. If $f \in L^p[0,\infty)$ for $p \geq 1$ and $|F(t)| < (1-\alpha)\eta^3$ with $0 < \eta < \sqrt{3}/3$ and $(1-\eta^2)^{3(p-1)} > \alpha$, then the solution of (17) with $x(0) = 0$ is in $L^{3p}[0,\infty)$ with $\|x\|_{3p} \leq L\|f\|_p^{1/3}$ for some constant $L > 0$.*

Proof. For $x(0) = 0$, we see that $z(t) = 0$ and (18) becomes

$$x(t) = \int_0^t R(t-s)[x(s) - x^3(s) + \frac{(J-a(s))}{J}x^3(s)]ds + F(t). \tag{19}$$

First, we claim that $|x(t)| < \eta$ for all $t \geq 0$. If there is a $\bar{t} > 0$ such that $|x(\bar{t})| = \eta$ and $|x(s)| < |x(\bar{t})|$ for all $0 \leq s < \bar{t}$, then

$$\begin{aligned}|x(\bar{t})| &\leq \int_0^{\bar{t}} R(\bar{t}-s)[(|x(s)| - |x(s)|^3) + \alpha|x(s)|^3]ds + |F(\bar{t})| \\ &\leq [|x(\bar{t})| - |x(\bar{t})|^3 + \alpha|x(\bar{t})|^3]\int_0^{\bar{t}} R(\bar{t}-s)ds + |F(\bar{t})|.\end{aligned}$$

Since $\eta - (1-\alpha)\eta^3 > 0$, this yields

$$(1-\alpha)\eta^3 \leq |F(\bar{t})|$$

which contradicts the condition on F. Thus, $|x(t)| < \eta$ for all $t \geq 0$.

Next, we set $\gamma = 3(p-1)$ and multiply (19) by $(|x(t)| - |x(t)|^3)^\gamma$ to obtain

$$\begin{aligned}&\left(|x(t)| - |x(t)|^3\right)^{\gamma+1} + |x(t)|^3\left(|x(t)| - |x(t)|^3\right)^\gamma \\ &\leq \int_0^t R(t-s)[|x(s)| - |x(s)|^3]ds\left(|x(t)| - |x(t)|^3\right)^\gamma \\ &+ \alpha\int_0^t R(t-s)|x(s)|^3 ds\left(|x(t)| - |x(t)|^3\right)^\gamma \\ &+ |F(t)|\left(|x(t)| - |x(t)|^3\right)^\gamma.\end{aligned} \tag{20}$$

Use Young's inequality to write

$$\begin{aligned}&\left(|x(s)| - |x(s)|^3\right)\left(|x(t)| - |x(t)|^3\right)^\gamma \\ &\leq \frac{1}{\gamma+1}\left(|x(s)| - |x(s)|^3\right)^{\gamma+1} + \frac{\gamma}{\gamma+1}\left(|x(t)| - |x(t)|^3\right)^{\gamma+1}\end{aligned}$$

and

$$|x(s)|^3 \left(|x(t)| - |x(t)|^3\right)^\gamma$$
$$\leq \frac{3}{\gamma+3}|x(s)|^{\gamma+3} + \frac{\gamma}{\gamma+3}\left(|x(t)| - |x(t)|^3\right)^{\gamma+3}$$

so that

$$|F(t)|\left(|x(t)| - |x(t)|^3\right)^\gamma \leq K|F(t)|^p + \delta\left(|x(t)| - |x(t)|^3\right)^{\gamma+3}$$

where $0 < \delta < (1-\eta^2)^\gamma - \alpha$ and $K = K(\delta)$ is a constant.

Substitute these inequalities into (20) and integrate on $[0, t]$ to obtain

$$\int_0^t \left(|x(u)| - |x(u)|^3\right)^{\gamma+1} du + \int_0^t |x(u)|^3 \left(|x(u)| - |x(u)|^3\right)^\gamma du$$
$$\leq \frac{1}{\gamma+1}\int_0^t \int_0^u R(u-s)\left(|x(s)| - |x(s)|^3\right)^{\gamma+1} dsdu$$
$$+ \frac{\gamma}{\gamma+1}\int_0^t \int_0^u R(u-s)\left(|x(u)| - |x(u)|^3\right)^{\gamma+1} dsdu$$
$$+ \frac{3\alpha}{\gamma+3}\int_0^t \int_0^u R(u-s)|x(s)|^{\gamma+3} dsdu$$
$$+ \frac{\gamma\alpha}{\gamma+3}\int_0^t \int_0^u R(u-s)\left(|x(u)| - |x(u)|^3\right)^{\gamma+3} dsdu$$
$$+ K\int_0^t |F(u)|^p du + \delta\int_0^t \left(|x(u)| - |x(u)|^3\right)^{\gamma+3} du.$$

Interchange the order of integration, use the property $\int_0^t R(u)du \leq 1$, and cancel terms, leaving us with

(21)
$$\int_0^t |x(u)|^3 \left(|x(u)| - |x(u)|^3\right)^\gamma du$$
$$\leq \frac{3\alpha}{\gamma+3}\int_0^t |x(s)|^{\gamma+3} ds + \frac{\gamma\alpha}{\gamma+3}\int_0^t \left(|x(u)| - |x(u)|^3\right)^{\gamma+3} du$$
$$+ K\int_0^t |F(u)|^p du + \delta\int_0^t \left(|x(u)| - |x(u)|^3\right)^{\gamma+3} du$$
$$\leq (\alpha+\delta)\int_0^t |x(s)|^{\gamma+3} ds + K\int_0^t |F(u)|^p du$$

The last line was obtained by noting that $|x(t)| > |x(t)|^3$. Since $|x(t)| < \eta$ for all $t \geq 0$, we have

$$(1-\eta^2)^\gamma \int_0^t |x(s)|^{\gamma+3} ds$$

$$\leq \int_0^t |x(u)|^{\gamma+3} \left(1-|x(u)|^2\right)^\gamma du$$

$$= \int_0^t |x(u)|^3 \left(|x(u)|-|x(u)|^3\right)^\gamma du$$

$$\leq (\alpha+\delta) \int_0^t |x(s)|^{\gamma+3} ds + K \int_0^t |F(u)|^p du$$

Note that $\gamma + 3 = 3p$ and $(1-\eta^2)^\gamma > \alpha + \delta$. Moving the next-to-last integral to the left-hand-side of the display, we see that $x \in L^{3p}[0,\infty)$ and $\|x\|_{3p} \leq L\|f\|_p^{1/3}$ for some constant $L > 0$. This completes the proof. □

Corollary 4.8. *Suppose (11) holds. If $f \in L^p[0,\infty)$ for $p \geq 1$, and if f is bounded on $[0,\infty)$, then the solution of (17) with $x(0) = 0$ is in $L^{3p}[0,\infty)$ with $\|x\|_{3p} \leq L\|f\|_p^{1/3}$ for some constant $L > 0$.*

Proof. Since f is bounded on $[0,\infty)$, there exists a constant $H > 0$ such that $|f(t)| \leq H$ for all $t \geq 0$. For the fixed $0 < \alpha < 1$ and $p \geq 1$, we choose $\eta > 0$ so that $0 < \eta < \sqrt{3}/3$ and $(1-\eta^2)^{3(p-1)} > \alpha$. We see from the proof of Lemma 4.1 that

$$|F(t)|^p \leq J^{-p} \int_0^t R(t-s)|f(s)|^p ds$$

$$\leq J^{-p} H^p \int_0^t R(t-s) ds \leq (H/J)^p \leq (2H/M)^p$$

by the definition of J, where M is defined in (11). This yields

$$|F(t)| \leq 2H/M.$$

We may choose M sufficiently large so that $2H/M < (1-\alpha)\eta^3$, and thus

$$|F(t)| \leq (1-\alpha)\eta^3.$$

Therefore, all conditions of Theorem 4.7 are satisfied, and the assertion of Corollary 4.8 follows. □

We now consider the case $x(0) \neq 0$.

Theorem 4.9. *Suppose (11) holds. If $f \in L^p[0,\infty)$ for $p > 1/q$ and $|F(t)| < (1-\alpha)\eta^3$ with $0 < \eta < \sqrt{3}/3$ and $(1-\eta^2)^{3(p-1)} > \alpha$, then the solution of (17) is in $L^{3p}[0,\infty)$ with*

$$\|x\|_{3p} \leq L\left[|x(0)|\|\widetilde{R}\|_p + \|f\|_p\right]^{1/3} \tag{22}$$

for some constant $L > 0$.

Proof. The proof is similar to that of Theorem 4.7 (with an additional term) by working with (18). Since $F(t) \to 0$ as $t \to \infty$, we see that $\|F\| < (1-\alpha)\eta^3$ and $|x(t)| < \eta$ for all $t \geq 0$ if

$$|x(0)| + \|F\| < (1-\alpha)\eta^3.$$

We now set $\gamma = 3(p-1)$, multiply (18) by $(|x(t)| - |x(t)|^3)^\gamma$, and follow through the calculations in (20) and (21) to obtain

$$\begin{aligned}
&\int_0^t |x(u)|^3 \left(|x(u)| - |x(u)|^3\right)^\gamma du \\
&\leq (\alpha+\delta)\int_0^t |x(s)|^{\gamma+3} ds + K\int_0^t |F(u)|^p du \\
&\quad + |x(0)|\int_0^t \widetilde{R}(u)\left(|x(u)| - |x(u)|^3\right)^\gamma du.
\end{aligned}$$

Next, find $0 < \delta_1 < (1-\eta^2)^\gamma - (\alpha+\delta)$ and $K_1 > 0$ such that

$$\begin{aligned}
&|x(0)|\int_0^t \widetilde{R}(u)\left(|x(u)| - |x(u)|^3\right)^\gamma du \\
&\leq |x(0)|^p K_1 \int_0^t |\widetilde{R}(u)|^p du + \delta_1 \int_0^t \left(|x(u)| - |x(u)|^3\right)^{\gamma+3} du.
\end{aligned}$$

Since $|x(t)| < \eta$ for all $t \geq 0$, we have

$$\begin{aligned}
&(1-\eta^2)^\gamma \int_0^t |x(s)|^{\gamma+3} ds \\
&\leq \int_0^t |x(u)|^{\gamma+3} \left(1-|x(u)|^2\right)^\gamma du \\
&= \int_0^t |x(u)|^3 \left(|x(u)|-|x(u)|^3\right)^\gamma du \\
&\leq (\alpha+\delta+\delta_1) \int_0^t |x(s)|^{\gamma+3} ds \\
&\quad + |x(0)|^p K_1 \int_0^t |\widetilde{R}(u)|^p du + K \int_0^t |F(u)|^p du.
\end{aligned}$$

This implies that there exists a constant $K_2 > 0$ such that

$$\int_0^t |x(s)|^{\gamma+3} ds \leq K_2 \left[|x(0)|^p \int_0^t |\widetilde{R}(u)|^p du + \int_0^t |F(u)|^p du \right] \tag{23}$$

for all $t \geq 0$. Raising both sides of (23) to the power $1/(\gamma+3)$ and taking into account that $\gamma + 3 = 3p$ and $\widetilde{R} \in L^p$, we see

$$\begin{aligned}
\|x\|_{3p} &\leq K_2^{\frac{1}{3p}} \left[|x(0)|^p \|\widetilde{R}\|_p^p + \|F\|_p^p \right]^{\frac{1}{3p}} \\
&\leq K_2^{\frac{1}{3p}} 2^{\frac{1}{3p}} \left[|x(0)| \|\widetilde{R}\|_p + \|f\|_p / J \right]^{\frac{1}{3}} \\
&\leq K_2^{\frac{1}{3p}} 2^{\frac{1}{3p}} (1+1/J)^{\frac{1}{3}} \left[|x(0)| \|\widetilde{R}\|_p + \|f\|_p \right]^{\frac{1}{3}} \\
&=: L \left[|x(0)| \|\widetilde{R}\|_p + \|f\|_p \right]^{\frac{1}{3}}
\end{aligned}$$

This completes the proof. □

Corollary 4.10. *Suppose (11) holds. If $f \in L^p[0,\infty)$ for $p > 1/q$, and if f is bounded on $[0,\infty)$, then the solution of (17) is in $L^{3p}[0,\infty)$, and it satisfies (22).*

Remark: If $f \equiv 0$, then by (22), the zero solution of (17) is L^{3p}-asymptotically stable. One may also note that Theorem 4.7 and Theorem 4.9 are general results. Everything would work for

$$^cD^q x = -a(t)x^{2n+1}(t) + f(t), \quad 0 < q < 1$$

for a positive integer n.

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References

[1] L. C. Becker, Resolvents and solutions of weakly singular linear Volterra integral equations, *Nonlinear Anal.*74(2011), 1892-1912.
[2] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, Mineola, New York, 2006.
[3] Kai Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Heidelberg, 2010.
[4] V. Lakshmikantham , S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cottenham, Cambridge, 2009.
[5] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.* 69(2008), 2677-2682.
[6] R. K. Miller, *Nonlinear Integral Equations*, Benjamin, Menlo Park, CA, 1971.
[7] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[8] A. Windsor, A contraction mapping proof of the smooth dependence on parameters of solutions to volterra integral equations, *Nonlinear Anal.* 72(2010), 3627-3634.

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