LIAPUNOV FUNCTIONALS, FIXED POINTS, AND STABILITY BY KRASNOSELSKII'S THEOREM

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Abstract. This is a paper in a series of investigations into the use of fixed point theorems to prove stability. Here, we use a modification of a fixed point theorem of Krasnoselskii. The work concerns a scalar functional differential equation $x' = -a(t)x^3 + b(t)x^3(t - r(t))$ where $r(t)$ need be neither bounded nor differentiable, while a and b can be unbounded. Such problems have proved very challenging in the theory of Liapunov's direct method. We show that it fits very nicely into the framework of the modified Krasnoselskii theorem so that asymptotic stability is readily concluded.

1. The equation and the fixed point theorem.

This paper addresses two problems. First, in the application of Liapunov's direct method to problems of stability in delay differential equations, serious difficulties occur if the functions in the equations are unbounded with time (see Hatvani [8]), if the delay is unbounded (see Seifert [12] which also is pertinent for problems with Razumikhin techniques), or if the derivative of the delay is not small (see [2,9,14]). Here, we demonstrate by means of a fully nonlinear example how all of those problems can be avoided by means of fixed point theory. A modern look at construction of Liapunov functionals for linear equations, with applications to nonlinear problems, is found in Zhang [15].

Next, this is a continuation of a series of papers in which we have addressed some of those problems by means of fixed point theory. To effectively use fixed point theory it is usually necessary to invert the differential equation, obtaining a mapping equation whose

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fixed points solve the original problem. In [4] we presented many examples when there was a nontrivial linear term so that the variation of parameters could be used to obtain an effective mapping equation. The present problem is totally nonlinear and we resort to a very old idea of adding and subtracting a linear term for the mapping. The interesting part is that the added term does two things. First, it destroys a contraction already present in part of the equation. But it replaces it with what we have called a large contraction that is fully effective in the fixed point theory.

When a function is written without its argument, that argument is t.

The following concise equation exhibits all of those problems. The solution offers a general guide as to how they may be met. We consider the equation

$$x' = -a(t)x^3 + b(t)x^3(t - r(t)) \tag{1}$$

in which $r(t) \geq 0$, a, b, r are continuous, $0 \leq r(t) \leq t$. In order to motivate our conditions and to put the problem in some context, we will look at some results in the greatly simplified case of r being a positive constant.

I. The simplest case occurs when both a and b are bounded, as may be seen in Hale [7; p. 117]. Thus, we start our motivation one step beyond by letting $a(t)$ be unbounded and asking that

$$b(t) \text{ is bounded }, r(t) = r \text{ a constant, and } a(t) \geq |b(t + r)| + k$$

for some $k > 0$. Define a Liapunov functional by

$$V(t, x_t) = |x(t)| + \int_{t-r}^{t} |b(s + r)|x^3(s)|ds$$

which has a derivative along solutions of (1) satisfying

$$V'(t, x_t) \leq -a(t)|x^3(t)| + |b(t)||x^3(t - r)| + |b(t + r)||x^3(t)| - |b(t)|x^3(t - r)| \leq -k|x^3|.$$

Then the argument in a result of Burton [1] yields uniform asymptotic stability of the zero solution.

II. We can let $b(t)$ be unbounded and retain the conclusion if we increase the conditions on $a(t)$; and this will lead to a very important compactness condition in our main result. We suppose that $r(t) = r$ is a positive constant and that there is a $k > 0$ with

$$-a(t) + (1/4)|b(t)|^2 + 1 \leq -k$$

for all $t \geq 0$. Then

$$V(t, x_t) = (1/4)x^4(t) + \int_{t-r}^{t} x^6(s)ds$$

yields

$$\begin{aligned} V'(t, x_t) &\leq -a(t)x^6 + (|b(t)|/\sqrt{2})|x|^3\sqrt{2}|x^3(t-r)| + x^6 - x^6(t-r) \\ &\leq -a(t)x^6 + (1/4)|b(t)|^2x^6 + x^6(t-r) + x^6 - x^6(t-r) \\ &\leq [-a(t) + (1/4)|b(t)|^2 + 1]x^6 \\ &\leq -kx^6. \end{aligned}$$

Again uniform asymptotic stability follows from the proof given in Burton [1].

In our problem we will allow both a and b to be unbounded and, consistent with II above, ask that $b(t)/a(t) \to 0$ as $t \to \infty$. In the next section we present a far more compelling reason for this assumption.

A few years ago we began investigating the possibility of circumventing such difficulties by fixed point theory. The problems in [4] depended, for the most part, on the differential equation having a nontrivial linear term so that the variation of parameters formula could be employed to construct a mapping equation suitable for the contraction mapping theorem.

But our equation (1) has no such linear term and here we resort to adding and subtracting a linear term. In principle one believes that this should not work because the extraneous linear term would strongly dominate these nonlinear terms near zero, effectively destroying the original stability properties. In fact, that does not happen and the problem seems made to order for the fixed point theorem.

According to Smart [13; p. 31], Krasnoselskii studied a 1932 paper of Schauder [11] on partial differential equations and formulated the working hypothesis: **The inversion of a perturbed differential operator yields the sum of a contraction and compact map**. Accordingly, he formulated the following theorem (cf. [10] or [13; p. 31]).

Theorem 1. *Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map M into S such that*

(i) $Ax + By \in M (\forall x, y \in M)$,

(ii) *A is continuous and AM is contained in a compact set,*

(iii) *B is a contraction with constant $\alpha < 1$.*

Then there is a $y \in M$ with $Ay + By = y$.

It turns out that the theorem can be more useful with certain changes. First, it seems that it is too much to ask that B be a contraction. Next, it is too much to ask that the work takes place in the full Banach space. We studied the problem in [3]. Here are the critical results from that paper.

Definition. Let (M, ρ) be a metric space and $B : M \to M$. B is said to be a *large contraction* if φ, $\psi \in M$, with $\varphi \neq \psi$ then $\rho(B\varphi, B\psi) < \rho(\varphi, \psi)$ and if $\forall \varepsilon > 0\ \exists \delta < 1$ such that $[\varphi, \psi \in M,\ \rho(\varphi, \psi) \geq \varepsilon] \Rightarrow \rho(B\varphi, B\psi) \leq \delta\rho(\varphi, \psi)$.

Theorem 2. *Let (M, ρ) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and an $L > 0$, such that $\rho(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.*

Theorem 3. *Let $(S, \|\cdot\|)$ be a Banach space, M a bounded, convex nonempty subset of S. Suppose that A, $B : M \to M$ and that*

(i) $$x, y \in M \Rightarrow Ax + By \in M,$$

(ii) *A is continuous and AM is contained in a compact subset of M,*

(iii) *B is a large contraction.*

Then $\exists\, y \in M$ *with* $Ay + By = y$.

We will use this theorem to obtain a stability result. In an earlier paper [5] we obtained stability by means of a different modification of Krasnoselskii's theorem. In that paper we integrated the equation directly, without adding a linear term, producing an entirely different technique.

The following example illustrates the definition and also provides the details of work taking place in an integrand in the proof of Lemma 2.

Example. If $\|\cdot\|$ is the supremum norm, if $M = \{\phi : [0,\infty) \to R | \phi \in C, \|\phi\| \le \sqrt{3}/3\}$, and if $(H\varphi)(t) = \varphi(t) - \varphi^3(t)$, then H is a large contraction of the set M.

Proof. In the following computation, φ, ψ are evaluated at each t. We have $D := \big|H\varphi - H\psi\big| = \big|\varphi - \varphi^3 - \psi + \psi^3\big| = \big|\varphi - \psi\big|\, \big|1 - (\varphi^2 + \varphi\psi + \psi^2)\big|$. Then for

$$|\varphi - \psi|^2 = \varphi^2 - 2\varphi\psi + \psi^2 \le 2(\varphi^2 + \psi^2)$$

and for $\varphi^2 + \psi^2 < 1$ we have

$$\begin{aligned} D &\le \big|\varphi - \psi\big| \left[1 + |\varphi\psi| - (\varphi^2 + \psi^2)\right] \\ &\le \big|\varphi - \psi\big| \left[1 + \frac{\varphi^2 + \psi^2}{2} - \big(\varphi^2 + \psi^2\big)\right] \\ &= \big|\varphi - \psi\big| \left[1 - \frac{\varphi^2 + \psi^2}{2}\right]. \end{aligned}$$

What we have shown is that pointwise we have a large contraction. It is easy to see that this implies a large contraction in the supremum norm.

For a given $\epsilon \in (0,1)$, let $\phi, \psi \in M$ with $\|\phi - \psi\| \ge \epsilon$.

a) Suppose that for some t we have

$$\epsilon/2 \le |\phi(t) - \psi(t)|$$

so that

$$(\epsilon/2)^2 \le |\phi(t) - \psi(t)|^2 \le 2(\phi^2(t) + \psi^2(t))$$

or

$$\phi^2(t) + \psi^2(t) \ge \epsilon^2/8.$$

For all such t we have

$$|(B\phi)(t) - (B\psi)(t)| \le |\phi(t) - \psi(t)|[1 - \frac{\epsilon^2}{8}]$$

$$\le \|\phi - \psi\|[1 - \frac{\epsilon^2}{8}].$$

b) Suppose that for some t we have $|\phi(t) - \psi(t)| \le \epsilon/2$. Then

$$|(B\phi)(t) - (B\psi)(t)| \le |\phi(t) - \psi(t)| \le (1/2)\|\phi - \psi\|.$$

Thus, for all t we have

$$|(B\phi)(t) - (B\psi)(t)| \le \min[1/2, 1 - \frac{\epsilon^2}{8}]\|\phi - \psi\|.$$

2. Convergence of solutions to zero

Consider again

$$x' = -a(t)x^3 + b(t)x^3(t - r(t)) \tag{1}$$

in which $r(t) \ge 0$, the functions a, b, r are continuous, and there is a $J > 1$ with

$$J|b(t)| \le a(t) \text{ and } \int_0^\infty a(s)ds = \infty. \tag{2}$$

In order to show that solutions of (1) with small initial functions tend to zero using Theorem 3, we define a mapping equation by first writing

$$x' + a(t)x = a(t)x - a(t)x^3(t) + b(t)x^3(t - r(t)) \tag{3}$$

and then using the variation of parameters formula to write

(4)

$$x(t) = x_0 e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} a(s)[x(s) - x^3(s)]ds + \int_0^t e^{-\int_s^t a(u)du} b(s)x^3(s - r(s))ds,$$

where we take the first two terms on the right as Bx and the last as Ax, so that (4) is expressed as

$$x = Bx + Ax. \tag{5}$$

We will define B and A more precisely in a moment. There is an assumed continuous initial function ψ on some initial interval $[-R, 0]$ with $\psi(0) = x_0$. Here, $[-R, 0] = \{u \leq 0 | u = t - r(t), t \geq 0\}$. A solution is then denoted by $x(t, 0, \psi)$ which is continuous, agrees with ψ on the initial interval, satisfies (1) for $t \geq 0$, and which may have a discontinuity in its derivative whenever $t - r(t) = 0$, as discussed later.

Definitions of S, M, A, and B

We will do the work in two steps. First, we show that solutions starting in a certain set tend to zero. That work will use the supremum norm and will assume that $b(t)/a(t) \to 0$ as $t \to \infty$ which allows us to show that a certain set is compact. Then we will work in a space with weighted norm and prove a Liapunov stability result without that condition on $b(t)/a(t)$ since compactness will rest on the equi-continuity.

With Theorem 3 in mind we let S be the Banach space of bounded continuous functions $\phi : [0, \infty) \to R$ with the supremum norm $\| \cdot \|$ and define the set

$$M = \{\phi \in S | |\phi(t)| \leq L, |\phi(t)| \to 0 \text{ as } t \to \infty\} \tag{6}$$

where $L = \sqrt{3}/3$. Denote the initial function by ψ and its maximum on $[-R, 0]$ by $\|\psi\|$, which should not cause confusion with the same symbol denoting the norm in S. Then we require that ψ be chosen so that

$$\|\psi\| + (2\sqrt{3}/9) + (\sqrt{3}/[9J]) \leq \sqrt{3}/3. \tag{8}$$

We will use this relation in Lemma 2 and define a $\delta > 0$ so that if $\epsilon = \sqrt{3}/3$ and if $\|\psi\| < \delta$, then the solution satisfies $|x(t, 0, \psi)| < \epsilon$. We treat the more general $\epsilon - \delta$ stability relation in the next section.

Now, with M defined, if $\phi \in M$ and if $x_0 = \psi(0)$ then

$$(B\phi)(t) = x_0 e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} a(s)(\phi(s) - \phi^3(s))ds.$$

Note that B would have been a contraction for small ϕ had $a(t)\phi(t)$ not been present. But we will show that B is a large contraction for small ϕ. The reader should take careful note of the changes in contraction properties; this is exactly the typical occurrence when a linear term is added to both sides of an equation which had a "stable" local contraction before the addition.

Next, for $\phi \in M$, extend ϕ to $[-R, \infty)$ by defining $\phi(t) = \psi(t)$ on $[-R, 0)$ so that

$$(A\phi)(t) = \int_0^t e^{-\int_s^t a(u)du} b(s)\phi^3(s - r(s))ds \tag{9}$$

is defined for $t \geq 0$.

If there is a fixed point ϕ for the mapping $P\phi = A\phi + B\phi$, then $x(t, 0, \psi) = \phi(t)$ for $t \geq 0$, $x(t, 0, \psi) = \psi(t)$ on $[-R, 0]$, and $x(t, 0, \psi)$ satisfies (1) for $t > 0$, whenever its derivative exists.

NOTE. For functional differential equations we always expect a discontinuity in the derivative of the continuous solution at $t = t_0$; but here we would expect a possible discontinuity in the derivative of the solution at each point where

$$t - r(t) = 0. \tag{10}$$

We say "possible discontinuity" because $r(t) = t$ for all t, for example, would not yield discontinuities for $t > 0$.

Fulfillment of (i), (ii), (iii) in Theorem 3

Here is our main result. It will require the additional assumption that

$$b(t)/a(t) \to 0 \text{ as } t \to \infty. \tag{11}$$

We had discussed this central requirement in the introduction and we now add some substance to it. In [12] Seifert studies asymptotic stability of functional differential equations by means of Razumikhin techniques which can be very fruitful in proving Liapunov stability for problems in the general class of (1). But he warns of the difficulties in proving asymptotic stability when $t - r(t)$ fails to tend to infinity; and his main result on asymptotic stability asks that along each solution the main action of the delay be on $[t - r, t]$ for r constant. He has an example of great difficulties encountered when $r(t) = t$. If we look at a linear problem parallel to (1) with $r(t) = t$ we would have

$$x' = -a(t)x + b(t)x(0)$$

with solution

$$x(t) = x(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} b(s)x(0)ds$$

$$= x(0)e^{-\int_0^t a(s)ds} + x(0)\int_0^t e^{-\int_s^t a(u)du} a(s)(b(s)/a(s))ds.$$

It is readily shown that $x(t) \to 0$ as $t \to \infty$ in case $b(t)/a(t) \to 0$ as $t \to \infty$. On the other hand, if $a(t)/b(t)$ is bounded away from zero, then $x(t)$ does not tend to zero. Hence, (11) is critical to our proof and it is critical to the validity of the theorem. There are cases in between: we might get by with $\liminf_{t\to\infty} b(t)/a(t) = 0$.

This is more than simply a justification of the condition. All of this work was motivated by the difficulties encountered in studying this problem via Liapunov's direct method. In that method the real difficulties occur because of unboundedness of the delay and the properties of the derivative of $r(t)$; those difficulties with the derivative are not seen at all in our work here.

Theorem 4. *Let (2) and (11) hold. If $L = \sqrt{3}/3$ and if ψ is the initial function satisfying (8), $\psi \in C$, then there is a solution $x(t, 0, \psi)$ of (1) with $|x(t, 0, \psi)| < L$ for $t \geq 0$ and $x(t, 0, \psi) \to 0$ as $t \to \infty$.*

The proof is based on four lemmas.

Lemma 1. *For A defined in (5), if $\phi \in M$ then $|(A\phi)(t)| \leq L^3/J \leq L$. Moreover, $(A\phi)(t) \to 0$ as $t \to \infty$.*

Proof. We have

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)||\phi^3(s - r(s))|ds$$

$$\leq (L^3/J)\int_0^t e^{-\int_s^t a(u)du}a(s)ds \leq L^3/J,$$

as required.

Let $\phi \in M$ be fixed. We will show that $(A\phi)(t) \to 0$. For a given $\epsilon > 0$ we can find T such that $|\phi^3(t - r(t))| < \epsilon$ for $t \geq T$. We then have

$$|(A\phi)(t)| \leq \int_0^T e^{-\int_s^t a(u)du}a(s)ds(L^3/J) + \int_T^t (\epsilon/J)e^{-\int_s^t a(u)du}a(s)ds$$

$$\leq (L^3/J)e^{-\int_T^t a(u)du} + (\epsilon/J).$$

The result follows from this.

Lemma 2. *For A, B defined in (5) and ψ satisfying (8), if $y \in M$ is fixed, but arbitrary, then the mapping $Bx + Ay : M \to M$ and B is a large contraction on M with a unique fixed point in M.*

Proof. Using the definition of B, the result of Lemma 1, and the fact that $|x| \leq \sqrt{3}/3$ implies $|x - x^3| \leq (2\sqrt{3})/9$ we obtain

$$|Bx + Ay| \leq |x_0| + (2\sqrt{3}/9)\int_0^t e^{-\int_s^t a(u)du}a(s)ds + L^3/J$$

$$\leq \|\psi\| + (2\sqrt{3}/9) + (\sqrt{3}/[9J]) \leq \sqrt{3}/3$$

by (8). Note that $0 \in M$ so we have proved that $B : M \to M$.

To see that $B : M \to M$ is a large contraction on M, we note that our example after Theorem 3 showed that $\phi - \phi^3$ is a large contraction within the integrand; when that term is taken outside the integral as a supremum, then the resulting integral is bounded by 1. Thus for the ϵ of the proof of that example, we have

$$\begin{aligned}\|B\phi_1 - B\phi_2\| &\leq \int_0^t e^{-\int_s^t a(u)du} a(s)ds \|\phi_1 - \phi_2\|\delta \\ &\leq \|\phi_1 - \phi_2\|\delta,\end{aligned}$$

as required.

We already know that $Ay \to 0$ as $t \to \infty$ and the proof that $Bx \to 0$ is just the same.

Lemma 3. *The mapping $A : M \to M$ is continuous in the supremum norm.*

Proof. If $\phi_1, \phi_2 \in M$ then there are positive constants J and K with

$$\begin{aligned}&|(A\phi_1)(t) - (A\phi_2)(t)| \\ &= \left| \int_0^t e^{-\int_s^t a(u)du} b(s)[\phi_1^3(s - r(s)) - \phi_2^3(s - r(s))]ds \right| \\ &\leq \|\phi_1 - \phi_2\|(1/J) \int_0^t e^{-\int_s^t a(u)du} a(s)ds \\ &\leq K\|\phi_1 - \phi_2\|.\end{aligned}$$

Lemma 4. *The operator A maps M into a compact subset of M.*

Proof. Let $\phi \in M$ and let $0 \leq t_1 < t_2$ so that

$$\begin{aligned}&|(A\phi)(t_2) - (A\phi)(t_1)| \\ &= \left| \int_0^{t_2} e^{-\int_s^{t_2} a(u)du} b(s)\phi^3(s - r(s))ds - \int_0^{t_1} e^{-\int_s^{t_1} a(u)du} b(s)\phi^3(s - r(s))ds \right| \\ &= \Bigg| \int_0^{t_1} e^{-\int_s^{t_1} a(u)du} b(s)\phi^3(s - r(s))[e^{-\int_{t_1}^{t_2} a(u)du} - 1]ds\end{aligned}$$

$$+\int_{t_1}^{t_2} e^{-\int_s^{t_2} a(u)du} b(s)\phi^3(s-r(s))ds\Big|$$

$$\leq (L^3/J)\int_0^{t_1} e^{-\int_s^{t_1} a(u)du} a(s)ds\Big|e^{-\int_{t_1}^{t_2} a(u)du}-1\Big| + (L^3/J)\Big|e^{-\int_{t_1}^{t_2} a(u)du}-1\Big|$$

$$\leq 2(L^3/J)\Big|e^{-\int_{t_1}^{t_2} a(u)du}-1\Big|.$$

Hence, AM is equi-continuous. Next, we notice that for arbitrary $\phi \in M$ we have

$$|(A\phi)(t)| \leq \int_0^t e^{-\int_s^t a(u)du} a(s)|b(s)/a(s)|L^3 ds$$

$$=: c(t)$$

where $c(t) \to 0$ as $t \to \infty$ by a proof like that of Lemma 1, because of the assumption that $|b(t)|/a(t) \to 0$ as $t \to \infty$. This, and the equi-continuity will imply the conclusion (see Burton-Furumochi [6]).

The conditions of Theorem 3 are satisfied and there is a $\phi \in M$ with $\phi = A\phi + B\phi$, a solution of (1).

NOTE. The set AM contains the solution and so from $c(t)$ we see that we actually prove equi-asymptotic stability. If $b(t)/a(t) \to 0$ in a certain uniform way, we could prove uniform asymptotic stability.

4. Stability and compactness

We have mentioned before that Razumikhin techniques can be very effective in proving Liapunov stability for problems like (1). While one may show that the zero solution of (1) is Liapunov stable using a Razumikhin technique, we know of no way to prove that solutions tend to zero except by the method presented. As we are trying to give a general discussion of how fixed point theory will yield stability, it is still worth the space to give a short discussion here.

If all we ask is stability (not asymptotic stability) then we can avoid (11) and still use Krasnoselskii's theorem. In that case we would achieve the compactness by using a different norm. Any C_g-space will work.

Let $g(t) = t + 1$ for $t \geq 0$ and define $(S, |\cdot|_g)$ to be the Banach space of continuous functions $\phi : [0, \infty) \to R$ for which

$$|\phi|_g := \sup_{t \geq 0} |\phi(t)|/g(t)$$

exists. Continue to use $\|\cdot\|$ as the supremum norm of any $\phi \in S$, provided ϕ is bounded. Also, continue to use $\|\psi\|$ as the bound on an initial function, as before.

Theorem 5. *If the conditions of Theorem 4 hold, except for (11), then the zero solution of (1) is stable.*

Proof. We give the proof for solutions starting at $t_0 = 0$. Refer to (8) and the proof of Lemma 2. If $\epsilon > 0$ is given with $0 < \epsilon < \sqrt{3}/3$, then for $|x| \leq \epsilon$, find a δ^* with $|x - x^3| \leq \delta^*$; then we will ask that

$$\|\psi\| + \delta^* + [\epsilon^3/J] \leq \epsilon.$$

To verify that the last inequality allows $\|\psi\| > 0$, note that for $0 \leq x \leq \epsilon < \sqrt{3}/3$, the function $x - x^3$ is increasing so $0 \leq x - x^3 \leq \epsilon - \epsilon^3 =: \delta^*$. This will yield a $\delta > 0$ so that $\|\psi\| < \delta$ is the requirement on the initial function.

We now construct a set

$$M = \{\phi \in S | \|\phi\| \leq \epsilon\}.$$

Define A and B as before. We easily verify that B is a large contraction on M and that $Ax + By : M \to M$, just as before. AM is an equi-continuous set and in the $g-norm$ it is contained in a compact subset of M. Moreover, both A and B are continuous in the $g-$norm. The mapping has a fixed point satisfying (1) and lying in M.

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