Communications in Applied Analysis **17** (2013), no. N, xxx–xxx

VOLTERRA'S LINEAR EQUATION AND KRASNOSELSKII'S HYPOTHESIS

T. A. BURTON[1] AND I. K. PURNARAS[2]

[1]Northwest Research Institute, 732 Caroline St., Port Angeles, WA 98362, USA
E-mail: taburton@olypen.com

[2]Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail: ipurnara@uoi.gr

Dedicated to the Memory of Professor V. Lakshmikantham

ABSTRACT. This paper is part of a larger study suggested by Krasnoselskii some sixty years ago concerning the unification of the broad area of differential equations. The larger study included fractional differential equations, neutral functional differential equations, and initial investigation of a problem discussed by Volterra in 1928 which is still used to describe many real-world problems. This paper continues that study of Volterra's problem in the linear case. Using two transformations we obtain an integral equation which defines a mapping that is automatically compact. Just like Brouwer's fixed point theorem for $\Re^n$, when this map is continuous and takes a closed, bounded, convex set $M = \{\phi \in BC([0,\infty)) : a \le \phi \le b\}$ for some $a < b$, into itself, then there is a fixed point.

AMS (MOS) Subject Classification. 34A08, 45J05, 34K40, 47H10.

Key words and phrases. fixed points, fractional differential equations, integral equations.

1. INTRODUCTION

This is the second paper in which we study an idea of Krasnoselskii concerning the unification of the broad area of differential equations. The study started in [5]. Our focus here concerns solutions to the forced linear integrodifferential equation

$$x'(t) = -\int_0^t D(t-s)x(s)ds + f(t), \ \ t \ge 0, \tag{E}$$

where $f, D : [0,\infty) \to \mathbb{R}$ are continuous with

$$D(t) > 0, \quad \int_0^\infty D(t)dt < \infty. \tag{1.1}$$

Received xxxx xx, 2013

In 1928 Volterra [18] noted that many real-world problems were being modeled by (E) with a convex or completely monotone kernel. That observation continues to this day and we see classical and modern problems including nuclear reactors, heat flow, fluid flow, as well as many other problems discussed in [11], [12], [13], [14], [15], and [17] all modeled by (E).

The case in which the kernel has compact support also has interesting applications to one-dimensional viscoelasticity and reactor dynamics [7, p. 121] and neural networks [1]. Volterra's paper contains an application to mathematical biology which continues to be interesting and a paper by Levin and Nohel [16] covers work of Ergen on nuclear reactors as well as several other important problems. The problems with compact support are not covered here, but are closely related to this work.

Volterra conjectured that if the kernel, D, is convex then a Liapunov functional could be constructed, which would establish stability and long-term qualitative behavior of solutions. Levin [11] constructed several forms of that Liapunov functional for convex kernels starting in 1963 and those are still widely used by investigators. The convexity facilitates such a totally elementary proof of boundedness and asymptotic properties of solutions that investigators have eagerly used it.

But convexity is an assumption which is virtually impossible to verify by inspection of real-world problems. We must verify that $D(t) \geq 0$, $D'(t) \leq 0$, and $D''(t) \geq 0$; if any of those conditions fail, the conclusion is completely lost. Meaningful results must rely on far less stringent assumptions, regardless of the effort required to supply proofs.

While Volterra's conjecture of the existence of a Liapunov functional was correct, the Liapunov functional was not robust in any sense. If we add perturbations to take into account errors in observation concerning the very delicate conditions on the derivatives of D, then the Liapunov arguments collapse.

In the present work we offer conditions resting only on the average value of D in terms of an integral.

It is worth taking a moment to contrast this with an old parallel problem. We employ Newton's second law of motion to model a spring-mass-dashpot system (or its counterpart electrical circuit) and obtain an equation

$$x'' + cx' + kx = 0,$$

where c is a positive coefficient of friction and k is a positive spring constant. Knowing that friction will vary with time, position, and velocity, while the restoring force will be a function of the position, in the 18^{th} century Lagrange considered a perturbed equation

$$x'' + f(t, x, x')x' + g(x) = 0,$$

where $f > 0$ and $xg(x) > 0$ when $x \neq 0$. He constructed what was, in effect, the first Liapunov function

$$V(x, x') = \frac{1}{2}(x')^2 + \int_0^x g(s)ds,$$

a positive definite function whose time derivative along solutions was non-positive. It is the sum of the kinetic and potential energy which is a minimum at zero. We continue to work with that Liapunov function to this day showing that the solutions of the nonlinear equation are well-represented by solutions of the linear equation. Most unfortunately, Levin's Liapunov functional does not help us in the same way with (E); some mild forcing functions have been allowed. One may consult [15] for a discussion of the potential and kinetic energy of (E).

The present work starts with an idea of Krasnoselskii [8] some sixty years ago. He studied an old paper by Schauder on elliptic partial differential equations and deduced a general hypothesis which we formalize as follows.

Krasnoselskii's Hypothesis *The inversion of a perturbed differential operator yields the sum of a contraction and a compact map.*

Lest we err in thinking that this concerns only the single elliptic equation, note that Krasnoselskii then obtained the following general result.

Theorem 1.1 (Krasnoselskii)**.** *Let $\|\cdot\|$ be a norm and $(\mathcal{S}, \|\cdot\|)$ be a Banach space. Let $M \subset \mathcal{S}$ be closed, convex, nonempty. Define A, $B : M \to \mathcal{S}$ with A continuous,*

$$x, y \in M \Rightarrow Ax + By \in M, \tag{i}$$

$$AM \text{ resides in a compact set }, \tag{ii}$$

$$B \text{ is a contraction} \tag{iii}$$

with constant $\alpha < 1$. Then $\exists\ y \in M$ with $Ay + By = y$.

The big problem is to invert the perturbed differential operator so as to fulfill the conditions of this theorem, and that has been very challenging in applications over the last many years. Condition (i) has proved to be difficult, while showing that A is a compact map has also often been difficult especially when we want a solution on an infinite interval. In a most interesting way, both difficulties vanish when we invert the operators in the way introduced in [2] and which was used in a number of subsequent papers.

In a recent paper [5] we began a study putting Krasnoselskii's Hypothesis to the test. We began with a scalar fractional differential equation of Caputo type

$${}^cD^q x(t) = u(t, x(t)), \quad 0 < q < 1, \quad x(0) \in \Re, \quad t \geq 0,$$

with u continuous. Using theory found in [9, p. 54] and [10], we inverted it as an integral equation with large kernel

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s, x(s)) ds, \quad t \geq 0.$$

That was then transformed into an integral equation

$$x(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[x(s) + \frac{u(s,x(s))}{J}\right] ds,$$

with a completely monotone kernel satisfying $\int_0^\infty R(t)dt = 1$. Under general conditions it turned out that the last integral equation defined a compact map which is so simple that it is an extended form of Brouwer's theorem [3]. Such fractional differential equations represent a myriad of real-world problems in the form of many types of ordinary and partial differential equations. In short, Krasnoselskii's Hypothesis seems to work very well for such equations.

We moved then to some general neutral functional differential equations widely used in the literature concerning problems in mathematical biology and showed that the same kind of transformations produced similar inversion, although both contractions and compact maps were used; again, the compactness of the maps was automatic, even on infinite time intervals. Krasnoselskii's Hypothesis continued to be valid.

We then turned to the problem of Volterra (E) and, again, the inversion led to the same type of problem as in the case of the fractional differential equations. For lack of space we considered only a simple form of (E). This paper is now devoted to finishing that problem in the linear case. The nonlinear case is also under investigation. With the context and history now complete, we return to (E).

Following the same steps as in [5], we integrate (E), divide and multiply by $J > 0$ (with J being an arbitrary positive number), add and subtract $x(s)$ to obtain

$$\begin{aligned} x(t) &= x(0) \\ &\quad - \int_0^t J\left[x(s) - x(s) - \frac{f(s)}{J} + \frac{\int_0^s D(s-u)x(u)du}{J}\right] ds, \quad t \geq 0. \end{aligned}$$

Write the linear part as

$$z(t) = x(0) - \int_0^t Jz(s)ds, \quad t \geq 0,$$

so that there is a resolvent equation

$$R(t) = J - \int_0^t JR(s)ds, \quad t \geq 0,$$

with solution

$$R(t) = Je^{-Jt}, \quad t \geq 0,$$

which is completely monotone and satisfies

$$\int_0^\infty R(s)ds = 1.$$

We then have

$$z(t) = x(0)\left[1 - \int_0^t R(s)ds\right], \quad t \geq 0,$$

and by a variation of parameters formula

$$\begin{aligned} x(t) &= z(t) + F(t) \\ &+ \int_0^t R(t-s)\left[x(s) - \frac{\int_0^s D(s-u)x(u)du}{J}\right] ds,\ t \geq 0, \end{aligned} \tag{1.2}$$

where

$$F(t) = \int_0^t R(t-s)[f(s)/J]ds, \quad t \geq 0.$$

Now, we prepare the integrand in (1.2):

$$\begin{aligned} &\int_0^t R(t-s)\int_0^s \frac{D(s-u)}{J}x(u)duds \\ &= \int_0^t \int_u^t R(t-s)\frac{D(s-u)}{J}x(u)dsdu \\ &= \int_0^t \int_u^t R(t-s)\frac{D(s-u)}{J}dsx(u)du, \end{aligned}$$

so (1.2) is written as

$$\begin{aligned} x(t) &= z(t) + F(t) \\ &+ \int_0^t \left[R(t-u)x(u) - \int_u^t R(t-s)\frac{D(s-u)}{J}dsx(u)\right] du, \quad t \geq 0. \end{aligned} \tag{1.3}$$

Let

$$\begin{aligned} Ax\,(t) &:= z(t) + F(t) \\ &+ \int_0^t \left[R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds\right] x(u)du,\ t \geq 0, \end{aligned} \tag{1.4}$$

and note that as $R(t) = Je^{-Jt}$, $t \geq 0$, we have

$$\begin{aligned} z\,(t) &= x(0)\left[1 - \int_0^t R(s)ds\right] \\ &= x(0)e^{-Jt}, \quad t \geq 0, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} F(t) &= \int_0^t R(t-s)[f(s)/J]ds = \int_0^t e^{-J(t-s)}f(s)ds \\ &= e^{-Jt}\int_0^t e^{Js}f(s)ds, \quad t \geq 0. \end{aligned} \tag{1.6}$$

The space used here is that of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm, $(BC, \|\cdot\|)$, and we denote by M the set $M = \{\phi \in BC : \|\phi\| \leq 1\}$. We make use of the following result used in [5] (see, also, [3] and its correction [4]). The correction concerns only the set M in Theorems 3.1 and 4.1 (iii). It asks that the set M be exactly as described in the result here. This corrects a statement in the middle of the proof of Theorem 4.1 (iii) stating that M is closed in the weighted space. With this change, the statement is true and the proof is elementary. Note that no compactness of the operator A is required but it depends on the mapping being defined by an integral equation with the kernel as described for (2.1).

Theorem 1.2 (Brouwer-Schauder type). *Let $(BC, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm and let $M = \{\phi \in BC : a \leq \phi(t) \leq b\}$ for some $a < b$ and all $t \in [0,\infty)$, and let $u(t,x)$ be the right-hand-side of (E). Suppose that $x(0)$ and M are chosen so that for A defined in (1.4) then $A : M \to M$. If A is continuous, if F is uniformly continuous, and if there is an $L > 0$ so that $|u(t,x)| \leq L$, for $t \geq 0$, $x \in M$, then A has a fixed point in M.*

In Section 2 we present the main results of the paper. Application of these results to the case of the unforced linear equation along with some remarks are given in Section 3. Section 4 consists of the proofs of Theorem 2.1 and Lemma 1.

2. THE MAIN RESULTS

In view of (1.5) and (1.6), equation (1.3) becomes

$$\begin{aligned} x(t) &= x(0)e^{-Jt} + e^{-Jt}\int_0^t e^{Js}f(s)ds \\ &+ \int_0^t \left[Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D(s-u)ds \right] x(u)du,\ t \geq 0. \end{aligned} \tag{2.1}$$

Main Assumption: With reference to Theorem 1.2 we always assume that F is uniformly continuous on $[0,\infty)$. This holds, for example, if f is bounded. A proof can be patterned after Theorem 5.1 of [6].

In addition, we will later be needing

$$e^{-Jt}\left| \int_0^t e^{Js}f(s)ds \right| \to 0. \tag{2.2}$$

Theorem 2.1. *If*

$$\begin{aligned} &\left(|x(0)| + \left| \int_0^t e^{Js}f(s)ds \right| \right) e^{-Jt} \\ &+ \int_0^t \left| Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D(s-u)ds \right| du \leq 1, \quad t \geq 0, \end{aligned} \tag{2.3}$$

then the mapping defined by (2.1) on the set M maps $M \to M$, and Th. 1.2 will give a fixed point in M.

Proof. The natural mapping of M into itself from (2.1) is

$$\begin{aligned}(P\phi)(t) &= x(0)e^{-Jt} + e^{-Jt}\int_0^t e^{Js} f(s)ds \\ &\quad + \int_0^t \left[Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u)ds \right] \phi(u)du, \quad t \geq 0.\end{aligned}$$

With (2.3) holding, it is easily verified that $\|P\phi\| \leq 1$. Then, in view of (2.2), one can see that for the right-hand-side u of (E) (hence, of (1.4)), we have that $|u(t,x)|$ is bounded for $t \geq 0$, $x \in M$. The existence of the fixed point is now exactly as in Theorem 1.2. □

The crucial part of Theorem 2.1, above, is inequality (2.3). The next lemma attempts to shed some light on that inequality. The proof of the lemma can be found in the Appendix.

Lemma 1. *Let $J > 0$ and suppose that*

$$\left| \int_0^t e^{Js} f(s)ds \right| \leq \int_0^t \int_u^t e^{Js} D(s-u)dsdu, \quad t \geq 0. \tag{2.4}$$

(i) If

$$\int_0^\infty e^{Jv} D(v)\, dv \leq J, \tag{2.5}$$

then (2.3) is always true.

(ii) If

$$J < \int_0^\infty e^{Jv} D(v)\, dv, \tag{2.6}$$

then (2.3) is equivalent with

$$\begin{aligned} &J\left(|x(0)| + \left| \int_0^t e^{Js} f(s)ds \right| \right) e^{-Jt} \\ &+ \int_k^t \left[1 - e^{-J(t-s)} \right] D(s)\, ds \leq \int_0^k D(s)\, ds, \quad t \geq k, \end{aligned} \tag{2.7}$$

where k is the unique solution of

$$J = \int_0^k e^{Jv} D(v)\, dv, \quad k \in (0, \infty). \tag{2.8}$$

By Lemma 1, if (2.4) and (2.5) hold, so does (2.3) and Theorem 1.2 gives a solution of (E). Proposition 1 below gives some sufficient conditions so that (2.5) is satisfied.

Proposition 1. *Assume that (2.4) holds true.*

(i) If there exists some $\mu > 2$ *such that* $D(t) \le e^{-\mu t}$, $t \ge 0$, *then (2.3) holds true for any* $J \in \left[\frac{\mu-\sqrt{\mu^2-4}}{2}, \frac{\mu+\sqrt{\mu^2-4}}{2}\right]$.

(ii) Let $J > 0$ *be given. Then (2.3) holds true for any* D *with* $D(t) \le e^{-\left(J+\frac{1}{J}\right)t}$, $t \ge 0$.

Proof. (i) For D with $D(t) \le e^{-\mu t}$, $t \ge 0$ where $\mu > 2$, we have for $t \ge 0$

$$\int_0^t e^{Jv} D(v)\, dv \le \int_0^t e^{(J-\mu)v} dv = \left[\frac{1}{J-\mu} e^{(J-\mu)v}\right]_0^t = \frac{1}{\mu - J}\left[1 - e^{(J-\mu)t}\right],$$

so for $J < \mu$

$$\lim_{t\to\infty} \int_0^t e^{Jv} D(v)\, dv \le \frac{1}{\mu - J}.$$

Thus, in order that (2.5) holds, it suffices to have

$$\frac{1}{\mu - J} \le J \Longleftrightarrow J^2 - \mu J + 1 \le 0,$$

which is satisfied if and only if $J \in \left[\frac{\mu-\sqrt{\mu^2-4}}{2}, \frac{\mu+\sqrt{\mu^2-4}}{2}\right]$.

(ii) Let $J > 0$ be given and D be such that $D(t) \le e^{-\left(J+\frac{1}{J}\right)t}$, $t \ge 0$. Then for $\mu = J + \frac{1}{J}$ we have $J^2 - \mu J + 1 = 0$, i.e., (2.5) is satisfied, so (2.3) holds true. Note that the greatest magnitude for D is $D(v) = e^{-2v}$ which is achieved for $J = 1$. □

For the case that (2.6) holds, Lemma 1 (ii) states that the crucial inequality (2.3) becomes (2.7). Under rather mild assumptions, the next proposition presents some necessary and some sufficient conditions so that (2.6) holds. We note that, in case that $f \equiv 0$, condition (2.9) holds by itself while condition (2.10) is not very difficult to be verified. The proof of this result is cited in the Appendix.

Proposition 2. *Let (2.4) and (2.6) hold and assume that*

$$\int_k^t e^{Js}\left[D(s) - J|f(s)|\right] ds + e^{Jt}|f(t)| \quad \text{is} \ \nearrow \ \text{on } [k,\infty), \tag{2.9}$$

and

$$J\left(|x(0)| + \left|\int_0^k e^{Js} f(s) ds\right|\right) e^{-Jk} \le \int_k^\infty D(s)\, ds. \tag{2.10}$$

(i) *If (2.2) holds and if k is defined in (2.8), then*

$$\frac{1}{2}\int_0^\infty D(v)\, dv \le \int_0^k D(v)\, dv, \tag{2.11}$$

is a sufficient condition so that (2.7) holds true.

(ii) *Condition (2.11) is necessary so that (2.7) holds true.*

Thus, if (2.2) holds then (2.11) is a necessary and sufficient condition so that (2.7) holds true.

In view of Proposition 2 and Lemma 1, we have the next result.

Proposition 2b. *Let (2.4), (2.6), (2.9) and (2.10) hold.*

(i) *If (2.2) holds, then (2.11) is a sufficient condition so that (2.3) holds true.*

(ii) Condition (2.11) is necessary so that (2.3) holds true.

(iii) If (2.2) holds then (2.11) is a necessary and sufficient condition so that (2.3) holds true.

A particular case of interest is when there exists, either some $\mu > 0$ such that the kernel D satisfies $D(t) \leq e^{-\mu t}$, $t \geq 0$, or some $\lambda > 0$ such that $D(t) \geq e^{-\lambda t}$, $t \geq 0$, then (2.11) can be replaced by a condition not containing the number k.

Before we present this result, we show that for a given $\lambda > 0$ there always exist a $J > 0$ such that the number

$$k_\lambda = \frac{\ln(J^2 - \lambda J + 1)}{J - \lambda},$$

is well defined and positive.

If $\lambda \in (0, 2)$, then the discriminant of $J^2 - \lambda J + 1$ is negative, so we have $J^2 - \lambda J + 1 > 0$, for all reals J. Moreover, $(J - \lambda)\ln(J^2 - \lambda J + 1) > 0$ for $J \neq \lambda$, thus k_λ is well defined and positive for any $J > 0$ with $J \neq \lambda$. Note that, for $\lambda = 2$, we have $(J - 2)\ln(J^2 - 2J + 1) = (J - 2)\ln[(J - 2)J + 1] > 0$ for any $J > 0$ with $J \neq 2$. It follows that for $\lambda \in (0, 2]$ the number k_λ is well defined and positive for any $J > 0$ with $J \neq \lambda$.

If $\lambda > 2$, then the discriminant is positive and $J^2 - \lambda J + 1$ has two positive roots, namely $\lambda_1 = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}$, $\lambda_2 = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}$, both roots being less than λ. Thus, for $J \in (0, \lambda_1) \cup (\lambda_2, \infty)$ we have $J^2 - \lambda J + 1 > 0$, so k_λ is well defined. Furthermore, choosing $J \in (0, \lambda_1)$ we have $\ln(J^2 - \lambda J + 1) = \ln[(J - \lambda)J + 1] < 0$ and $J - \lambda < 0$, while choosing $J > \lambda$ we have $\ln(J^2 - \lambda J + 1) > 0$ and $J - \lambda > 0$, hence, in both cases k_λ is positive. It follows that for $\lambda > 2$ the number k_λ is well defined and positive for any $J \in \left(0, \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}\right) \cup (\lambda, \infty)$.

In view of the above, we conclude that we may always choose J such that k_λ and k_μ are well defined and positive, and such a choice is assumed to hold in the proof of the proposition below.

Next, we note that, when $D(t) = e^{-\lambda t}$, then the positive number k_λ is the solution of

$$J = \int_0^{k_\lambda} e^{-\lambda s} e^{Js} ds.$$

Indeed, as

$$J=\int_0^{k_\lambda} e^{-\lambda s}e^{Js}ds=\frac{1}{J-\lambda}\left[e^{k_\lambda(J-\lambda)}-1\right],$$

solving for k_λ we take

$$\begin{aligned} J^2-\lambda J+1 &= e^{k_\lambda(J-\lambda)}, \\ k_\lambda(J-\lambda) &= \ln\left(J^2-\lambda J+1\right), \\ k_\lambda &= \frac{\ln(J^2-\lambda J+1)}{J-\lambda}. \end{aligned}$$

Proposition 3. *Let (2.6), (2.9) and (2.10) hold.*

(i) Assume that for some $\mu>0$ we have

$$D(t)\le e^{-\mu t},\quad t\ge 0,$$

and J is such that the number $k_\mu=\frac{\ln\left(J^2-\mu J+1\right)}{J-\mu}$ is well defined and positive. If (2.2) holds, then

$$\frac{1}{2}\int_0^\infty D(v)\,dv\le\int_0^{k_\mu} D(v)\,dv, \tag{2.12}$$

is a sufficient condition so that (2.7) holds true.

(ii) Assume that for some $\lambda>0$ we have

$$e^{-\lambda t}\le D(t),\quad t\ge 0,$$

and J is such that the number $k_\lambda=\frac{\ln\left(J^2-\lambda J+1\right)}{J-\lambda}$ is well defined and positive. Then

$$\int_0^{k_\lambda} D(v)\,dv\ge\frac{1}{2}\int_0^\infty D(v)\,dv, \tag{2.13}$$

is a necessary condition so that (2.7) holds true.

Proof. It suffices to show that (2.11) is satisfied.

(i) Assume that for some $\mu>0$ we have

$$D(s)\le e^{-\mu s},\quad s\ge 0.$$

By the discussion before Proposition 3, we have that the number k_μ is well defined, positive, and it is the solution of

$$J=\int_0^{k_\mu} e^{-\mu s}e^{Js}ds.$$

Furthermore, as k is given by (2.8) and $0\le D(s)\le e^{-\mu s}$, $s\ge 0$, we have

$$\int_0^k e^{Js}D(s)\,ds=J=\int_0^{k_\mu} e^{Js}e^{-\mu s}ds,$$

and we see that

$$k_\mu\le k,$$

which implies

$$\int_0^{k_\mu} D(v)\,dv \le \int_0^k D(v)\,dv.$$

From this inequality and (2.12) it follows that (2.11) is satisfied and the result follows from Proposition 2b (i).

(ii) As k_λ is the solution of $J = \int_0^{k_\lambda} e^{-\lambda s} e^{Js} ds$ and k is given by (2.8), in view of $e^{-\lambda s} \le D(s)$, $s \ge 0$ and

$$\int_0^k e^{Js} D(s)\,ds = J = \int_0^{k_\lambda} e^{Js} e^{-\lambda s} ds,$$

we have

$$k \le k_\lambda.$$

This implies

$$\int_0^k D(v)\,dv \le \int_0^{k_\lambda} D(v)\,dv,$$

from which it follows that, if (2.11) is true, then (2.13) must hold. We conclude that (2.13) is a necessary condition so that (2.11) holds true. The result follows from Proposition 2b (ii). □

We note that if (2.2) holds and

$$D(t) = e^{-\sigma t}, \quad t \ge 0,$$

then a necessary and sufficient condition so that (2.7) holds true is

$$\int_0^{k_\sigma} e^{-\sigma v} dv \ge \frac{1}{2} \int_0^\infty e^{-\sigma v} dv,$$

from which we take

$$\frac{1}{2} \le e^{-\sigma k_\sigma} \quad \text{or} \quad k_\sigma \le \frac{\ln 2}{\sigma},$$

and

$$\frac{\ln(J^2 - \sigma J + 1)}{J - \sigma} = k_\sigma \le \frac{\ln 2}{\sigma}.$$

It is easily verified that this inequality is satisfied with $J = 2\sigma$ and $2\sigma^2 + 1 = 2$, i.e., $\sigma = \frac{\sqrt{2}}{2}$ and $J = \sqrt{2}$.

3. APPLICATIONS AND COMMENTS

For $f \equiv 0$ equation (E) reduces to the linear equation

$$x'(t) = \int_0^t D(t-s)x(s)ds, \quad t \ge 0. \tag{E$_0$}$$

Existence of solutions to (E_0) was discussed in [5] and the existence result obtained in that paper may be considered as a special case of Theorem 2.1. Indeed, for $f \equiv 0$ equation (1.3) is

$$x(t) = x(0)e^{-Jt} + \int_0^t \left[R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds \right] x(u)du, t \geq 0. \tag{3.1}$$

In this case (2.2), (2.4), and (2.9) automatically hold, $F \equiv 0$, while (2.3) becomes

$$|x(0)|\, e^{-Jt} + \int_0^t \left| Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u)ds \right| du \leq 1, \quad t \geq 0. \tag{3.2}$$

Then Theorem 2.1 reduces to the next result (see, also, [5]).

Theorem 3.1. *If (3.2) holds, then the natural mapping defined by (3.1) on the set M of all bounded continuous functions maps $M \to M$ and Theorem 1.2 will give a fixed point in M.*

Since (2.4) holds for $f \equiv 0$, from Lemma 1 we have the following:

Lemma 2. *Let $J > 0$ and suppose that D satisfies (1.1).*
(i) If (2.5) holds then (3.2) is always true.
(ii) If (2.6) holds then (3.2) is equivalent with

$$J\,|x(0)|\, e^{-Jt} + \int_k^t \left[1 - e^{-J(t-s)}\right] D(s)\, ds \leq \int_0^k D(s)\, ds, \quad t \geq k, \tag{3.3}$$

where k is the unique solution of (2.8).

Part (i) of Lemma 2 is Lemma 5.2 in [5]. Furthermore, as for $f \equiv 0$ assumptions (2.2), (2.4) and (2.9) hold by themselves, from Propositions 1, 2b and 3 we have the following results.

Proposition 4 *(i) If there exists some $\mu > 2$ such that $D(t) \leq e^{-\mu t}$, $t \geq 0$, then (3.2) holds true for any $J \in \left[\frac{\mu - \sqrt{\mu^2-4}}{2}, \frac{\mu+\sqrt{\mu^2-4}}{2}\right]$.*

(ii) Let $J > 0$ be given. Then (3.2) holds true for any D with $D(t) \leq e^{-\left(J+\frac{1}{J}\right)t}$, $t \geq 0$.

Proposition 5. *Let (2.6) hold for some $J > 0$, and assume that*

$$J\,|x(0)|\, e^{-Jk} < \int_k^\infty D(v)\, dv. \tag{3.4}$$

Then inequality (2.11) is necessary and sufficient condition so that (3.3) hold true.

Proposition 6. *(i) Assume that for some $\mu > 0$ we have*

$$D(t) \leq e^{-\mu t}, \quad t \geq 0,$$

and J is such that the number $k_\mu = \frac{\ln\left(J^2 - \mu J + 1\right)}{J - \mu}$ is well defined and positive. Then (2.12) is a sufficient condition so that (3.3) holds true.

(ii) Assume that for some $\lambda > 0$ we have

$$e^{-\lambda t} \leq D(t), \quad t \geq 0,$$

and J is such that the number $k_\lambda = \frac{\ln\left(J^2 - \lambda J + 1\right)}{J - \lambda}$ is well defined and positive. Then (2.13) is a necessary condition so that (3.3) holds true.

REMARK 1. Assume that x is a solution of (E) which tends to zero. If f has a limit at infinity then this limit must be zero. For if $f \to m \neq 0$, then as the integral in (E) tends to zero (as the convolution of an L^1 function with a continuous function tending to zero), then we have that $x' \to m \neq 0$ which implies that $x \to \pm\infty$, a contradiction to $x \to 0$. Hence, if f has a limit at $+\infty$ this limit must be zero.

Now assume that f does not have a limit at $+\infty$. Then, as x satisfies (2.1), engaging the arguments in the proof of Theorem 2.1 we see that the last integral in (2.1) tends to zero as $t \to \infty$. Then from (2.1) we have that $e^{-Jt} \int_0^t e^{Js} f(s) ds \to 0$, so that (2.2) holds true. On the other hand, integrating (E) we have

$$x(t) = x(0) - \int_0^t \left[\int_0^s D(s-u)\, x(u)\, du \right] ds + \int_0^t f(u)\, du, \; t \geq 0; \tag{3.5}$$

hence for $t \to \infty$ the two integrals on the right either both converge or both diverge.

In view of the above we have the following corollary.

Corollary 1. *Assume that* (E) *has a solution x which tends to zero at infinity. Then*

(i) either (2.2) holds or $f \to 0$.

(ii) $f \in L^1[0, \infty) \Longleftrightarrow \int_0^s D(s-u)\, x(u)\, du \in L^1[0, \infty)$.

REMARK 2. Clearly, as

$$\begin{aligned}
& \int_0^t \int_0^s e^{Js} D(s-u) du ds - \left| \int_0^t e^{Js} f(s) ds \right| \\
\geq \; & \int_0^t \int_0^s e^{Js} D(s-u) du ds - \int_0^t e^{Js} \left| f(s) \right| ds \\
= \; & \int_0^t e^{Js} \left[\int_0^s D(s-u) du - \left| f(s) \right| \right] ds \\
= \; & \int_0^t e^{Js} \left[\int_0^s D(u) du - \left| f(s) \right| \right] ds,
\end{aligned}$$

it follows that *a sufficient condition so that (2.4) holds is*

$$|f(t)| \leq \int_0^t D(u)du, \quad t \geq 0.$$

Since $D(t) > 0$ and $\int_0^\infty e^{Jv} D(v)\, dv \leq J$ imply $\int_0^\infty D(v)\, dv < \infty$ (i.e., (1.1) holds true) we have the following result.

Let $J > 0$ and suppose that D satisfies

$$D(t) > 0 \quad \textit{and} \quad \int_0^\infty e^{Jv} D(v)\, dv \leq J. \tag{3.6}$$

If

$$|f(t)| \leq \int_0^t D(u)du, \quad t \geq 0, \tag{3.7}$$

then (2.3) is always true.

We close this remark by showing that the assumption (2.9) in Proposition 2b is not unrealistic. To this end it suffices to show that for a given D satisfying (1.1), there always exist nontrivial functions f with (2.9) holding, i.e., the function $g : [k, \infty) \to \mathbb{R}$ with

$$g(t) = \int_k^t e^{Js} [D(s) - J|f(s)|]\, ds + Je^{Jt} |f(t)|, \ \ t \geq k,$$

is nondecreasing on $[k, \infty)$. Indeed, taking

$$f(t) = \alpha D(t), \ \ t \geq k,$$

with $\alpha > 0$, we have

$$\begin{aligned} g'(t) &= \left[\int_k^t e^{Js} [D(s) - Jf(s)]\, ds + Je^{Jt} f(t) \right]' \\ &= e^{Jt} [D(t) - Jf(t)] + J^2 e^{Jt} f(t) + Je^{Jt} f'(t) \\ &= e^{Jt} [D(t) - \alpha J D(t)] + J^2 e^{Jt} \alpha D(t) + Je^{Jt} \alpha D'(t) \\ &= e^{Jt} \left[D(t) - \alpha J D(t) + J^2 \alpha D(t) + J \alpha D'(t) \right] \\ &= e^{Jt} D(t) \left\{ 1 + J\alpha \left[J + \frac{D'(t)}{D(t)} - 1 \right] \right\}, \end{aligned}$$

from which it follows that, if D' is bounded, then for sufficiently small values of α we have $g'(t) > 0$ for all $t \geq 0$. In particular, if $D(t) = e^{-mt}$, $t \geq k$, for some $m > 0$ then

$$g'(t) = e^{Jt} e^{-mt} \left[1 - \alpha J + J^2 \alpha - J\alpha m \right], \ \ t \geq k.$$

Thus, in order that g be nondecreasing, it suffices to have $1 - \alpha J + J^2 \alpha - J\alpha m \geq 0$, or

$$1 \geq \alpha J (m + 1 - J);$$

i.e., for given J and m, there always exists a (sufficiently small) $\alpha > 0$ such that (2.9) holds.

REMARK 3. A closer look at the inequality (2.4) reveals that the integral on the right hand side is not bounded. Changing the order of integration in the double integral, we may write (2.4) as

$$\left|\int_0^t e^{Js} f(s)ds\right| \le \int_0^t \int_0^s e^{Js} D(s-u)duds, \quad t \ge 0,$$

or

$$\left|\int_0^t e^{Js} f(s)ds\right| \le \int_0^t e^{Js} D^*(s)ds, \quad t \ge 0, \tag{3.8}$$

with $D^*(s) = \int_0^s D(v)dv$, $s \ge 0$. As D is positive, then D^* is increasing and the assertion is immediate. However, there is an unpleasant requirement on f resulting from (2.4) (see, also, (2.4)). As $D^*(0) = 0$, it is not difficult to see that if $f(0) \ne 0$, then (3.8) does not hold for values of t close to zero. Clearly, requiring $f(0) = 0$ is a disadvantage of condition (2.4) in Lemma 1, which, however, is not necessary in (2.3). The following example will show that inequality (2.3) is not superfluous.

Take $J \ge 1$ and

$$D(t) = e^{-2Jt}, \quad t \ge 0.$$

Then (2.3) is

$$|x(0)| + \left|\int_0^t e^{Js} f(s)ds\right| + \int_0^t \left|Je^{Ju} - \int_u^t e^{Js} e^{-2J(s-u)}ds\right| du \le e^{Jt}, \quad t \ge 0.$$

We have

$$\begin{aligned}
\int_u^t e^{Js} e^{-2J(s-u)} ds &= \int_u^t e^{-Js} e^{2Ju} ds \\
&= e^{2Ju}\left[\frac{e^{-Js}}{-J}\right]_u^t \\
&= -\frac{1}{J} e^{2Ju}\left[e^{-Jt} - e^{-Ju}\right] \\
&= \frac{1}{J} e^{Ju} - \frac{1}{J} e^{2Ju} e^{-Jt},
\end{aligned}$$

and, noting that $J \ge 1$ implies that the integrand in the absolute value is nonnegative, we obtain

$$\begin{aligned}
&\int_0^t \left|Je^{Ju} - \int_u^t e^{Js} e^{-2J(s-u)} ds\right| du \\
&= \int_0^t \left|Je^{Ju} - \frac{1}{J} e^{Ju} + \frac{1}{J} e^{2Ju} e^{-Jt}\right| du \\
&= \int_0^t \left|\left(J - \frac{1}{J}\right) e^{Ju} + \frac{1}{J} e^{2Ju} e^{-Jt}\right| du \\
&= \int_0^t \left[\left(J - \frac{1}{J}\right) e^{Ju} + \frac{1}{J} e^{2Ju} e^{-Jt}\right] du \\
&= \left(J - \frac{1}{J}\right) \frac{1}{J}\left(e^{Jt} - 1\right) + \frac{1}{2J^2} e^{-Jt}\left(e^{2Jt} - 1\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{J}\left[\left(J-\frac{1}{J}\right)\left(e^{Jt}-1\right)+\frac{1}{2J}\left(e^{Jt}-e^{-Jt}\right)\right] \\
&= \frac{1}{J}\left[\left(J-\frac{1}{J}\right)e^{Jt}-\left(J-\frac{1}{J}\right)+\frac{1}{2J}e^{Jt}-\frac{1}{2J}e^{-Jt}\right] \\
&= \frac{1}{J}\left[\left(J-\frac{1}{2J}\right)e^{Jt}-\left(J-\frac{1}{J}\right)-\frac{1}{2J}e^{-Jt}\right] \\
&< \frac{1}{J}\left(J-\frac{1}{2J}\right)e^{Jt} \\
&= \left(1-\frac{1}{2J^2}\right)e^{Jt}.
\end{aligned}
$$

It follows that for any choice of $|x(0)|$ and f with

$$
|x(0)|+\left|\int_0^t e^{Js}f(s)ds\right| \le \frac{1}{2J^2}e^{Jt}, \quad t\ge 0, \tag{3.9}
$$

the inequality (2.3) holds true for all $t \ge 0$. For example, it is not difficult to verify that the choice

$$
|x(0)| \le \frac{1}{4J^2} \quad \text{and} \quad |f(t)| \le \frac{1}{4J}e^{-2Jt}, \quad t \ge 0,
$$

satisfies (3.9). Furthermore, it is interesting to see that (3.9) is satisfied with

$$
|x(0)| \le \frac{1}{2J^2} \quad \text{and} \quad |f(t)| \le J\,|x(0)|\,, \quad t \ge 0. \tag{3.10}
$$

Indeed, for $t \ge 0$ we have

$$
\begin{aligned}
|x(0)|+\left|\int_0^t e^{Js}f(s)ds\right| &\le |x(0)|+\int_0^t e^{Js}\,|f(s)|\,ds \\
&\le |x(0)|+J\,|x(0)|\int_0^t e^{Js}ds \\
&= |x(0)|+J\,|x(0)|\,\frac{1}{J}\left(e^{Jt}-1\right) \\
&= |x(0)|+|x(0)|\,e^{Jt}-|x(0)| \\
&= |x(0)|\,e^{Jt} \\
&\le \frac{1}{2J^2}e^{Jt},
\end{aligned}
$$

so that (3.9) is satisfied, and this proves our assertion. *With* $D(t)=e^{-2Jt}$, $t \ge 0$ *for* $J \ge 1$, $f:[0,+\infty)\to\mathbb{R}$ *satisfying (3.9) and* $\lim_{t\to+\infty} f(t)=0$, *we conclude that (E) has a solution.* Moreover, from (3.9) we see that *if* $f \in L^1[0,+\infty)$, *with* $f^* := \int_0^{+\infty}|f(s)|\,ds < \frac{1}{2J^2}$, *then* (E) *has a solution for any initial value* $x(0)$ *with* $|x(0)| \le \frac{1}{2J^2}-f^*$.

4. APPENDIX

Proof. (*of Lemma 1*) Fix an arbitrary $t > 0$. For $0 \leq u \leq t$ we set

$$\begin{aligned} R(u;t) &= R(t-u) = Je^{-J(t-u)}, \\ T(u;t) &= \int_u^t R(t-s)\frac{D(s-u)}{J}ds = \frac{1}{J}\int_u^t Je^{-J(t-s)}D(s-u)\,ds \\ &= \int_u^t e^{-J(t-s)}D(s-u)\,ds = e^{-Jt}\int_u^t e^{Js}D(s-u)\,ds. \end{aligned}$$

Then

$$\begin{aligned} R(0;t) &= R(t) = Je^{-Jt}, \\ T(0;t) &= \int_0^t e^{-J(t-s)}D(s)\,ds = e^{-Jt}\int_0^t e^{Js}D(s)\,ds, \end{aligned}$$

while

$$\begin{aligned} R(t;t) &= R(t-t) = J, \\ T(t;t) &= 0. \end{aligned}$$

Denote by $\mathcal{C}_R$ and $\mathcal{C}_T$ the graphs of R and T, respectively, on $[0,t]$. If the two graphs meet at some $u_0 \in (0,t)$, then

$$\begin{aligned} Je^{-J(t-u_0)} &= \int_{u_0}^t e^{-J(t-s)}D(s-u_0)\,ds, \\ J &= \int_{u_0}^t e^{J(t-u_0)}e^{-J(t-s)}D(s-u_0)\,ds = \int_{u_0}^t e^{J(s-u_0)}D(s-u_0)\,ds; \end{aligned}$$

that is,

$$J = \int_0^{t-u_0} e^{Jv}D(v)\,ds.$$

Because of the positivity of the integrand $e^{Jv}D(v)$, the integral $\int_0^u e^{Jv}D(v)\,dv$ is an increasing function in u, so if (2.8) has a solution, then this solution is unique. (It is worth noticing that, the distance between t and the point u_0 where the two graphs meet is $t-u_0 = k$, thus this distance is independent of t). Clearly, if for some $t \geq 0$ we have $\int_0^t e^{Jv}D(v)\,dv \leq J$, then $\mathcal{C}_R$ and $\mathcal{C}_T$ do not meet on $(0,t)$. As $T(t;t) = 0 < J = R(t;t)$, it follows that $T(u;t) \leq R(u;t)$ for all $0 \leq u \leq t$, i.e.,

$$\int_u^t e^{-J(t-s)}D(s-u)ds \leq Je^{-J(t-u)}, 0 \leq u \leq t,$$

and so, the argument of the absolute value in the second integral in (2.3) is non-negative for such t. Thus, in view of (1.6) and using $|x(0)| \leq 1$, taking (2.4) into consideration for the expression in the left-hand side of (2.3) we have for $t \geq 0$

$$\left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt} + \int_0^t \left|Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D(s-u)ds\right|du$$

$$
\begin{aligned}
&= |x(0)|\, e^{-Jt} + \left| \int_0^t e^{Js} f(s) ds \right| e^{-Jt} + \int_0^t \left[Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u) ds \right] du \\
&\leq 1 \cdot e^{-Jt} + \left| \int_0^t e^{Js} f(s) ds \right| e^{-Jt} + \int_0^t Je^{-J(t-u)} du - \int_0^t \int_u^t e^{-J(t-s)} D(s-u) ds du \\
&= e^{-Jt} + \left| \int_0^t e^{Js} f(s) ds \right| e^{-Jt} + 1 - e^{-Jt} - \int_0^t \int_u^t e^{-J(t-s)} D(s-u) ds du \\
&= e^{-Jt} + 1 - e^{-Jt} - \left[\int_0^t \int_u^t e^{-J(t-s)} D(s-u) ds du - \left| \int_0^t e^{Js} f(s) ds \right| e^{-Jt} \right] \\
&= 1 - e^{-Jt} \left[\int_0^t \int_u^t e^{Js} D(s-u) ds du - \left| \int_0^t e^{Js} f(s) ds \right| \right] \\
&\leq 1.
\end{aligned}
$$

In other words, if $t \geq 0$ is such that $\int_0^t e^{Jv} D(v)\, dv < J$, then (2.3) holds true.

In view of the above, we proceed to the proof of (i) and (ii).

(i) If (2.5) holds true it follows that $\int_0^t e^{Jv} D(v)\, dv < J$ for all $t \geq 0$, so (2.3) holds true for all $t \geq 0$.

(ii) Assume now that (2.6) holds. For t with $\int_0^t e^{Jv} D(v)\, dv \leq J$ we have seen that (2.3) holds true.

Now, let us consider a t with $\int_0^t e^{Jv} D(v)\, dv > J$, thus $t > k$. We see that $T(0;t) = e^{-Jt} \int_0^t e^{Js} D(s)\, ds > Je^{-Jt} = R(0;t)$ while $T(t;t) = 0 < R(t;t) = J$, hence the graphs of $\mathcal{C}_R$ and $\mathcal{C}_T$ meet exactly once at $u_0 = t - k$. Moreover, $\mathcal{C}_R$ is below $\mathcal{C}_T$ on $[0, u_0]$ and $\mathcal{C}_R$ is above $\mathcal{C}_T$ on $[u_0, t]$, i.e., the argument in the absolute value in the second integral is nonnegative on $[0, u_0]$ and nonpositive on $[u_0, t]$. Hence, for the second integral on the left hand-side part of (2.3) we have for $t > k$

$$
\begin{aligned}
& \int_0^t \left| Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u)\, ds \right| du \\
= & \int_0^{u_0} \left| Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u)\, ds \right| du \\
& + \int_{u_0}^t \left| Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u)\, ds \right| du \\
= & \int_0^{u_0} \left[\int_u^t e^{-J(t-s)} D(s-u)\, ds - Je^{-J(t-u)} \right] du \\
& + \int_{u_0}^t \left[Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u)\, ds \right] du \\
= & \; e^{-Jt} \int_0^{u_0} \left[\int_u^t e^{Js} D(s-u)\, ds - Je^{Ju} \right] du \\
& + e^{-Jt} \int_{u_0}^t \left[Je^{Ju} - \int_u^t e^{Js} D(s-u)\, ds \right] du
\end{aligned}
$$

$$\begin{aligned}
&= e^{-Jt}\int_0^{u_0}\left[\int_u^t e^{Js}D(s-u)\,ds\right]du - e^{-Jt}\int_0^{u_0} Je^{Ju}du \\
&\quad + e^{-Jt}\int_{u_0}^t Je^{Ju}du - e^{-Jt}\int_{u_0}^t\int_u^t e^{Js}D(s-u)\,dsdu \\
&= e^{-Jt}\int_0^{u_0}\left[\int_u^t e^{Js}D(s-u)\,ds\right]du - e^{-Jt}\left(e^{Ju_0}-1\right) \\
&\quad + e^{-Jt}\left(e^{Jt}-e^{Ju_0}\right) - e^{-Jt}\int_{u_0}^t\int_u^t e^{Js}D(s-u)\,dsdu \\
&= e^{-Jt}\int_0^{u_0}\left[\int_u^t e^{Js}D(s-u)\,ds\right]du - e^{-J(t-u_0)} + e^{-Jt} \\
&\quad + 1 - e^{-J(t-u_0)} - e^{-Jt}\int_{u_0}^t\int_u^t e^{Js}D(s-u)\,dsdu \\
&= e^{-Jt}\int_0^{u_0}\left[\int_u^t e^{Js}D(s-u)\,ds\right]du - 2e^{-J(t-u_0)} + e^{-Jt} \\
&\quad + 1 - e^{-Jt}\int_{u_0}^t\int_u^t e^{Js}D(s-u)\,dsdu,
\end{aligned}$$

so, for $t>k$, inequality (2.3) is equivalent to

$$\begin{aligned}
&\left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt} + e^{-Jt}\int_0^{u_0}\left[\int_u^t e^{Js}D(s-u)\,ds\right]du \\
&\qquad - 2e^{-J(t-u_0)} + e^{-Jt} - e^{-Jt}\int_{u_0}^t\int_u^t e^{Js}D(s-u)\,dsdu \le 0.
\end{aligned}$$

Multiplying by e^{Jt} we see that, for $t>k$, inequality (2.7) is written as

$$\begin{aligned}
&|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right| + \int_0^{u_0}\left[\int_u^t e^{Js}D(s-u)\,ds\right]du \\
&\qquad - 2e^{Ju_0} + 1 - \int_{u_0}^t\int_u^t e^{Js}D(s-u)\,dsdu \le 0,
\end{aligned}$$

or (recall that $u_0 = t-k$)

$$\begin{aligned}
&|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right| + \int_0^{t-k}\left[\int_u^t e^{Js}D(s-u)\,ds\right]du + 1 \qquad (4.1)\\
\le\ & 2e^{J(t-k)} + \int_{t-k}^t\left[\int_u^t e^{Js}D(s-u)\,ds\right]du,\ t>k.
\end{aligned}$$

Since

$$\int_u^t e^{Js}D(s-u)\,ds = \int_0^{t-u} e^{J(s+u)}D(s)\,ds = e^{Ju}\int_0^{t-u} e^{Js}D(s)\,ds,$$

we see that (4.1) becomes

$$|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right| + \int_0^{t-k}\left[e^{Ju}\int_0^{t-u} e^{Js}D(s)\,ds\right]du + 1 \qquad (4.2)$$

$$\leq \ 2e^{J(t-k)} + \int_{t-k}^{t} \left[e^{Ju} \int_0^{t-u} e^{Js} D(s)\, ds \right] du, \ t > k.$$

Using integration by parts and taking into consideration the definition of k for the term with the double integral on the left-hand side of (4.2) we have

$$\begin{aligned}
&\int_0^{t-k} \left[e^{Ju} \int_0^{t-u} e^{Js} D(s)\, ds \right] du \\
= \ & \left[\frac{1}{J} e^{Ju} \int_0^{t-u} e^{Js} D(s)\, ds \right]_{u=0}^{u=t-k} - \frac{1}{J} \int_0^{t-k} e^{Ju} \left[e^{J(t-u)} D(t-u) \right] (-1)\, du \\
= \ & \frac{1}{J} e^{J(t-k)} \int_0^k e^{Js} D(s)\, ds - \frac{1}{J} \int_0^t e^{Js} D(s)\, ds + \frac{1}{J} \int_0^{t-k} e^{Jt} D(t-u)\, du \\
= \ & \frac{1}{J} e^{J(t-k)} \int_0^k e^{Js} D(s)\, ds - \frac{1}{J} \int_0^t e^{Js} D(s)\, ds + \frac{1}{J} e^{Jt} \int_0^{t-k} D(t-u)\, du \\
= \ & \frac{1}{J} e^{J(t-k)} J - \frac{1}{J} \int_0^t e^{Js} D(s)\, ds + \frac{1}{J} e^{Jt} \int_0^{t-k} D(t-u)\, du \\
= \ & e^{J(t-k)} - \frac{1}{J} \int_0^t e^{Js} D(s)\, ds + \frac{1}{J} e^{Jt} \int_0^{t-k} D(t-u)\, du,
\end{aligned}$$

and (4.2) becomes

$$\begin{aligned}
|x(0)| + \left| \int_0^t e^{Js} f(s) ds \right| + e^{J(t-k)} - \frac{1}{J} \int_0^t e^{Js} D(s)\, ds + \frac{1}{J} e^{Jt} \int_0^{t-k} D(t-u)\, du + 1 \\
\leq 2e^{J(t-k)} + \int_{t-k}^{t} \left[e^{Ju} \int_0^{t-u} e^{Js} D(s)\, ds \right] du,
\end{aligned}$$

or

$$\begin{aligned}
& |x(0)| + \left| \int_0^t e^{Js} f(s) ds \right| \\
& -\frac{1}{J} \int_0^t e^{Js} D(s)\, ds + \frac{1}{J} e^{Jt} \int_0^{t-k} D(t-u)\, du + 1 \\
\leq \ & e^{J(t-k)} + \int_{t-k}^{t} \left[e^{Ju} \int_0^{t-u} e^{Js} D(s)\, ds \right] du, \ t > k.
\end{aligned} \tag{4.3}$$

In a similar manner, integrating the integral on the right-hand side in (4.2) by parts, we find

$$\begin{aligned}
&\int_{t-k}^{t} \left[e^{Ju} \int_0^{t-u} e^{Js} D(s)\, ds \right] du \\
= \ & \left[\frac{1}{J} e^{Ju} \int_0^{t-u} e^{Js} D(s)\, ds \right]_{u=t-k}^{u=t} - \int_{t-k}^{t} \frac{1}{J} e^{Ju} \left[e^{J(t-u)} D(t-u) \right] (-1)\, du \\
= \ & 0 - \frac{1}{J} e^{J(t-k)} \int_0^k e^{Js} D(s)\, ds + \frac{1}{J} e^{Jt} \int_{t-k}^{t} D(t-u)\, du \\
= \ & -\frac{1}{J} e^{J(t-k)} J + \frac{1}{J} e^{Jt} \int_{t-k}^{t} D(t-u)\, du
\end{aligned}$$

$$= \quad -e^{J(t-k)} + \frac{1}{J} e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du,$$

so (4.3) is

$$\begin{aligned} &|x(0)| + \left| \int_0^t e^{Js} f(s) ds \right| - \frac{1}{J} \int_0^t e^{Js} D\,(s)\,ds + \frac{1}{J} e^{Jt} \int_0^{t-k} D\,(t-u)\,du + 1 \\ &\qquad \le e^{J(t-k)} - e^{J(t-k)} + \frac{1}{J} e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du, \quad t > k, \end{aligned}$$

i.e.,

$$\begin{aligned} &|x(0)| + \left| \int_0^t e^{Js} f(s) ds \right| - \frac{1}{J} \int_0^t e^{Js} D\,(s)\,ds + \frac{1}{J} e^{Jt} \int_0^{t-k} D\,(t-u)\,du + 1 \\ &\qquad \le \frac{1}{J} e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du,\ t > k, \end{aligned}$$

or

$$\begin{aligned} &J \left(|x(0)| + \left| \int_0^t e^{Js} f(s) ds \right| \right) - \int_0^t e^{Js} D\,(s)\,ds \\ &\qquad + e^{Jt} \int_0^{t-k} D\,(t-u)\,du + J \le e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du,\ t > k. \end{aligned}$$

That is,

$$\begin{aligned} & J \left(|x(0)| + \left| \int_0^t e^{Js} f(s) ds \right| \right) + e^{Jt} \int_0^{t-k} D\,(t-u)\,du + J \\ \le \quad & e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du + \int_0^t e^{Js} D\,(s)\,ds, \quad t > k. \end{aligned} \tag{4.4}$$

As $k < t$, in view of the definitions of J and k, the right-hand side of (4.4) gives

$$\begin{aligned} & e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du + \int_0^t e^{Js} D\,(s)\,ds \\ = \quad & e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du + \int_0^k e^{Js} D\,(s)\,ds + \int_k^t e^{Js} D\,(s)\,ds \\ = \quad & e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du + J + \int_k^t e^{Js} D\,(s)\,ds, \end{aligned}$$

and so (4.4) may be written as

$$\begin{aligned} &J \left(|x(0)| + \left| \int_0^t e^{Js} f(s) ds \right| \right) + e^{Jt} \int_0^{t-k} D\,(t-u)\,du \\ &\qquad \le e^{Jt} \int_{t-k}^{t} D\,(t-u)\,du + \int_k^t e^{Js} D\,(s)\,ds,\ t > k, \end{aligned}$$

or

$$J \left(|x(0)| + \left| \int_0^t e^{Js} f(s) ds \right| \right) e^{-Jt} + \int_0^{t-k} D\,(t-u)\,du$$

$$\leq \int_{t-k}^{t} D(t-u)\,du + e^{-Jt}\int_{k}^{t} e^{Js} D(s)\,ds,\ t>k,$$

from which for $v = t-u$ we have

$$J\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_t^k D(v)(-dv)$$
$$\leq \int_k^0 D(v)(-dv) + \int_k^t e^{-J(t-s)} D(s)\,ds, \quad t>k,$$

or

$$J\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_k^t D(v)\,dv$$
$$\leq \int_0^k D(v)\,dv + \int_k^t e^{-J(t-s)} D(s)\,ds,$$

or

$$J\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_k^t \left[1 - e^{-J(t-s)}\right] D(s)\,ds$$
$$\leq \int_0^k D(s)\,ds, \quad t>k.$$

Due to the continuity (in t) of the left-hand-side of the last inequality, we see that it still holds for $t = k$, i.e.,

$$J\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_k^t \left[1 - e^{-J(t-s)}\right] D(s)\,ds$$
$$\leq \int_0^k D(s)\,ds, \quad t \geq k.$$

which is (2.7). The proof of Lemma 1 is complete. □

Before we prove Proposition 2, we establish that

$$\lim_{t\to\infty} \int_k^t \left[1 - e^{-Jt}e^{Js}\right] D(s)\,ds = \int_k^\infty D(v)\,dv. \tag{***}$$

To this end we consider an arbitrary $t_1 \geq k$. Then for any $t \geq t_1$ we have

$$\begin{aligned}
&\left[1 - e^{-Jt}e^{Jt_1}\right] \int_k^{t_1} D(s)\,ds \\
= &\int_k^{t_1} \left[1 - e^{-Jt}e^{Jt_1}\right] D(s)\,ds \\
\leq &\int_k^{t_1} \left[1 - e^{-Jt}e^{Js}\right] D(s)\,ds + \int_{t_1}^{t} \left[1 - e^{-Jt}e^{Js}\right] D(s)\,ds \\
= &\int_k^{t} \left[1 - e^{-Jt}e^{Js}\right] D(s)\,ds
\end{aligned}$$

$$\leq \int_k^\infty D(v)\,dv;$$

that is, for an arbitrary $t_1 \geq k$ and any $t \geq t_1$ we obtain

$$\left[1 - e^{-Jt}e^{Jt_1}\right]\int_k^{t_1} D(s)\,ds \leq \int_k^t \left[1 - e^{-Jt}e^{Js}\right] D(s)\,ds \leq \int_k^\infty D(v)\,dv.$$

Letting $t \to \infty$ we see that for any $t_1 \geq k$ we have

$$\int_k^{t_1} D(s)\,ds \leq \lim_{t\to\infty}\int_k^t \left[1 - e^{-Jt}e^{Js}\right] D(s)\,ds \leq \int_k^\infty D(v)\,dv.$$

As the last inequality holds true for any arbitrary $t_1 \geq k$ we conclude that (***) holds true.

Proof. (*of Proposition 2.*) (i) We will prove that (2.11) implies (2.7).

Denote by G the left hand side of (2.7), i.e., let

$$G(t) = J\left(|x(0)| + \left|\int_0^t e^{Js} f(s)ds\right|\right)e^{-Jt} + \int_k^t \left[1 - e^{-J(t-s)}\right] D(s)\,ds, \quad t \geq k,$$

and set

$$G_1(t) = J\left(|x(0)| + \int_0^t e^{Js}\,|f(s)|\,ds\right)e^{-Jt} + \int_k^t \left[1 - e^{-J(t-s)}\right] D(s)\,ds, \quad t \geq k.$$

Clearly

$$G(t) \leq G_1(t), \quad t \geq k. \tag{4.5}$$

Noting that (2.11) is equivalent to

$$\int_k^\infty D(v)\,dv \leq \int_0^k D(v)\,dv, \tag{4.6}$$

we see that all we have to show is that (4.6) implies (2.7), i.e., that

$$(4.6) \Longrightarrow G(t) \leq \int_0^k D(v)\,dv.$$

To this end, in view of (4.6) and (4.5) it is sufficient to show that

$$G_1(t) \leq \int_k^\infty D(v)\,dv, \quad t \geq k. \tag{4.7}$$

We have for $t \geq k$,

$$\begin{aligned} G_1'(t) &= -JJ\left(|x(0)| + \int_0^t e^{Js}\,|f(s)|\,ds\right)e^{-Jt} + Je^{Jt}\,|f(t)|\,e^{-Jt} \\ &\quad + \left[1 - e^{-J(t-t)}\right] D(t) + \int_k^t \left[Je^{-J(t-s)}\right] D(s)\,ds \end{aligned}$$

$$\begin{aligned}
&= Je^{-Jt}\left\{-J\left(|x(0)| + \int_0^t e^{Js}|f(s)|\,ds\right)\right.\\
&\quad \left. + \int_k^t e^{Js}D(s)\,ds + e^{Jt}|f(t)|\right\}\\
&= Je^{-Jt}\left\{-J\left(|x(0)| + \int_0^k e^{Js}|f(s)|\,ds\right)\right.\\
&\quad \left. -J\int_k^t e^{Js}|f(s)|\,ds + \int_k^t e^{Js}D(s)\,ds + e^{Jt}|f(t)|\right\},
\end{aligned}$$

i.e.,

$$G_1'(t) = Je^{-Jt}\left\{c + \int_k^t e^{Js}\left[D(s) - J|f(s)|\right]ds + e^{Jt}|f(t)|\right\},\ t \geq k,$$

with $c = -J\left(|x(0)| + \int_0^k e^{Js}|f(s)|\,ds\right)$. Taking (2.9) into consideration, we see that for the derivative G_1' exactly one of the following can hold:

(I) either $G_1'(t) \leq 0$ for all $t \geq k$ (i.e., G_1 is nonincreasing on $[k,\infty)$), or

(II) there exists some $t_0 \in (k,\infty)$ such that $G_1'(t) \leq 0$ on $[k,t_0)$ and $G_1'(t) \geq 0$ on (t_0,∞).

We observe that

$$G_1(k) = J\left(|x(0)| + \left|\int_0^k e^{Js}f(s)ds\right|\right)e^{-Jk},$$

while by (***) and (2.2) it follows that

$$\begin{aligned}
&G_1(\infty)\\
&= \lim_{t\to\infty}\left\{J\left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt} + \int_k^t\left[1 - e^{-J(t-s)}\right]D(s)\,ds\right\}\\
&= 0 + 0 + \lim_{t\to\infty}\int_k^t\left[1 - e^{-J(t-s)}\right]D(s)\,ds = \int_k^\infty D(s)\,ds.
\end{aligned}$$

Thus, from (2.10) we have

$$G_1(k) = J\left(|x(0)| + \left|\int_0^k e^{Js}f(s)ds\right|\right)e^{-Jk} \leq \int_k^\infty D(s)\,ds = G_1(\infty),$$

which means that G_1 cannot be nonincreasing on the whole interval $[k,\infty)$, i.e., (I) is not true. Hence (II) does hold true, and this means that G_1 is decreasing on $[k,t_0]$ and nondecreasing on $[t_0,\infty)$ for some $t_0 \in (k,\infty)$, and so

$$G_1(t) \leq \max\{G_1(k), G_1(\infty)\} = G_1(\infty) = \int_k^\infty D(s)\,ds,\ \ t \in [k,\infty),$$

i.e., (4.7) is true.

(ii) As (2.11) is equivalent to (4.6), it suffices to prove that (2.7) implies (4.6). For the sake of contradiction we assume that (2.7) holds but (4.6) is not true, i.e., that $\int_k^\infty D(v)\,dv > \int_0^k D(v)\,dv$. Then, in view of (***), we see that there exists a $t_1 > 0$ such that

$$\int_k^{t_1} \left[1 - e^{-J(t_1 - s)}\right] D(s)\,ds = \int_k^{t_1} \left[1 - e^{-Jt_1} e^{Js}\right] D(s)\,ds > \int_0^k D(v)\,dv,$$

an immediate contradiction to (2.7). We conclude that (4.6) is a necessary condition so that (2.7) holds. □

REFERENCES

[1] T. A. Burton, Averaged neural networks, *Neural Networks* **6** (1993), 667–680.

[2] T. A. Burton, Fractional differential equations and Lyapunov functionals, *Nonlinear Anal.: TMA* **74** (2011), 5648–5662.

[3] T. A. Burton, Fractional equations and a theorem of Brouwer-Schauder type, *Fixed Point Theory* **14** Nr. 1 (2013), 91-96.

[4] T. A. Burton, Corrigendum on "Fractional equations and a theorem of Brouwer-Schauder type, *Fixed Point Theory* **14** Nr. 1 (2013), 91-96", submitted.

[5] T. A. Burton, and I. K. Purnaras, A unification theory of Krasnoselskii for differential equations, *Nonlinear Anal.: TMA* **89** (2013), 121-133

[6] Burton, T. A. and Zhang, Bo, Fixed points and fractional differential equations: Examples, *Fixed Point Theory*, accepted January 19, 2012.

[7] Jack Hale, *Theory of Functional Differential Equations*, Springer, New York, 1977.

[8] Krasnoselskii, M. A., *Amer. Math. Soc. Transl. (2)* **10** (1958), 345–409.

[9] V. Lakshmikantham , S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cottenham, Cambridge, 2009.

[10] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.:TMA* **69** (2008), 2677-2682.

[11] J. J. Levin, The asymptotic behavior of a Volterra equation, *Proc. Amer. Math. Soc.* **14** (1963), 434-451.

[12] J. J. Levin, The qualitative behavior of a nonlinear Volterra equation, *Proc. Amer. Math. Soc.* **16** (1965), 711-718.

[13] J. J. Levin, On a nonlinear Volterra equation, *J. Math. Anal. Appl.* **39** (1972), 458-476.

[14] J. J. Levin and J. A. Nohel, On a system of integrodifferential equations occurring in reactor dynamics, *J. Math. Mech.* **9** (1960), 347-368.

[15] J. J. Levin and J. A. Nohel, Note on a nonlinear Volterra equation, *Proc. Amer. Math. Soc.* **14** (1963), 924-929.

[16] J. J. Levin and J. A. Nohel, On a nonlinear delay equation, *J. Math. Anal. Appl.* **8** (1964), 31-44.

[17] Stig-Olof Londen, On the solution of a nonlinear Volterra equation, *J. Math. Anal. Appl.* **39** (1972), 564-573.

[18] V. Volterra, Sur la théorie mathématique des phénomès héréditaires. *J. Math. Pur. Appl.***7** (1928), 249–298.