INTEGRAL EQUATIONS, L^p-FORCING, REMARKABLE RESOLVENT: LIAPUNOV FUNCTIONALS

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ABSTRACT. In this paper we study an integral equation of the form $x(t) = a(t) - \int_0^t C(t,s)x(s)ds$ with resolvent $R(t,s)$ and variation-of-parameters formula $x(t) = a(t) - \int_0^t R(t,s)a(s)ds$. We give a variety of conditions under which the mapping $(P\phi)(t) = \phi(t) - \int_0^t R(t,s)\phi(s)ds$ maps a vector space containing unbounded functions into an L^p space. It is known from the ideal theory of Ritt that $R(t,s)$ is arbitrarily complicated. Thus, it is widely supposed that this integral is also extremely complicated. In fact, it is not. That integral can be a very close approximation to ϕ even when ϕ is unbounded. These unbounded functions are essentially harmless perturbations.

1. Introduction

An integral equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds$$

has a resolvent, $R(t,s)$, and a variation-of-parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds.$$

Ideal theory of Ritt [13] shows that $R(t,s)$ is arbitrarily complicated even for relatively simple equations arising from $x'' + tx = 0$.

By studying the method of undetermined coefficients, among other things, investigators work from two informal hypotheses:

(i) If $C(t,s)$ is well-behaved, then the solution follows $a(t)$. See, for example, Corduneanu [5; pp. 127-129].

(ii) Since $R(t,s)$ is arbitrarily complicated, so is $\int_0^t R(t,s)a(s)ds$.

In a recent paper [4] we cast doubt on these ideas. Under a variety of general conditions, the integral in (ii) strips away the complications and has a value "closely" approximating $a(t)$. Indeed, the mapping

$$(P\phi)(t) = \phi(t) - \int_0^t R(t,s)\phi(s)ds$$

1991 *Mathematics Subject Classification.* Primary: 45A05, 45J05, 45M10.
Key words and phrases. Integral equations, resolvents, Liapnov functionals.

can map a vector space with unbounded functions into the vector space of bounded continuous functions or even into a space of functions tending to zero. It is a remarkable simplification.

In this paper we continue that study and sharpen the results by showing $P\phi \in L^p$ for p a positive integer. Thus, on average that integral is close to ϕ and converges to ϕ in the sense of L^p.

In [4] all the work was by fixed point theory which is particularly well suited to L^∞ problems. Here, we focus entirely on Liapunov techniques. In addition to the main results on the resolvent we offer a brief introduction to Liapunov theory for integral equations. Sections 3 and 4 offer analogues between Liapunov functionals for integral equations and parallel differential equations.

REMARK ON NOTATION. In all of our work, when we write that a function $\phi \in L^p$, we will mean that it is also continuous.

2. Liapunov functionals

Liapunov's direct method is now 114 years old and most investigators in differential equations are thoroughly familiar with its application to differential equations. However, very few investigators are familiar with the application to integral equations. We hope in this short paper to give an introduction to that application and to present ready comparisons with differential equations. In preparation for that, we believe it is appropriate to take the space to explain the basic ideas, particularly as applied to integral equations. Classical theory and many examples of Liapunov's direct method are found, for example, in [2, 3].

A Liapunov function has its existence totally independent of any integral or differential equation. The Liapunov function is easily united to the differential equation in the very first step, usually by means of the chain rule. That process will be seen in each proof in Section 4. The Liapunov function is united to the integral equation in the last step, usually by means of an inequality, but sometimes by direct substitution. The process will be shown in each proof in Section 3.

A Liapunov function is a measuring device which allows us to measure the distance from a solution to the origin. It is a generalized metric. A trivial example is given by the scalar equation $x' = -x$ and the Liapunov function $V(x) = x^2$. We invoke existence theory and say that the differential equation has a solution, say $x(t)$, so that we could form $V(x(t))$ and differentiate by the chain rule to obtain $\frac{dV(x(t))}{dt} = 2x\frac{dx}{dt} = -2x^2 \leq 0$. This yields $V(x(t)) = x^2(t) \leq V(x(0)) = x^2(0)$ for $t \geq 0$. It is a perfect relation of $x^2(t) \leq x^2(0)$.

It is critical to notice throughout this elementary discussion that the differential equation and the Liapunov function are united in the very first step. This is where we will see the big change when we study Liapunov functionals for integral equations.

A Liapunov functional does the same and one more very important thing. If we were to use a Liapunov function on a functional equation then we would compare functions with functionals. That can be done and it is called a Razumikhin technique and it is preferred by many to a Liapunov functional. A Liapunov functional can reduce the problem by replacing the functionals in such a way that we need only compare functions with functions. Here is the simplest idea of the process. Suppose now that we have

$$x'(t) = -2x(t) + x(t-1)$$

and we try the same Liapunov function, $V(x) = x^2$, with derivative along solutions of the equation given by

$$\frac{dV(x(t))}{dt} = 2x(t)[-2x(t) + x(t-1)] = -4x^2(t) + 2x(t)x(t-1).$$

We struggle to see how this might be made negative, but cannot because we are comparing $x(t)$ with $x(t-1)$. The mighty mathematician, Krasovskii, planted the seed some 65 years ago and it is still growing. Define a Liapunov function plus a Liapunov functional in the form

$$V(x(\cdot)) = x^2 + \int_{t-1}^{t} x^2(u)du$$

so that by the chain rule

$$\begin{aligned}\frac{dV(x(\cdot))}{dt} &= (2x(t))(-2x(t) + x(t-1)) + x^2(t) - x^2(t-1)\\ &= -4x^2(t) + 2x(t)x(t-1) + x^2(t) - x^2(t-1)\\ &\le -2x^2(t) + x^2(t) + x^2(t-1) - x^2(t-1)\\ &= -x^2(t).\end{aligned}$$

We now have $x^2(t) \le V(x(t)) \le V(x(0))$ so $x(t)$ is bounded. Krasovskii has succeeded in reducing the problem to one in which functions are compared with functions. Investigators, including the author, will frequently call the Liapunov function plus the functional simply a Liapunov functional. But it is critical to think of it as the sum of two things, each of which has a drastically different role.

A Liapunov function measures the distance from a solution to the origin. A Liapunov functional converts a functional to a function. That is exactly what we have seen above. In the same way, consider an integral equation $x(t) = a(t) - \int_0^t C(t,s)x(s)ds$. If only that integral were a function! We would have an algebraic relation which we might successfully solve for x, thereby obtaining a bound on x. What we need is a Liapunov functional to convert that integral into a function. Our goal is to construct a double integral as the Liapunov functional in such a shrewd way that when we differentiate it we will have two parts; one is that integral and the other is a function. We will then substitute the integral equation into the derivative of the Liapunov functional and have the derivative of the Liapunov functional as an algebraic relation. We will see this in several problems in the next section.

3. Results of Classical Type

Much of what we do can be done for systems, but the ideas will be conveyed so much more clearly for scalar equations of the form

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds \tag{1}$$

where $a : [0,\infty) \to \Re$ is continuous, as is $C : \Re \times \Re \to \Re$. The standard resolvent equation for (1) may be found in Burton [2; Chapter 7] or Miller [12; especially p. 190] and it is given as

$$R(t,s) = C(t,s) - \int_s^t R(t,u)C(u,s)du \tag{2}$$

with solution $R(t,s)$. There is then the variation-of-parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{3}$$

Much about the resolvent can be found in [1, 2, 5, 6, 7, 9, 10, 14].

In this section $a(t)$ will be small and we will see that $a(t)$, $x(t)$, and $\int_0^t R(t,s)a(s)ds$ are always in the same space. Thus, the classical view that x follows a is natural. But in the next section $a(t)$ may be large and we then see that it is $a(t)$ and $\int_0^t R(t,s)a(s)ds$ which are in the same space and very close together; in (3), then, x must be "content" with whatever is left over and it is simply not in the same space as those other two functions. The function $a(t)$ may be very large, yet $\int_0^t R(t,s)a(s)ds$ is almost an exact copy of $a(t)$. By (3), then, this large function $a(t)$ does not give rise to a large solution and is essentially a harmless perturbation.

Our work here involves studying the effect of integrating $R(t,s)$ against $a(s)$. That effect is described by the following definition.

DEFINITION. A function $H : \Re \times \Re \to \Re$ is said to generate an L^p approximate identity on a vector space W if there is a positive integer p and for $\phi \in W$ then

$$(P\phi)(t) := \phi(t) - \int_0^t H(t,s)\phi(s)ds \in L^p. \tag{4}$$

Other definitions with the L^∞ norm are given in Burton [4].

Theorem 3.1. *Suppose that $a \in L^1[0,\infty)$ and $\int_{t-s}^\infty |C(u+s,s)|du$ exists for $0 \le s \le t < \infty$. If there is an $\alpha < 1$ with $\int_0^\infty |C(u+t,t)|du \le \alpha$ then the solution $x(t)$ of (1) is in $L^1[0,\infty)$ and the resolvent, $R(t,s)$, of (2) generates an L^1 approximate identity on L^1.*

Proof. Define a Liapunov functional

$$V(t) = \int_0^t \int_{t-s}^\infty |C(u+s,s)|du|x(s)|ds.$$

We will take the derivative of V along the unique solution of (1) and need to unite the Liapunov functional with the integral equation. The inequality accomplishing this will now be prepared. From (1) we have

$$|x(t)| \le |a(t)| + \int_0^t |C(t,s)x(s)|ds$$

or

$$-\int_0^t |C(t,s)x(s)|ds \le |a(t)| - |x(t)|.$$

Now

$$\begin{aligned} V'(t) &= \int_0^\infty |C(u+t,t)|du|x(t)| - \int_0^t |C(t,s)x(s)|ds \\ &\le \alpha|x(t)| - |x(t)| + |a(t)| \\ &= (\alpha - 1)|x(t)| + |a(t)|. \end{aligned}$$

An integration from 0 to t, use of $V(t) \ge 0$, and use of $a \in L^1$ will yield $x \in L^1$. If we look at (3) and (4) we see that $P\phi \in L^1$ for each $\phi \in L^1$ and that completes the proof. □

We will now add to the conditions of Theorem 3.1 the classical condition that $\int_0^t |C(t,s)|ds \le \beta < 1$ for $t \ge 0$ to ensure boundedness of solutions of (1). Notice how a change in the Liapunov functional so that x^2 is in the integrand will yield the solution in L^2.

Theorem 3.2. *Suppose there exist $\alpha < 1$ and $\beta < 1$ with $\int_0^\infty |C(u+t,t)|du \le \alpha$ and $\sup_{t\ge 0} \int_0^t |C(t,s)|ds \le \beta$. Then $R(t,s)$ generates an L^1, an L^2, and an L^∞ approximate identity on the spaces L^1, L^2, and L^∞ respectively. If $\phi = \phi_1 + \phi_2 + \phi_3$ where $\phi_1 \in L^1$, $\phi_2 \in L^2$, and $\phi_3 \in L^\infty$, then $P\phi = \psi_1 + \psi_2 + \psi_3$ where $\psi_1 \in L^1$, $\psi_2 \in L^2$, and $\psi_3 \in L^\infty$.*

Proof. In Theorem 3.1 it was shown that the resolvent $R(t,s)$ of (2) generates an L^1 approximate identity on L^1 when $a \in L^1[0,\infty)$. We will now show that $R(t,s)$ generates an L^2 approximate identity on L^2 when $a \in L^2[0,\infty)$. First we prepare the inequality which will unite the integral equation to the Liapunov functional. Notice that for any $\epsilon > 0$ there is an $M > 0$ so that by squaring both sides of (1) we can say that

$$\begin{aligned}
x^2(t) &\le Ma^2(t) + (1+\epsilon)\left(\int_0^t C(t,s)x(s)ds\right)^2 \\
&\le Ma^2(t) + (1+\epsilon)\int_0^t |C(t,s)|ds \int_0^t |C(t,s)|x^2(s)ds \\
&\le Ma^2(t) + (1+\epsilon)\beta \int_0^t |C(t,s)|x^2(s)ds \\
&= Ma^2(t) + \int_0^t |C(t,s)|x^2(s)ds
\end{aligned}$$

where we choose ϵ so that $(1+\epsilon)\beta = 1$. This means that

$$-\int_0^t |C(t,s)|x^2(s)ds \le Ma^2(t) - x^2(t)$$

which will be our fundamental uniting inequality.

Next, define a Liapunov functional by

$$V(t) = \int_0^t \int_{t-s}^\infty |C(u+s,s)|dux^2(s)ds$$

so that

$$\begin{aligned}
V'(t) &= \int_0^\infty |C(u+t,t)|dux^2(t) - \int_0^t |C(t,s)|x^2(s)ds \\
&\le Ma^2(t) - x^2 + \int_0^\infty |C(u+t,t)|dux^2 \\
&\le Ma^2(t) - (1-\alpha)x^2(t).
\end{aligned}$$

Hence, an integration yields

$$(1-\alpha)\int_0^t x^2(s)ds \le M\int_0^t a^2(s)ds$$

and $x \in L^2[0,\infty)$.

Thirdly, there is the classical result (Burton [4] or Corduneanu [5; p. 127]) which shows that $\int_0^t |C(t,s)|ds \le \beta < 1$ and $a(t)$ bounded yields x bounded. These facts will prove the result. □

Our next result employs a very different kind of Liapunov functional. We will see it in three different contexts. It relies on sign conditions rather than on size conditions in Theorems 3.1 and 3.2. This Liapunov functional is a modified form of one by Levin [11] for an integrodifferential equation.

Theorem 3.3. *If $C(t,s) \ge 0$, $C_s(t,s) \ge 0$, $C_{st}(t,s) \le 0$, $C_t(t,0) \le 0$, and if $a \in L^2[0,\infty)$, then the unique solution x of (1) is in $L^2[0,\infty)$. That is, $R(t,s)$ generates an L^2 approximate identity on L^2.*

Proof. Define a Liapunov functional

$$V(t) = \int_0^t C_s(t,s)\Big(\int_s^t x(u)du\Big)^2 ds + C(t,0)\Big(\int_0^t x(s)ds\Big)^2$$

and differentiate along the unique solution of (1) to obtain

$$V'(t) = \int_0^t C_{st}(t,s)\Big(\int_s^t x(u)du\Big)^2 ds + 2x\int_0^t C_s(t,s)\int_s^t x(u)duds$$
$$+ C_t(t,0)\Big(\int_0^t x(s)ds\Big)^2 + 2xC(t,0)\int_0^t x(s)ds.$$

We now integrate the third-to-last term by parts to obtain

$$2x\Big[C(t,s)\int_s^t x(u)du\Big|_0^t + \int_0^t C(t,s)x(s)ds\Big]$$
$$= 2x\Big[-C(t,0)\int_0^t x(u)du + \int_0^t C(t,s)x(s)ds\Big].$$

Cancel terms, use the sign conditions, and use (1) in the last step of the process to unite the Liapunov functional and the equation obtaining

$$\begin{aligned} V'(t) &= \int_0^t C_{st}(t,s)\Big(\int_s^t x(u)du\Big)^2 ds + C_t(t,0)\Big(\int_0^t x(s)ds\Big)^2 \\ &\quad + 2x[a(t) - x(t)] \\ &\le 2xa(t) - 2x^2(t) \\ &\le a^2(t) - x^2(t). \end{aligned}$$

From this we obtain $x \in L^2$ whenever $a \in L^2$. This completes the proof. □

REMARK. All of these results are of the classical type. The solution is following $a(t)$. In the next section we will see that this is something of a coincidence. The difference between $a(t)$ and $\int_0^t R(t,s)a(s)ds$ turns out to be about the same as $a(t)$ so we perceive that $x(t)$ is following $a(t)$. In the next section we let $a(t)$ grow very rapidly and find that $x(t)$ still stays about the same size as when $a(t)$ was small.

4. One step up

We now suppose that both C_t and $a'(t)$ are continuous so that we can write (1) as

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds. \tag{5}$$

Properties of $x(t)$ (which is still a solution of (1)) will be determined and interpreted in (3) and (4). But since we are dealing with a Volterra integrodifferential equation it is well to present the resolvent theory for it so that similar problems can be considered which have no relation to (1).

The resolvent equation of Becker [1] (or Burton [2; Chapter 7]) for (5) is

$$Z_t(t,s) = -C(t,t)Z(t,s) - \int_s^t C_1(t,u)Z(u,s)du, \quad Z(s,s)=1, \tag{6}$$

where $C_1(t,s) = \partial C(t,s)/\partial t$, and the corresponding variation of parameters formula is

$$x(t) = Z(t,0)x(0) + \int_0^t Z(t,s)a'(s)ds. \tag{7}$$

It turns out that we may want to integrate that equation by parts so it is important to know that Z_s is continuous. It is continuous because there is an equivalent resolvent equation with the same $Z(t,s)$ as a solution which is written as

$$Z_s(t,s) = Z(t,s)C(s,s) + \int_s^t Z(t,u)C_1(u,s)du, \quad Z(t,t)=1, \tag{8}$$

and having the same variation of parameters formula. If we integrate (7) by parts and relate it to (1) so that $x(0)=a(0)$ we obtain

$$x(t) = a(t) - \int_0^t Z_s(t,s)a(s)ds \tag{9}$$

which is parallel to (3).

REMARK. Here is the typical objective. If we can show that $x \in L^2$ for every $a' \in L^2$ then $R(t,s)$ (or $Z_s(t,s)$) generates an L^2 approximate identity on the vector space W of function $a(t)$ for which $a'(t)$ is continuous and L^2. Thus, for example, $a(t) = \ln(t+1)$ qualifies and the solution of (1) is L^2, but it does not follow $a(t)$ in any sense at all. Our kernel can be arbitrarily well-behaved and the solution simply does not follow $a(t)$. In the result below, notice that the solution $x(t)$ of (1) is in L^1, but $a(t)$ is merely bounded.

Theorem 4.1. *If there is an $\alpha > 0$ such that*

$$-C(t,t) + \int_0^\infty |C_1(u+t,t)|du \le -\alpha$$

then $a' \in L^1$ implies that the solution $x(t)$ of (1) is in L^1 and is bounded. Hence, $x \in L^p$ for $p \in [1,\infty]$ and $R(t,s)$ generates an L^p approximate identity on the vector space W of functions ϕ with $\phi' \in L^1$.

Proof. Define

$$V(t) = |x(t)| + \int_0^t \int_{t-s}^\infty |C_1(u+s,s)|du|x(s)|ds.$$

Compare closely (1) with (5) and this functional with that of Theorem 3.1. We have

$$\begin{aligned} V'(t) &\leq |a'(t)| - C(t,t)|x(t)| + \int_0^t |C_1(t,s)x(s)|ds \\ &+ \int_0^\infty |C_1(u+t,t)|du|x(t)| - \int_0^t |C_1(t,s)x(s)|ds \\ &= \left[-C(t,t) + \int_0^\infty |C_1(u+t,t)|du\right]|x(t)| + |a'(t)| \\ &\leq -\alpha|x(t)| + |a'(t)|. \end{aligned}$$

An integration and use of V yields

$$|x(t)| \leq V(t) \leq V(0) + \int_0^\infty |a'(s)|ds - \alpha \int_0^\infty |x(s)|ds.$$

The result follows from this. □

Next, we consider a quadratic Liapunov functional.

Theorem 4.2. *If there exists $\alpha > 0$ such that*

$$-2C(t,t) + \int_0^\infty |C_1(u+t,t)|du + \int_0^t |C_1(t,s)|ds \leq -\alpha$$

and if $a' \in L^2$, then the solution $x(t)$ of (1) satisfies $x \in L^p$ for $p \in [2,\infty]$. Thus, $R(t,s)$ generates an L^p approximate identity on the vector space W of functions ϕ with $\phi' \in L^2$.

Proof. Define

$$V(t) = x^2(t) + \int_0^t \int_{t-s}^\infty |C_1(u+s,s)|dux^2(s)ds.$$

For any $\epsilon > 0$ there is an $M > 0$ with $2a'(t)x \leq Ma'(t)^2 + \epsilon x^2$. Thus

$$\begin{aligned} V'(t) &= 2a'(t)x(t) - 2C(t,t)x^2(t) - 2x(t)\int_0^t C_1(t,s)x(s)ds \\ &+ \int_0^\infty |C_1(u+t,t)|dux^2(t) - \int_0^t |C_1(t,s)|x^2(s)ds \\ &\leq 2a'(t)x(t) - 2C(t,t)x^2(t) + \int_0^t |C_t(t,s)|(x^2(t) + x^2(s))ds \\ &+ \int_0^\infty |C_1(u+t,t)|dux^2(t) - \int_0^t |C_1(t,s)|x^2(s)ds \\ &\leq Ma'(t)^2 + \epsilon x^2(t) - 2C(t,t)x^2(t) \\ &+ \int_0^t |C_t(t,s)|dsx^2(t) + \int_0^\infty |C_1(u+t,t)|dux^2 \\ &\leq -(\alpha/2)x^2(t) + Ma'(t)^2. \end{aligned}$$

It readily follows that $x \in L^2$ and is bounded so the conclusion follows. □

In the equation (1) the quantity $\int_0^t x(s)ds$ is frequently of interest, as may be seen, for example, in Feller [8]. We can write (1) as

$$x(t) = a(t) - C(t,s)\int_0^s x(u)du\Big|_0^t + \int_0^t C_s(t,s)\int_0^s x(u)du$$
$$= a(t) - C(t,t)\int_0^t x(u)du + \int_0^t C_s(t,s)\int_0^s x(u)duds$$

or with $w(t) := \int_0^t x(u)du$ we have

$$w'(t) = a(t) - C(t,t)w(t) + \int_0^t C_s(t,s)w(s)ds. \tag{10}$$

There is the resolvent equation drawn from (10) and there is the variation of parameters formula. By noting that $w(0) = 0$ we obtain

$$w(t) = Z(t,0)w(0) + \int_0^t Z(t,s)a(s)ds$$
$$= \int_0^t Z(t,s)a(s)ds$$
$$= Z(t,s)\int_0^s a(u)du\Big|_0^t - \int_0^t Z_s(t,s)\int_0^s a(u)duds$$

or

$$w(t) = \int_0^t a(u)du - \int_0^t Z_s(t,s)\int_0^s a(u)duds. \tag{11}$$

In the next result, $C_2(t,s) = \partial C(t,s)/\partial s$.

Theorem 4.3. *If there is a positive number α such that*

$$\int_0^\infty |C_2(u+s,s)|du - C(t,t) \le -\alpha$$

and if $a \in L^1$, then $w \in L^1$ for any solution of (11). Hence

$$w(t) = \int_0^t a(u)du - \int_0^t Z_s(t,s)\int_0^s a(u)duds \in L^1.$$

Therefore,

$$(P\phi)(t) = \int_0^t \phi(s)ds - \int_0^t Z_s(t,s)\int_0^s \phi(u)duds$$

maps $W \to L^1$ where $\phi \in W$ if $\phi \in L^1$.

Proof. Let

$$V(t) = |w(t)| + \int_0^t \int_{t-s}^\infty |C_2(u+s,s)|du|w(s)|ds$$

so that

$$V'(t) \le |a(t)| - C(t,t)|w(t)| + \int_0^t |C_s(t,s)||w(s))|ds$$
$$+ \int_0^\infty |C_2(u+t,t)|du|w(t)| - \int_0^t |C_2(t,s)w(s)|ds$$
$$\le |a(t)| + \left[-C(t,t) + \int_0^\infty |C_2(u+t,t)|du\right]|w(t)|$$
$$\le |a(t)| - \alpha|w(t)|.$$

Hence, $a \in L^1$ implies $w \in L^1$. □

We have seen with integral equations how to change the Liapunov functional to obtain L^2 results and they work in exactly the same way here.

Theorem 4.4. *If there exists $\alpha > 0$ such that*

$$\int_0^\infty |C_2(u+s,s)|du + \int_0^t |C_2(t,s)|ds - 2C(t,t) \le -\alpha$$

and if $a \in L^2$ then any solution $w(t)$ of (10) is also in L^2. Notice that for $a(t) = 1/(t+1)$, then $\int_0^t a(u)du = \ln(t+1)$ so the terms in the variation of parameter formula (11) tend to ∞ and, yet, the difference in the terms is in L^2.

Proof. Let

$$V(t) = w^2(t) + \int_0^t \int_{t-s}^\infty |C_2(u+s,s)|du w^2(s)ds$$

so that for $\alpha/2 > 0$ there is a positive number M with $2w(t)a(t) \le (\alpha/2)w^2(t) + Ma^2(t)$ and

$$V'(t) = 2w(t)a(t) - 2C(t,t)w^2(t) + 2w(t)\int_0^t C_s(t,s)w(s)ds$$
$$+ \int_0^\infty |C_2(u+t,t)|du w^2(t) - \int_0^t |C_2(t,s)|w^2(s)ds$$
$$\le Ma^2(t) + (\alpha/2)w^2(t) - 2C(t,t)w^2(t) + \int_0^t |C_s(t,s)|(w^2(t) + w^2(s))ds$$
$$+ \int_0^\infty |C_2(u+t,t)|du w^2(t) - \int_0^t |C_2(t,s)|w^2(s)ds$$
$$= Ma^2(t) + \left[(\alpha/2) - 2C(t,t) + \int_0^\infty |C_2(u+t,t)|du + \int_0^t |C_s(t,s)|ds\right]w^2(t)$$
$$\le -(\alpha/2)w^2 + Ma^2(t).$$

This yields $w \in L^2$. □

We return now to (5) and to the Levin functional.

Theorem 4.5. *Suppose that $C(t,t) \geq \alpha > 0$ and for $H(t,s) = C_t(t,s)$ we suppose that*

$$H(t,s) \geq 0,\ H_s(t,s) \geq 0,\ H_{st}(t,s) \leq 0,\ H_t(t,0) \leq 0.$$

If, in addition, $a' \in L^2$, then any solution $x(t)$ of (5) is also in L^2. Thus, $R(t,s)$ generates an L^2 approximate identity on the space of functions ϕ with $\phi' \in L^2$.

Proof. We have

$$x'(t) = a'(t) - C(t,t)x - \int_0^t H(t,s)x(s)ds$$

and we define

$$V(t) = x^2(t) + \int_0^t H_s(t,s)\Big(\int_s^t x(u)du\Big)^2 ds + H(t,0)\Big(\int_0^t x(s)ds\Big)^2$$

so that

$$\begin{aligned} V'(t) = &\int_0^t H_{st}(t,s)\Big(\int_s^t x(u)du\Big)^2 ds + 2x\int_0^t H_s(t,s)\int_s^t x(u)duds \\ &H_t(t,0)\Big(\int_0^t x(s)ds\Big)^2 + 2xH(t,0)\int_0^t x(s)ds \\ &+ 2x(t)a'(t) - 2C(t,t)x^2(t) - 2x(t)\int_0^t H(t,s)x(s)ds. \end{aligned}$$

Integrate the second term on the right-hand-side and obtain

$$\begin{aligned} &2x\Big[H(t,s)\int_s^t x(u)du\Big|_0^t + \int_0^t H(t,s)x(s)ds\Big] \\ &= 2x\Big[-H(t,0)\int_0^t x(u)du + \int_0^t H(t,s)x(s)ds\Big]. \end{aligned}$$

This yields

$$V'(t) \leq 2x(t)a'(t) - 2C(t,t)x^2(t) \leq Ma'(t)^2 - kx^2(t)$$

for appropriate positive numbers M and k. Noting that $x^2 \leq V(t)$ and integrating we obtain $x \in L^\infty$, $x \in L^2$. Thus, as $a' \in L^2$ we obtain $a(t) - \int_0^t R(t,s)a(s)ds \in L^2$. □

5. Two steps up and, perhaps, more

We started with

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds$$

and

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds.$$

If we prove that $x \in L^p$ for every $a \in L^q$ then

$$(P\phi)(t) = \phi(t) - \int_0^t R(t,s)\phi(s)ds$$

maps L^q into L^p and $R(t,s)$ generates an L^p approximate identity on the vector space L^q. The resolvent $R(t,s)$ is determined from $C(t,s)$ alone.

If $x \in L^p$ for each a with $a^{(q)} \in L^d$, then $R(t,s)$ generates an L^p approximate identity on the vector space W of functions ϕ such that $\phi^{(q)} \in L^d$. Thus, if $\phi^{(q)} \in L^d$, then ϕ can have arbitrarily rapid growth as $q \to \infty$.

We have seen resolvents generate L^p approximate identities on spaces of functions which grow almost as fast as $t^{1/2}$. Can we continue and obtain L^p approximate identities on spaces of functions with arbitrarily rapid growth? Our ability to prove such behavior is limited only by our ability to prove that $x \in L^p$ when $a^{(q)} \in L^d$. That is simply a technical problem. Here, we contrive such a problem, allowing a with $a'' \in L^1$. At the same time we show how the Levin functional can be used in a variety of ways.

The following result concerns (1) in which

$$C(t,t) = \alpha > 0,\ C_1(t,t) = \beta > 0,$$

$$C_{11}(t,s) =: f(t-s) < 0,\ \beta + \int_0^\infty f(u)du > 0, \tag{12}$$

and

$$\int_{t-s}^\infty f(u)du =: F(t-s),\ F(t-s) < 0,$$

$$F_s(t-s) < 0,\ F_{st}(t-s) > 0,\ F(t) < 0,\ F_t(t) > 0. \tag{13}$$

A function satisfying these conditions is

$$C(t,s) = 2 + 3(t-s) - e^{-(t-s)}.$$

Moreover, taking $a(t) = t$ will show the desired behavior and will avoid much of the work below. We have shown before how changing the Liapunov functional will allow $a' \in L^2$. In the result below we can make changes which will allow $a'' \in L^2$. Such changes are left to the reader.

Theorem 5.1. *If (12) and (13) hold, then the solution x of (1) is in L^2 whenever $a'' \in L^1$. Thus, $R(t,s)$ generates an L^2 approximate identity on the vector space of functions $\phi : [0,\infty) \to \Re$ with $\phi''(t) \in L^1[0,\infty)$.*

Proof. We have

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_1(t,s)x(s)ds$$

and

$$x''(t) = a''(t) - \alpha x'(t) - C_1(t,t)x(t) - \int_0^t C_{11}(t,s)x(s)ds$$
$$= a''(t) - \alpha x'(t) - \beta x(t) - \int_0^t f(t-s)x(s)ds.$$

Write

$$x'(t) = y(t) - \alpha x(t) + \int_0^t \int_{t-s}^\infty f(u)dux(s)ds$$

so

$$x''(t) = y' - \alpha x'(t) + \int_0^\infty f(u)dux(t) - \int_0^t f(t-s)x(s)ds$$
$$= a''(t) - \alpha x'(t) - \beta x(t) - \int_0^t f(t-s)x(s)ds$$

or

$$y' = a''(t) - \beta x(t) - \int_0^\infty f(u)dux(t).$$

This yields the system

$$x'(t) = y(t) - \alpha x(t) + \int_0^t F(t-s)x(s)ds$$
$$y'(t) = a''(t) - \left(\beta + \int_0^\infty f(u)du\right)x(t).$$

A suitable Liapunov functional is

$$V(t) = \frac{x^2(t)}{2} + \frac{y^2(t)}{2\left(\beta + \int_0^\infty f(u)du\right)}$$
$$- (1/2)\int_0^t F_s(t-s)\left(\int_s^t x(u)du\right)^2 ds - (1/2)F(t)\left(\int_0^t x(u)du\right)^2$$

so that

$$V'(t) = xy - \alpha x^2 + x\int_0^t F(t-s)x(s)ds + \frac{a''(t)y}{\beta + \int_0^\infty f(u)du}$$
$$- xy - (1/2)\int_0^t F_{st}(t-s)\left(\int_s^t x(u)du\right)^2 ds$$
$$- x\int_0^t F_s(t-s)\int_s^t x(u)duds$$
$$- (1/2)F_t(t)\left(\int_0^t x(u)du\right)^2 - xF(t)\int_0^t x(u)du.$$

Integrating the third-to-last term by parts yields

$$-x\left[F(t-s)\int_s^t x(u)du\Big|_0^t + \int_0^t F(t-s)x(s)ds\right]$$
$$= -x\left[-F(t)\int_0^t x(u)du + \int_0^t F(t-s)x(s)ds\right].$$

Hence,

$$\begin{aligned} V'(t) &= -\alpha x^2 + \frac{a''(t)y}{\beta + \int_0^\infty f(u)du} \\ &\quad - (1/2)\int_0^t F_{st}(t-s)\left(\int_s^t x(u)du\right)^2 ds - (1/2)F_t(t)\left(\int_0^t x(u)du\right)^2 \\ &\le -\alpha x^2 + \frac{a''(t)y}{\beta + \int_0^\infty f(u)du} \\ &\le -\alpha x^2 + K|a''(t)|[V(t)+1] \end{aligned}$$

for an appropriate constant K.

We integrate that differential inequality and obtain

$$V(t) \le V(0)e^{\int_0^t K|a''(s)|ds} - \int_0^t \alpha x^2(s)ds + e^{\int_0^t K|a''(s)|ds}.$$

This yields $x \in L^2$ whenever $a'' \in L^1$. □

In conclusion, we have portrayed large spaces of large functions, $a(t)$, which are essentially harmless. There is great incentive for further study.

6. Acknowledgments

We are very grateful to Professors L. Becker and Bo Zhang for reading portions of the manuscript and making corrections and very useful suggestions.

7. References

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