INTEGRAL EQUATIONS, LARGE AND SMALL FORCING FUNCTIONS: PERIODICITY

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Abstract.
The defining property of an integral equation with resolvent $R(t,s)$ is the relation between $a(t)$ and $\int_0^t R(t,s)a(s)ds$ for functions $a(t)$ in a given vector space. We study the behavior of a solution of an integral equation

$$x(t) = a_1(t) + a_2(t) - \int_0^t C(t,s)x(s)ds$$

when $a_1(t)$ is periodic, $C(t+T, s+T) = C(t,s)$, while $a_2(t)$ is typified by $(t+1)^\beta$ with $0 < \beta < 1$. There is a resolvent, $R(t,s)$, so that

$$x(t) = a_1(t) + a_2(t) - \int_0^t R(t,s)[a_1(s) + a_2(s)]ds.$$

We show that the integral $\int_0^t R(t,s)a_2(s)ds$ so closely approximates $a_2(t)$ that the only trace of that large function, $a_2(t)$, in the solution is an L^p-function, $p < \infty$. In short, that large function $a_2(t)$ has essentially no long term effect on the solution which turns out to be the sum of a periodic function, a function tending to zero, and an L^p-function. The noteworthy property here is that with great precision the integral $\int_0^t R(t,s)a(s)ds$ can duplicate vector spaces of functions both large and small, both monotone and oscillatory; however, it cannot duplicate a given nontrivial periodic function $a(t)$ other than $k\left[1 + \int_{-\infty}^t C(t,s)ds\right]$ where k is constant. The integral $\int_0^t R(t,s)\sin(s+1)^\beta ds$ is an L^p approximation to $\sin(t+1)^\beta$ for $0 < \beta < 1$, but contraction mappings show us that precisely at $\beta = 1$ that approximation fails and $\sin(t+1) - \int_0^t R(t,s)\sin(s+1)ds$ approaches a nontrivial periodic function.
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1. Introduction

It is a classical, but elementary, exercise to show that if $\int_0^t |C(t,s)|ds \leq \alpha < 1$ then every solution of $x(t) = a(t) - \int_0^t C(t,s)x(s)ds$ is bounded for every $a \in BC$, the space of bounded continuous functions. Moreover, it is a simple consequence of Perron's theorem that every solution of that equation is bounded for every $a \in BC$ if and only if $\sup_{t\geq 0} \int_0^t |R(t,s)|ds < \infty$ where $R(t,s)$ is the resolvent. Such facts promote the conventional idea that when an integral equation has a nice kernel the solution follows the forcing function. To motivate our work, if we encounter

$$x(t) = \sin t + (t+1)^\beta - \int_0^t C(t,s)x(s)ds$$

with $\int_0^t |C(t,s)|ds \leq \alpha < 1$, it might never occur to us to ask if there is an asymptotically periodic solution following $\sin t$ since $(t+1)^\beta$, for $0 < \beta < 1$, should vastly dominate everything else in the equation. Yet, under a variety of generous conditions, it is true that $(t+1)^\beta$ is totally ignored and the solution follows $\sin t$. It is a classic David and Goliath scenerio which should signal important properties to be investigated.

In our work here we consider an integral equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds,$$

a resolvent $R(t,s)$, and a variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds.$$

Continuing work in [3] and [5] we use a Liapunov functional and Young's inequality to obtain conditions showing that $a' \in L^p$, $0 < p < \infty$, implies that the solution $x \in L^p[0,\infty)$. While this fact in the equation itself seems most unremarkable, when we look at the variation of parameters formula it is remarkable indeed. This means that $a(t) - \int_0^t R(t,s)a(s)ds \in L^p$ for all functions a in a vector space W of functions having a derivative in L^p. For every one of those functions, it is true that $\int_0^t R(t,s)a(s)ds$ so closely approximates $a(t)$ that it is virtually the identity map for large t on that whole vector space. Thus, we may think of R as being stable since the result of the operation $\int_0^t R(t,s)a(s)ds$ remains close to $a(t)$.

Such behavior is a special surprise since the ideal theory of Ritt [12] shows that $R(t,s)$ is arbitrarily complicated. Yet, $\int_0^t R(t,s)a(s)ds$

strips away all that complication and leaves us with $a(t)$ virtually unscathed.

The new direction here concerns the question of what happens to $a(t) - \int_0^t R(t,s)a(s)ds$ at what we might call boundary points of that vector space W. Obviously, with no topology given for W there is no boundary in standard mathematical terminology. To be definite, if $0 < \beta_i < 1$ then we have a $p > 0$ with

$$\sin(t+1)^{\beta_1} + (t+1)^{\beta_2} - \int_0^t R(t,s)[\sin(s+1)^{\beta_1} + (s+1)^{\beta_2}]ds \in L^p.$$

It is shown in [3] that when $\beta_2 = 1$ then the relation is unchanged; thus, the so-called boundary point $(t+1)$ is still faithfully duplicated by that integral.

The problem of this paper begins with the challenge of just what happens when $\beta_1 = 1$. Two sections of this paper are devoted to it using contractions. Once we have determined the proper space and the proper mapping, then it becomes a simple problem with contractions. It turns out that the point $\sin(t+1)$ generates a major instability in the variation of parameters relation. That resolvent which could so faithfully duplicate a when $a' \in L^p$ is totally unable to duplicate $\sin(t+1)$. As β_1 leaves $(0,1)$ and takes on the value 1, the solution loses its L^p property and becomes the sum of a periodic function, a function tending to zero (both from $\sin(t+1)$), and an L^p-function (from (t+1)). The surprising fact is that the large function $t+1$ is essentially ignored as part of the solution and the small periodic function exerts a permanent and large influence on the solution.

Thus, some boundary points are stable and some are not. This introduces an important inquiry. Which properties of $a(t)$ are essentially unimportant and which properties are of lasting and fundamental significance for the solution?

This paper is one in a long series in which we strive to construct spaces and mappings to study the fundamental properties of solutions of differential and integral equations in a concise and elementary way. In a recent monograph [4] we collect roughly 100 such presentations.

Most of this work can be done for systems, but all the ideas are present in the scalar case. General theory of integral equations can be found in [1], [6-8], and [11].

PART I: SMALL KERNELS, PERFECT HARMONY

In the next two sections we develop material which shows us that under appropriate smallness condiitons on $a(t)$ and $C(t,s)$ there is perfect correspondence between the solution $x(t)$ and the forcing function $a(t)$. Notice that every one of these theorems is true for $C(t,s) \equiv 0$; thus, if we expect the solution to be bounded we must ask $a(t)$ to be bounded. In Part II we allow $a(t)$ to be unbounded.

2. Asymptotically Periodic Forcing

First, we study a known example in which neither the mapping nor the space is elusive, but it directs our thinking for future problems. We consider a scalar integral equation

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)x(s)ds \tag{1}$$

in which a is continuous on R, while $C(t,s)$ is continuous on $R \times R$, and there is a positive constant T with

$$a(t+T) = a(t) \text{ and } C(t+T, s+T) = C(t,s). \tag{2}$$

Theorem 2.1. *Suppose that (2) holds, that $\int_{-\infty}^{t} |C(t,s)|ds$ is continuous and*

$$\sup_{0 \le t \le T} \int_{-\infty}^{t} |C(t,s)|ds \le \alpha < 1.$$

Then (1) has a T-periodic solution.

Proof. Let $(X, \|\cdot\|)$ be the Banach space of continuous T-periodic functions ϕ with the supremum norm. Define $P : X \to X$ by $\phi \in X$ implies

$$(P\phi)(t) = a(t) - \int_{-\infty}^{t} C(t,s)\phi(s)ds.$$

A translation using (2) readily establishes that $P\phi$ is T-periodic. Next, if $\phi, \eta \in X$ then for $0 \le t \le T$ we have

$$\begin{aligned} |(P\phi)(t) - (P\eta)(t)| &\le \int_{-\infty}^{t} |C(t,s)||\phi(s) - \eta(s)|ds \\ &\le \|\phi - \eta\| \int_{-\infty}^{t} |C(t,s)|ds \\ &\le \alpha\|\phi - \eta|. \end{aligned}$$

Hence, P is a contraction and there is a unique fixed point solving (1) and residing in X. ☐

The result is certainly known, but we want to focus on the forms being used here.

With this as our guide we consider a scalar equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds \tag{3}$$

where C is continuous on $R \times R$ and a is continuous on R.

When (2) is satisfied, then (3) can have a periodic solution for certain rare functions $a(t)$, as is noted in [2; p. 95] and [10; p. 121] for a differentiated form. But it depends on an orthogonal relation between the resolvent and $a(t)$. The difficulty is that the lower limit on the integral prevents the right-hand-side from being periodic when $x(t)$ is periodic. In those same references, as well as Hino and Murakami [9], one finds cases in which limiting equations can be used to prove the existence of asymptotically periodic solutions, but the details are buried deeply inside the theory. All of that work can be avoided with a simple contraction argument.

If we write (3) as

$$x(t) = a(t) - \int_{-\infty}^t C(t,s)x(s)ds + \int_{-\infty}^0 C(t,s)x(s)ds$$

then

$$a(t) - \int_{-\infty}^t C(t,s)x(s)ds$$

suggests the periodic function of our last problem, while

$$\int_{-\infty}^0 C(t,s)x(s)ds$$

can readily be expected to tend to zero for any bounded function x. It is then natural to expect a solution $x = p + q$ where p is periodic and q tends to zero. Moreover, a space of such functions with the supremum norm is a Banach space, $(Y, \|\cdot\|)$. We note that the natural mapping defined from (3) will map $Y \to Y$.

Let $\mathcal{P}_T$ be the set of continuous T-periodic functions and suppose that for $\phi \in \mathcal{P}_T$ then

$$\int_{-\infty}^0 C(t,s)\phi(s)ds \to 0 \text{ as } t \to \infty. \tag{4}$$

Let Q be the set of continuous functions $q : [0, \infty) \to R$ such that $q(t) \to 0$ as $t \to \infty$. For each $q \in Q$ let

$$\int_0^t C(t,s)q(s)ds \to 0 \text{ as } t \to \infty. \tag{5}$$

We will need the follwing lemma in the proof of the next result.

Lemma 2.2. *Let $(Y, \|\cdot\|)$ be the space of continuous functions $\phi : [0, \infty) \to R$ with the supremum norm such that $\phi \in Y$ implies there is a $p \in \mathcal{P}_T$ and $q \in Q$ with $\phi = p + q$. Then $(Y, \|\cdot\|)$ is a complete metric space.*

Proof. Let $\{p_n + q_n\}$ be a Cauchy sequence in $(Y, \|\cdot\|)$. Now for each $\epsilon > 0$ and each $q \in Q$ there is an $L > 0$ such that $t \geq L$ implies that $|q(t)| < \epsilon/4$. Given $\epsilon > 0$ there is an N such that for n, $m \geq N$ then

$$|p_n(t) + q_n(t) - p_m(t) - q_m(t)| < \epsilon/2$$

for all $t \geq 0$. Fix n, $m \geq N$; for $\epsilon/4$ find L such that $t \geq L$ implies that both $|q_n(t)| < \epsilon/4$ and $|q_m(t)| < \epsilon/4$. Now

$$|p_n(t) - p_m(t)| - |q_m(t) - q_n(t)| \leq |p_n(t) + q_n(t) - p_m(t) - q_m(t)| < \epsilon/2$$

so $t \geq L$ implies that

$$|p_n(t) - p_m(t)| < (\epsilon/2) + |q_n(t)| + |q_m(t)| < \epsilon.$$

But p_n and p_m are periodic so the left and right sides of the last inequality hold for all t. As this is true for every pair (m, n) with m, $n \geq N$, it follows that $\{p_n\}$ is a Cauchy sequence. This, in turn, shows the same for $\{q_n\}$. As both $\mathcal{P}_T$ and Q are complete in the supremum norm, Y is complete. □

Thanks to colleagues Geza Makay and Bo Zhang, one can say that the representation of $\phi \in Y$ as $\phi = p+q$ is unique. For if $p_1+q_1 = p_2+q_2$ then $p_1 - p_2 = q_2 - q_1$. The right-hand-side tends to zero, but the left is periodic so the left is zero.

Theorem 2.3. *Let $C(t + T, s + T) = C(t, s)$ and let (4) and (5) hold. Suppose also that there is an $\alpha < 1$ with $\int_0^t |C(t, s)|ds \leq \alpha$. If $a \in Y$ so is x, the unique solution of (3).*

Proof. Let $(Y, \|\cdot\|)$ be the Banach space of functions $\phi = p + q$ where $p \in \mathcal{P}_T$ and $q \in Q$ with the supremum norm. Also, let $a = p^* + q^* \in Y$.

Define a mapping $P : Y \to Y$ by $\phi = p + q \in Y$ implies that

$$\begin{aligned}(P\phi)(t) &= a(t) - \int_0^t C(t,s)[p(s)+q(s)]ds \\ &= \left[p^*(t) - \int_{-\infty}^t C(t,s)p(s)ds\right] \\ &+ \left[q^*(t) + \int_{-\infty}^0 C(t,s)p(s)ds - \int_0^t C(t,s)q(s)ds\right] \\ &=: B\phi + A\phi.\end{aligned}$$

This defines operators A and B on Y. Note that $B : Y \to \mathcal{P}_T \subset Y$ and $A : Y \to Q \subset Y$.

But from the first line of this array we see that P is a contraction with unique fixed point $\phi \in Y$ and that proves the result. □

Remark 2.4. Our use of contractions here allows us to show that the fixed point is not constant. In Section 4 we will see that when $a' \in L^{2n}$ then $x(t) = a(t) - \int_0^t R(t,s)a(s)ds \in L^{2n}$ so that the integral is faithfully duplicating $a(t)$. Notice that in the present case if $a \in \mathcal{P}_T$ and if $x(t) = p(t) + q(t)$ then $p(t)$ really is a nontrivial periodic function; the integral is not duplicating $a(t)$. For suppose p is constant. From the proof we see that $p = a(t) - \int_{-\infty}^t C(t,s)pds$ or $p[1 + \int_{-\infty}^t C(t,s)ds] = a(t)$. This is the only form for $a(t)$ for which that duplication is possible. For all other $a(t) \in \mathcal{P}_T$ there is a nontrivial asymptotically periodic solution. In that case $\int_0^t R(t,s)a(s)ds$ misses $a(t)$ by a nontrivial periodic function plus a function tending to zero.

3. An inequality and Liapunov functional

There is a set of results which the reader should have in mind as we proceed. Throughout this discussion, notice which variable in $C(t,s)$ is being integrated.

The first result is an ancient theorem and it has many generalizations. Here, BC is the Banach space of bounded continuous functions with the supremum norm.

Theorem 3.1. *If $a \in BC$ and C is continuous with*

$$\int_0^t |C(t,s)|ds \le \alpha < 1$$

then the solution x of (3) is also in BC.

Simply use (3) to define a mapping which is a contraction.

Theorem 3.2. *If a and C are continuous and if*

$$\int_0^\infty |C(u+t,t)|du \le \beta < 1$$

then $a \in L^1[0,\infty)$ implies that $x \in L^1[0,\infty)$.

This is proved in [3] and is a quick consequence of a Liapunov functional $V(t) = \int_0^t \int_{t-s}^\infty |C(u+s,s)|du|x(s)|ds$.

Theorem 3.3. *If both the integral conditions of the last two results hold, then $a \in L^2[0,\infty)$ implies that $x \in L^2[0,\infty)$.*

This is also proved in [3] and we note here that an inequality can be developed to yield a general result.

Lemma 3.4. *Suppose there is an $\alpha < 1$ with*

$$\sup_{t\ge 0} \int_0^t |C(t,s)|ds \le \alpha.$$

Consider equation (3). There is an $M > 0$ and for each integer $n > 0$ we have

$$x^{2^n}(t) \le M^{2^n-1} a^{2^n}(t) + \int_0^t |C(t,s)| x^{2^n}(s)ds.$$

Proof. In (3) we take absolute values and square both sides to obtain

$$x^2(t) \le a^2(t) - 2a(t)\int_0^t C(t,s)x(s)ds + \left(\int_0^t C(t,s)x(s)ds\right)^2.$$

Find $\epsilon > 0$ with $(1+\epsilon)\alpha = 1$ and then find $M > 1$ with $2|a(t)||y| \le (M-1)a^2(t) + \epsilon y^2$. Thus,

$$\begin{aligned} x^2(t) &\le Ma^2(t) + (1+\epsilon)\left(\int_0^t C(t,s)x(s)ds\right)^2 \\ &\le Ma^2(t) + (1+\epsilon)\int_0^t |C(t,s)|ds \int_0^t |C(t,s)|x^2(s)ds \\ &\le Ma^2(t) + \int_0^t |C(t,s)|x^2(s)ds \end{aligned}$$

where we have used the Schwarz inequality. Next, suppose there is a positive integer k with

$$x^{2k}(t) \le M^{2k-1}a^{2k}(t) + \int_0^t |C(t,s)|x^{2k}(s)ds.$$

Squaring yields

$$\begin{aligned}
x^{4k}(t) &\leq M^{4k-2}a^{4k}(t)\\
&\quad + 2M^{2k-1}a^{2k}(t)\int_0^t |C(t,s)|x^{2k}(s)ds\\
&\quad + \left(\int_0^t |C(t,s)|x^{2k}(s)ds\right)^2\\
&\leq M^{4k-2}a^{4k}(t) + (M-1)\Big[M^{2k-1}a^{2k}(t)\Big]^2\\
&\quad + (1+\epsilon)\left(\int_0^t |C(t,s)|x^{2k}(s)ds\right)^2\\
&\leq M^{4k-2}a^{4k}(t)(M-1+1)\\
&\quad + (1+\epsilon)\int_0^t |C(t,s)|ds\int_0^t |C(t,s)|x^{4k}(s)ds\\
&\leq M^{4k-1}a^{4k}(t) + \int_0^t |C(t,s)|x^{4k}(s)ds.
\end{aligned}$$

As we repeatedly apply these inequalities our exponents are the ones stated in the theorem. □

Theorem 3.5. *Let a and C be continuous. Suppose there are constants $\alpha < 1$ and $\beta < 1$ with*

$$\int_0^t |C(t,s)|ds \leq \alpha \text{ and } \int_0^\infty |C(u+t,t)|du \leq \beta.$$

If there is an $n > 0$ with $a \in L^{2^n}[0,\infty)$ then the solution of (3) satisfies $x \in L^{2^n}[0,\infty)$.

Proof. Define a Liapunov functional

$$V(t) = \int_0^t \int_{t-s}^\infty |C(u+s,s)|du x^{2^n}(s)ds$$

so that

$$\begin{aligned}
V'(t) &= \int_0^\infty |C(u+t,t)|du x^{2^n}(t) - \int_0^t |C(t,s)|x^{2^n}(s)ds\\
&\leq \beta x^{2^n}(t) - x^{2^n}(t) + M^{2^n-1}a^{2^n}(t).
\end{aligned}$$

Thus, an integration yields

$$0 \leq V(t) \leq V(0) - (1-\beta)\int_0^t x^{2^n}(s)ds + M^{2^n-1}\int_0^\infty a^{2^n}(t)dt,$$

as required. □

Remark 3.6. As we go through the previous results we see that there is absolute correspondence between $a(t)$ and $x(t)$ so it is easy to see how we would believe that with a nice kernel, then $x(t)$ always follows $a(t)$. But in the next part we will see that this is something of an illusion. The solution can be expressed as $x(t) = a(t) - \int_0^t R(t,s)a(s)ds$ and that integral faithfully duplicates $a(t)$. Thus, $a \in L^1$ will yield that the integral is in L^1 so x will be in L^1 by default. The function x was simply in the wrong place at the wrong time.

PART II: LARGE KERNELS, GREAT DISCORD

In this part we deal with kernels which could not be integrable and find that there is great disparity between $a(t)$ and the solution of (3). The fundamental difference between Parts I and II is that $C(t,s) \equiv 0$ is permitted in every result in Part I, but in no result in Part II.

4. A Remarkable Resolvent

All of the remaining theorems concern the case in which there is either a positive additive constant, an additive function of t, or an additive function of s which would be cleansed by differentiation with respect to either s or t. Our first result is new in both generality and use of the Hölder-Young's inequality. We mention Young here since it should be possible to extend this result to nonlinear equations and Young's inequality would be used instead of Hölder's.

The periodicity of $a(t)$ is a very special property which enables the forcing function to exert continued influence over the solution, no matter how small $a(t)$ may be. We now give a concise proof using Liapunov theory to show that this property is, indeed, rare.

Thus, we again consider the scalar equation (3) where we now suppose a' and C_t are continuous so that it can be written as

$$x'(t) = a'(t) - C(t,t)x - \int_0^t C_t(t,s)x(s)ds. \tag{7}$$

The resolvent equation for (3) is

$$R(t,s) = C(t,s) - \int_s^t R(t,u)C(u,s)du \tag{8}$$

and the variation-of-parameters formula for (3) is

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{9}$$

These relations can be found in Chapter 7 of [1] and [11; p. 190], for example.

We will see that for $a' \in L^p$ then $x \in L^p$ so that (9) will assure us that the integral so faithfully duplicates $a(t)$ that the error in that duplication is an L^p function. As time goes on the duplication becomes so precise on average that we can hardly tell the difference between the integral in (9) and $a(t)$. That large function, $a(t) = (t+1)^\beta$, has such a small effect on the solution of (3) it is almost as if it were absent. The same is true for $a(t) = \sin(t+1)^\beta$ when $0 < \beta < 1$.

In preparation for the next result, we note that the polynomial form of Young's inequality states that if p and q are numbers with $p > 0$, $p \neq 1$, and $(1/p) + (1/q) = 1$, then

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

For our repeated application below we will have n a positive integer, $p = \frac{2n}{2n-1}$, then $q = 2n$.

Moreover, in the subsequent work C_1 denotes the partial derivative with respect to the first argument, while C_2 denotes differentiation with respect to the second argument. The fundamental inequality in Theorem 4.1 below asks that integration of $C_t(t,s)$ fails to recover all of $C(t,s)$. Something, perhap a constant, is lost in forming $C_t(t,s)$. We have results in [3] which do not require this. However, this result is given mainly to work with our later results on asymptotically periodic solutions and all of those results require a similar condition.

Theorem 4.1. *Suppose there is a positive integer n with $a'(t) \in L^{2n}[0,\infty)$, a constant $\alpha > 0$, and a constant $N > 0$ with*

$$\frac{2n-1}{2nN^{\frac{2n}{2n-1}}} - C(t,t) + \frac{2n-1}{2n}\int_0^t |C_t(t,s)|ds + \frac{1}{2n}\int_0^\infty |C_1(u+t,t)|du \leq -\alpha.$$

Then the unique solution x of (3) is bounded and $x \in L^{2n}[0,\infty)$. From (9) we then have $x(t) = a(t) - \int_0^t R(t,s)a(s)ds \in L^{2n}$.

Proof. For a fixed solution of (7) we define the function

$$V(t) = \frac{x^{2n}(t)}{2n} + \frac{1}{2n}\int_0^t \int_{t-s}^\infty |C_1(u+s,s)|du\, x^{2n}(s)ds.$$

Compute the derivative along a solution of (7) by the chain rule as

$$V'(t) = -C(t,t)x^{2n} - \int_0^t C_t(t,s)x(s)x^{2n-1}(t)ds + x^{2n-1}(t)a'(t)$$
$$+ \frac{1}{2n}\int_0^\infty |C_1(u+t,t)|du x^{2n}(t) - \frac{1}{2n}\int_0^t |C_1(t,s)|x^{2n}(s)ds$$

(Use Young's inequality on the second and third terms on the R-H-S.)

$$\le \frac{(2n-1)x^{2n}(t)}{2nN^k} + \frac{(Na'(t))^{2n}}{2n} - C(t,t)x^{2n}$$
$$+ \int_0^t |C_t(t,s)|\left[\frac{(2n-1)x^{2n}(t)}{2n} + \frac{x^{2n}(s)}{2n}\right]ds$$
$$+ \frac{1}{2n}\int_0^\infty |C_1(u+t,t)|du x^{2n}(t) - \frac{1}{2n}\int_0^t |C_1(t,s)|x^{2n}(s)ds$$
$$= \frac{(Na'(t))^{2n}}{2n} + x^{2n}(t)\left[\frac{(2n-1)}{2nN^k} - C(t,t)\right.$$
$$\left.+ \frac{2n-1}{2n}\int_0^t |C_t(t,s)|ds + \frac{1}{2n}\int_0^\infty |C_1(u+t,t)|du\right]$$
$$\le -\alpha x^{2n}(t) + \frac{N^{2n}}{2n}|a'(t)|^{2n}$$

for large N and for $k = \frac{2n}{2n-1}$.

It follows that

$$\frac{x^{2n}(t)}{2n} \le V(t) \le V(0) - \alpha\int_0^t x^{2n}(s)ds + k^*\int_0^\infty (a'(s))^{2n}ds$$

for some $k^* > 0$. This is true for every solution of (7) and, hence, for (3). $\square$

As we continue to study the behavior of the resolvent, it increasingly seems to be a question of stability. In earlier work [3] and [5] we said that $R(t,s)$ generated an L^p approximate identity on a space W if $\phi \in W$ implies that $\phi(t) - \int_0^t R(t,s)\phi(s)ds \in L^p$.

It will be convenient in the remainder of the paper to use the following terminology.

Definition 4.2. A function R mapping $[0,\infty)\times[0,\infty)$ into the reals is said to be L^N-stable with respect to a vector space W of specified continuous functions ϕ mapping $[0,\infty)$ into the reals if for each $\phi \in W$ there is an integer n with

$$\phi(t) - \int_0^t R(t,s)\phi(s)ds \in L^n[0,\infty).$$

In our last theorem, the vector space W consisted of those functions ϕ such that $\phi' \in L^p$ for some $p \in (0,\infty)$.

Remark 4.3. Of course, this is a classical stability concept. For suppose that ϕ_1 and ϕ_2 are functions with $\phi_1 - \phi_2 \in W$. Then for

$$x_{\phi_1}(t) = \phi_1(t) - \int_0^t C(t,s)x(s)ds$$

and

$$x_{\phi_2}(t) = \phi_2(t) - \int_0^t C(t,s)x(s)ds$$

we have

$$x_{\phi_1} - x_{\phi_2} = \phi_1 - \phi_2 - \int_0^t R(t,s)[\phi_1(s) - \phi_2(s)]ds \in L^n[0,\infty).$$

We are saying that if $\phi_1 - \phi_2 \in W$ then they are "close" and the solutions generated are"close." In our examples we find that the functions $\sin(t+1)^\beta$, $(t+1)^\beta$, and $(t+1)$ are "close." Notice that if our examples are based on Theorem 4.1 with $\phi_1' \in L^p$ and $\phi_2' \in L^q$ where $p < q$, since $x^{2n}/2n \le V(t)$ we have $x(t) = \phi_1(t) - \int_0^t R(t,s)\phi_1 s)ds \in L^p$ and also $x \in L^q$.

5. Periodicity revisited

Theorem 2.3 and Lemma 2.2 enabled us to show the existence of asymptotically periodic solutions by an elementary fixed point argument instead of the traditional long and difficult process involving limiting equations, as may be seen for example in Burton [2; p. 105] or Lakshmikantham and Rao [10; pp. 115-122]. We continue that study here.

Our asymptotic periodic result in Theorem 2.3 asks that $\int_0^t |C(t,s)|ds \le \alpha < 1$. Differentiation of (3) can produce two simplifications; one is expected, the other is a pleasant surprise.

We continue to use the same notation here as in Theorem 2.3 for $\mathcal{P}_T$, Q and Y.

Suppose that $C(t,s)$ has an additive function of s, perhaps a constant, so that the inequality $\int_0^t |C(t,s)|ds \le \alpha < 1$ fails. When we write (7) as

$$x' = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds,$$

that offending function of s has been cleansed from the kernel. But, fortuitously, it has been transferred over to a place in the equation where it can be used to actually increase the size of the new requirement on the kernel, $C_t(t,s)$.

From (7) and the variation of parameters formula we have a new integral equation with $x(0) = a(0)$ in the form

(10)
$$x(t) = x(0)e^{-\int_0^t C(s,s)ds} + \int_0^t e^{-\int_u^t C(s,s)ds}\left[a'(u) - \int_0^u C_1(u,s)x(s)ds\right]du.$$

Notice that (2) will bestow many properties on (10). For example, $C(t,t)$ is periodic since $C(t+T, t+T) = C(t,t)$. Thus,

$$\int_{u+T}^{t+T} C(s,s)ds = \int_u^t C(s,s)ds,$$

as will be needed to show periodicity later. We suppose that there is a number $c^* > 0$ with

$$C(t,t) \geq c^* \tag{11}$$

and an $\alpha < 1$ with

$$\int_0^t e^{-\int_u^t C(s,s)ds} \int_0^u |C_1(u,s)|dsdu \leq \alpha \tag{12}$$

which will make the mapping defined from (10) be a contraction on any space with the supremum norm. Notice again that (12) is almost like the inequality in Theorem 4.1. We are assuming that when we integrate $C_1(t,s)$ we get less than $C(t,s)$; something is lost in the differentiation, possibly a constant.

Remark 5.1. We could replace (11) with

$$\int_0^T C(t,t)dt \geq c^* > 0 \tag{11*}$$

if we ask that (16) and (17) hold.

In order to prove that (10) has an asymptotically periodic solution in the same way we proved Theorem 2.3, (10) must be decomposed into the mappings A and B as in the proof of Theorem 2.3. Thus, we begin by writing $a'(t) = p^*(t) + q^*(t) \in Y$ and define a mapping from (10) by $\phi = p + q \in Y$ implies that

(13)
$$(P\phi)(t) = a(0)e^{-\int_0^t C(s,s)ds} + \int_0^t e^{-\int_u^t C(s,s)ds}\left[p^*(u) + q^*(u) - \int_0^u C_1(u,s)[p(s)+q(s)]ds\right]du.$$

The decomposition will be done in the proof of Theorem 5.3.

Parallel to (4) and (5) we now ask that

$$\int_{-\infty}^{0} |C_1(t,s)|ds \to 0 \text{ as } t \to \infty \tag{14}$$

and for $q \in Q$ then

$$\int_{0}^{t} C_1(t,s)q(s)ds \to 0 \text{ as } t \to \infty. \tag{15}$$

It may help to understand these by noting that if C were of convolution type then (14) would say that $C_1 \in L^1[0,\infty)$, while (15) would then be the classical theorem that the convolution of an L^1 function with a function tending to zero does, itself, tend to zero as $t \to \infty$.

Lemma 5.2. *If (11) holds then*

$$\int_{-\infty}^{0} e^{-\int_u^t C(s,s)ds} du \to 0 \ \textit{as } t \to \infty \tag{16}$$

and for $q \in Q$ then

$$\int_{0}^{t} e^{-\int_u^t C(s,s)ds} q(u)du \to 0 \ \textit{as } t \to \infty. \tag{17}$$

Proof. We have $c^* = \min C(t,t)$ and $C(t,t) \in \mathcal{P}_T$. Thus,

$$\begin{aligned}\int_{-\infty}^{0} e^{-\int_u^t C(s,s)ds} du &\le (1/c^*) \int_{-\infty}^{0} C(u,u) e^{-\int_u^t C(s,s)ds} du \\ &= (1/c^*) e^{-\int_u^t C(s,s)ds} \Big|_{-\infty}^{0} \\ &= (1/c^*) e^{\int_0^t -C(s,s)ds}\end{aligned}$$

which tends to zero as $t \to \infty$.

Next,

$$\int_{0}^{t} e^{-\int_u^t C(s,s)ds} |q(u)|du \le \int_{0}^{t} e^{-c^*(t-u)} |q(u)|du$$

which is the convolution of an L^1-function with a function tending to zero so it tends to zero. □

Theorem 5.3. *In (3) let a' and $C_1(t,s)$ be continuous. Let (11)-(12), and (14)-(15) hold. Suppose, in addition, that*

$$\int_{-\infty}^{t} |C_1(t,s)|ds \tag{18}$$

is bounded and $C(t+T, s+T) = C(t,s)$. If $a' \in Y$ so is x, the unique solution of (3).

Proof. Using (10) we define a mapping $P : Y \to Y$ by $\phi = p + q \in Y$ implies that

$$(P\phi)(t) = a(0)e^{-\int_0^t C(s,s)ds} + \int_0^t e^{-\int_u^t C(s,s)ds}\left[a'(u) - \int_0^u C_1(u,s)\phi(s)ds\right]du.$$

By (12) it is clearly a contraction, but we must show that $P : Y \to Y$. Write $a' = p^* + q^*$ and then

$$\begin{aligned}(P\phi)(t) = &\int_{-\infty}^t e^{-\int_u^t C(s,s)ds}\left[p^*(u) - \int_{-\infty}^u C_1(u,s)p(s)ds\right]du \\ &- \int_0^t e^{-\int_u^t C(s,s)ds}\int_0^u C_1(u,s)q(s)dsdu \\ &+ a(0)e^{-\int_0^t C(s,s)ds} - \int_{-\infty}^0 e^{-\int_u^t C(s,s)ds}\left[p^*(u) - \int_{-\infty}^u C_1(u,s)p(s)ds\right]du \\ &+ \int_0^t e^{-\int_u^t C(s,s)ds}\int_{-\infty}^0 C_1(u,s)p(s)dsdu \\ &+ \int_0^t e^{-\int_u^t C(s,s)ds}q^*(u)du.\end{aligned}$$

The first term on the right-hand-side is clearly in $\mathcal{P}_T$. In the second term, $\int_0^u C_1(u,s)q(s)ds \in Q$ by (15). Hence the second term is in Q by (17). The third term is in Q by (11). The fourth term is in Q by (16), (18), and the fact that $p^* \in \mathcal{P}_T$ and, hence, is bounded. The next to last term is in Q because of (14) followed by (15). The last term is in Q by (17). This completes the proof □

Remark 5.4. Notice in the last result that a significant instability can occur at $\beta = 1$. Under conditions on $C(t,s)$ of Theorem 4.1 the integral of that resolvent has been faithfully following $\sin(t+1)^\beta$ so that the difference is an L^p function. Suddenly, that relationship breaks completely and the integral with the resolvent seems to "struggle along trying to catch up with $\sin(t+1)$" but always is out of step, lagging by a nontrivial periodic function plus a function tending to zero.

Corollary 5.5. *If the conditions of Theorem 4.1 hold and if* $0 < \beta < 1$ *then* $\sin(t+1)^\beta - \int_0^t R(t,s)\sin(s+1)^\beta ds \in L^p$ *for some* $p < \infty$. *But at* $\beta = 1$, *under conditions on* $C(t,s)$ *of Theorem 5.3 then* $p = \infty$ *and that difference approaches a periodic function.*

We can now state the promised result, a corollary of Theorems 5.3 and 4.1.

Corollary 5.6. *Let the conditions on* $C(t,s)$ *of Theorems 5.3 and 4.1 hold. For fixed* $\beta \in (0,1)$ *there is a* $p \in \mathcal{P}_T$, $q \in Q$, *and* $u \in L^p[0,\infty)$

so that the solution of

$$x(t) = \sin t + (t+1)^\beta - \int_0^t C(t,s)x(s)ds$$

may be written as

$$x(t) = p(t) + q(t) + u(t).$$

Proof. The solution is

$$x(t) = \sin t + (t+1)^\beta - \int_0^t R(t,s)[\sin s + (s+1)^\beta]ds.$$

But

$$(t+1)^\beta - \int_0^t R(t,s)(s+1)^\beta ds =: u(t) \in L^p[0,\infty),$$

while

$$\sin t - \int_0^t R(t,s)\sin s ds$$

is the solution described in Theorem 5.3 and it has the required form of $p+q$. □

Remark 5.7. For a transparent example linking Theorem 4.1 and Theorem 5.3, let $k > 0$,

$$C(t,s) = k + \sin^2 s + D(t-s),\ D(t) > 0,\ D'(t) \leq 0.$$

We then have

$$\int_0^t -D'(s)ds = D(0) - D(t) < D(0)$$

and $C(t,t) = C(0) = k + \sin^2 t + D(0)$ and so we readily verify the inequality in Theorem 4.1 holds for large N and n. To satisfy (12) we have

$$\int_0^t e^{-\int_u^t [k+\sin^2 s + D(0)]ds} \int_0^u -D'(u-s)dsdu < \frac{D(0)}{k+D(0)}$$

and conditions of Theorem 5.3 are satisfied.

Notice in Theorem 5.3 that we work with $C_1(t,s)$ and that conditions (14), (15), and (18) all concern $C_1(t,s)$. There is a seemingly little known transformation which allows us to obtain a completely parallel result by working with $C_2(t,s)$ and to avoid differentiating $a(t)$. This time one might think of $C(t,s) = k + \sin^2 t + D(t-s)$ and note that

$C_s(t,s) = -D'(t-s)$; the kernel has been cleansed of $k + \sin^2 t$ so that conditions of Theorem 4.1 can possibly be satisfied. Write (3) as

$$\begin{aligned} x(t) &= a(t) - C(t,s)\int_0^s x(u)du\Big|_0^t + \int_0^t C_s(t,s)\int_0^s x(u)duds \\ &= a(t) - C(t,t)\int_0^t x(u)du + \int_0^t C_s(t,s)\int_0^s x(u)duds. \end{aligned}$$

Let $y(t) := \int_0^t x(u)du$ so that $y(0) = 0$ and we have

$$y'(t) = a(t) - C(t,t)y(t) + \int_0^t C_s(t,s)y(s)ds. \tag{7*}$$

By the variation of parameters formula we have

$$y(t) = \int_0^t e^{-\int_u^t C(s,s)ds}\left[a(u) + \int_0^u C_s(u,s)y(s)ds\right]du \tag{10*}$$

and we will need $\alpha < 1$ with

$$\int_0^t e^{-\int_u^t C(s,s)ds}\int_0^u |C_s(u,s)|dsdu \le \alpha. \tag{12*}$$

Parallel to (14) and (15) we ask that

$$\int_{-\infty}^0 |C_s(t,s)|ds \to 0 \text{ as } t \to \infty \tag{14*}$$

and

$$\int_0^t C_s(t,s)q(s)ds \to 0 \text{ as } t \to \infty \text{ for } q \in Q. \tag{15*}$$

Conditions (11), (16), (17), and Lemma 5.2 will be the same for both (7) and (7*), while (18) will be replaced by the conditon that

$$\int_{-\infty}^t |C_s(t,s)|ds \tag{18*}$$

is bounded.

Theorem 5.8. *In (3) let $a(t)$ and $C_s(t,s)$ be continuous. Let (11), (12*), (14*), (15*), and (18*) hold. Let $(C(t+T, s+T) = C(t,s)$ for some $T > 0$. If $a \in Y$ so is the unique solution of (10*) and of (3).*

Proof. The proof that $y \in Y$ is completely parallel to that of Theorem 5.3. Then consider (7^*) with $y \in Y$ so that $y = p + q$. We have

$$\int_0^t C_s(t,s)[p(s)+q(s)]ds = \int_{-\infty}^t C_s(t,s)p(s)ds - \int_{-\infty}^0 C_s(t,s)px(s)ds + \int_0^t C_s(t,s)q(s)ds \in Y.$$

It follows that $y' \in Y$. □

Remark 5.9. These results represent an introduction into some of the very interesting properties of the resolvent. One can hardly help but believe there are many more surprises about the effects of $\int_0^t R(t,s)a(s)ds$. It would be most interesting to come to understand how the resolvent can be so complicated and yet the integral so closely duplicate $a(t)$ for such a great variety of functions. But it does seem clear that we had come to believe that the solution follows $a(t)$ because we had looked at too few problems and functions $a(t)$ which were too small. We looked at small functions, found that $x(t)$ remained small, and erroneously concluded that $x(t)$ followed $a(t)$.

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