KRASNOSELSKII'S UNIFICATION, VOLTERRA'S INTEGRODIFFERENTIAL EQUATION, AND THE METHOD OF AIZERMAN

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Abstract. In this paper we continue the study of an idea of Krasnoselskii for a unifying theory of differential equations of many types based on a certain kind of inversion of the perturbed differential operator and a fixed point theorem. Earlier we had studied the idea for fractional differential equations, neutral differential equations, and for a linear form of an integrodifferential equation studied by Volterra. Here, we continue that study of Volterra's problem for nonlinear equations based on the global linearization technique of Aizerman.

1. Introduction

This is the third paper in which we study an idea of Krasnoselskii [8] concerning the unification of a broad area of differential equations. We have described that theory in some detail in [4] and that will not be repeated here. Briefly, Krasnoselskii's idea was that by careful inversion of a perturbed differential operator we obtain the sum of a contraction and compact map. He then offered a fixed point theorem which covered such cases. In [6] we simplified that theorem and used it to put the theory to the test in [4] and [5] concerning fractional differential equations, neutral functional differential equations, and a linear problem of Volterra. Here, we continue the study of Volterra's problem for the nonlinear case.

The problem of Volterra [16] is one of prime importance as it is used to model many classical and modern problems of applied mathematics. Our focus here is on the nonlinear integrodifferential equation

$$x'(t) = -\int_0^t D(t-s)g(x(s))ds + f(t), \quad t \geq 0, \tag{1.1}$$

where $f, D : [0,\infty) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous with

$$D(t) > 0, \quad \int_0^\infty D(t)dt < \infty, \tag{1.2}$$

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and

$$xg(x) > 0 \quad \text{for } x \neq 0. \tag{1.3}$$

The problem has been studied mainly by means of Liapunov's direct method under the assumption that D is convex: $D(t) > 0$, $D'(t) \leq 0$, and $D''(t) \geq 0$. See [10], [11], [12], [15], [13], and [14]. As those conditions are virtually impossible to verify in real-world problems, we study the problem here under assumptions on the average value of D.

We assume that there exist $G_1 \geq 0$ and $G_2 > 0$ with

$$G_1 \leq \frac{g(x)}{x} \leq G_2, \quad x \neq 0. \tag{1.4}$$

Note that from (1.4), (1.3) and the continuity of g it follows that

$$G_1 |x| \leq |g(x)| \leq G_2 |x|, \quad x \in \mathbb{R}. \tag{1.5}$$

In the special case when

$$G_1 = 1 = G_2,$$

equation (1.1) reduces to the forced linear equation

$$x'(t) = -\int_0^t D(t-s)x(s)ds + f(t), \quad t \geq 0. \tag{1.6}$$

The reader may note that this process, in effect, is an extension of the classical Aizerman [1] problem from ordinary differential equations to integrodifferential equations. Aizerman had an idea for global linearization of an ordinary differential equation containing a single nonlinear function, $g(x)$.

The first step in his process is to determine positive constants $\alpha < \beta$ so that if $\alpha < a < \beta$ and if $g(x)$ is replaced by ax in the differential equation (making it linear) then the zero solution of that linear equation is globally asymptotically stable. If the equation is autonomous, then that stability is immediately established using the Routh-Hurwicz criterion. Accordingly, the first step is quite trivial in the study of ordinary differential equations.

For the second step consider a class of admissible functions, $g(x)$, for which

$$\alpha x^2 < xg(x) < \beta x^2, \quad x \neq 0.$$

This expression is our (1.4) when we multiply by x^2. The Aizerman problem is to find conditions so that for every $g(x)$ in that admissible class then the zero solution is globally asymptotically stable. While the sector may turn out to be narrow, notice that this linearization does not require differentiability of $g(x)$ and it admits a solution for $0 \leq t < \infty$, together with qualitative properties of the solution, which is a much superior situation to that found in classical linearization using differentiability.

Aizerman's work was motived in large measure by control theory where $g(x)$ might be a control function. He was finding an admissible set of controls. Investigators have been very successful in solving the Aizerman problem for many important systems. Easily accessible discussions, together with many references, are found in Hahn [2, p. 42], Krasovskii [9, pp. 110-114], and Lefschetz [3, p. 42].

Our work here goes well beyond the Aizerman problem in that we consider an integrodifferential equation which has a forcing function. When we replace $g(x)$ by x in (1.1) we obtain

$$x'(t) = f(t) + \int_0^t R(t-s)x(s)ds,$$

which is the linear equation considered in [5]. In this paper we take the second step and show boundedness in case g is nonlinear and f is not zero.

The convexity of Volterra is replaced by (1.2) because of the essential impossibility of observing that behavior in real-world problems. In the same way, treating any function $g(x)$ satisfying (1.4) takes into account uncertainties and difficulty in measurements.

Having given the history and motivation of the project, we now begin the analysis. Concerning existence of solutions to (1.6), the results below cited as Theorem A and Lemma A have recently been obtained in [5]. We note that the assumption

$$e^{-Jt}\left|\int_0^t e^{Js}f(s)ds\right| \to 0, \tag{1.7}$$

found in [5] is not needed here. The reason is that condition (1.7) is used in obtaining Propositions 2 and 2b in [5] but results parallel to these are not given here.

Theorem A. *If F is uniformly continuous and*

$$\begin{aligned}\left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt}\\ + \int_0^t \left|Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D(s-u)ds\right| du \le 1, \quad t \ge 0,\end{aligned} \tag{1.8}$$

then there exists a bounded solution of equation (1.6).

Lemma A. *Let $J > 0$ and suppose that D satisfies (1.2).*
(i) If

$$\left|\int_0^t e^{Js}f(s)ds\right| \le \int_0^t \int_u^t e^{Js}D(s-u)dsdu, \quad t \ge 0, \tag{1.9}$$

is satisfied and

$$\int_0^\infty e^{Jv}D(v)\,dv \le J, \tag{1.10}$$

then (1.8) is always true.

(ii) If

$$\int_0^\infty e^{Jv} D(v)\, dv > J, \tag{1.11}$$

then (1.8) is equivalent with

$$J\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt}$$
$$+\int_k^t \left[1 - e^{-J(t-s)}\right] D(s)\, ds \le \int_0^k D(s)\, ds, \quad t \ge k,$$

where k is the unique solution of

$$J = \int_0^k e^{Jv} D(v)\, ds. \tag{1.12}$$

In this paper, our aim is to extend Theorem A, to the case of nonlinear equation (1.1).

Following the same steps as in [4], (see, also, [5]), we integrate (1.1), divide and multiply by $J > 0$ (with J being an arbitrary positive number), add and subtract $x(s)$ to obtain

$$x(t) = x(0) - \int_0^t J\left[x(s) - x(s) - \frac{f(s)}{J} + \frac{\int_0^s D(s-u) g(x(u)) du}{J}\right] ds, \quad t \ge 0.$$

Write the linear part as

$$z(t) = x(0) - \int_0^t Jz(s) ds, \quad t \ge 0,$$

so that there is a resolvent equation

$$R(t) = J - \int_0^t JR(s) ds, \quad t \ge 0,$$

with solution

$$R(t) = Je^{-Jt}, \quad t \ge 0,$$

which is completely monotone and satisfies

$$\int_0^\infty R(s) ds = 1.$$

We then have

$$z(t) = x(0)\left[1 - \int_0^t R(s) ds\right] = x(0) e^{-Jt}, \quad t \ge 0,$$

and by a variation of parameters formula

$$x(t) = z(t) + F(t) \tag{1.13}$$
$$+\int_0^t R(t-s)\left[x(s) - \frac{\int_0^s D(s-u) g(x(u)) du}{J}\right] ds,\ t \ge 0,$$

where

$$\begin{aligned} F(t) &= \int_0^t R(t-s)[f(s)/J]ds = \int_0^t e^{-J(t-s)} f(s)ds \\ &= e^{-Jt} \int_0^t e^{Js} f(s)ds, \quad t \geq 0. \end{aligned}$$

Changing the order of integration in the double integral, (1.13) is written as

(1.14)

$$x(t) = z(t) + F(t) + \int_0^t \left[R(t-u)x(u) - \int_u^t R(t-s)\frac{D(s-u)}{J} ds g(x(u)) \right] du, \; t \geq 0.$$

Let $(BC, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm and let

$$M = \{\phi \in BC : \|\phi\| \leq 1\}.$$

Let $A : M \to M$ be defined by

(1.15)

$$(Ax)(t) := z(t) + F(t) + \int_0^t \left[R(t-u)x(u) - \int_u^t R(t-s)\frac{D(s-u)}{J} ds g(x(u)) \right] du, \; t \geq 0.$$

For our purposes we make use of the the following fixed point theorem used in [5] (see, also, [4], [6] and its correction [7]). Note that *no compactness of the operator A is required.* Here is the reason. In the proof of the theorem we had an operator of the form

$$\int_0^t R(t-s)u(s, x(\cdot))ds,$$

and it was assumed that $|u(t, x(\cdot))|$ is bounded when $0 \leq t < \infty$ and $x(t)$ is a bounded function on $[0,\infty)$. That enabled us to show that we were dealing with an equi-continuous set of functions. By going to a weighted space we could then show that the resulting mapping is compact. In (1.15) we see that $u(s, x(\cdot))$ is a function which is bounded when x is bounded. Thus, we do not even mention u in the theorem and we do not mention compactness. All we are asking is continuity of A. The correction noted that we need M to be the closed ball.

Theorem 1.1 (Brouwer-Schauder type)**.** *Let $M = \{\phi \in BC : a \leq \phi \leq b\}$ for some $a < b$ and $t \geq 0$. Suppose that $x(0)$ and M are chosen so that for A defined in (1.15) then $A : M \to M$. If A is continuous then A has a fixed point in M.*

2. THE MAIN RESULTS

By continuity of g, from (1.4) we have that $g(0)=0$, thus for u_0 with $x(u_0)=0$ the integrand in (1.14) is zero. In view of (1.4), for this integrand we get the same result if we assume that the fraction $\frac{g(x)}{x}$ is meaningful at $x=0$ (with possible value any number in $[G_1,G_2]$) and write (1.14) as

(2.1)

$$x(t)=z(t)+F(t)$$
$$+\int_0^t\left[R(t-u)-\int_u^t R(t-s)\frac{D(s-u)}{J}ds\frac{g(x(u))}{x(u)}\right]x(u)du,\quad t\geq 0.$$

So, for the rest of the paper, without loss of generality, we may use the convention that the fraction $\frac{g(x)}{x}$ is meaningful at $x=0$ with (2.1) being well defined, regardless of u with $x(u)=0$.

Let

$$D_i(t)=G_iD(t),\quad t\geq 0,\ (i=1,2),$$

and set

$$m(u,t):=\max_{i=1,2}\left\{\left|Je^{-J(t-u)}-\int_u^t e^{-J(t-s)}D_i(s-u)ds\right|\right\},0\leq u\leq t.$$

Our first result is the next theorem which generalizes Theorem A.

Theorem 2.1. *If F is uniformly continuous and*

$$\left(|x(0)|+\left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt}+\int_0^t m(u,t)\,du\leq 1,\quad t\geq 0,\tag{2.2}$$

then the natural mapping defined by (2.1) on the set M maps M into M and Theorem 1.1 will give a fixed point in M.

Proof. From (2.1) we have for $x\in M$ and any $t\geq 0$

$$|x(t)|\leq|z(t)|+|F(t)|$$
$$+\int_0^t\left|R(t-u)-\int_u^t R(t-s)\frac{D(s-u)}{J}ds\frac{g(x(u))}{x(u)}\right||x(u)|\,du$$

and

$$|x(t)|\leq|z(t)|+|F(t)|\tag{2.3}$$
$$+\int_0^t\left|R(t-u)-\int_u^t R(t-s)\frac{D(s-u)}{J}ds\frac{g(x(u))}{x(u)}\right|du.$$

from which, in view of (1.4), (1.3) and the nonnegativity of the integrands, we have for $0 \le u \le t$

$$
\begin{aligned}
R(t-u) &- \int_u^t R(t-s)\frac{D(s-u)}{J}ds G_2 \\
&\le R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds\frac{g(x(u))}{x(u)} \\
&\qquad \le R(t-u) - G_1\int_u^t R(t-s)\frac{D(s-u)}{J}ds,
\end{aligned}
$$

or

$$
\begin{aligned}
R(t-u) &- \int_u^t R(t-s)\frac{D_2(s-u)}{J}ds \\
&\le R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds\frac{g(x(u))}{x(u)} \\
&\qquad \le R(t-u) - \int_u^t R(t-s)\frac{D_1(s-u)}{J}ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left|R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds\frac{g(x(u))}{x(u)}\right| \\
\le\ & \max\left\{\left|R(t-u) - \int_u^t R(t-s)\frac{D_1(s-u)}{J}ds\right|,\right. \\
& \left.\left|R(t-u) - \int_u^t R(t-s)\frac{D_2(s-u)}{J}ds\right|\right\} \\
=\ & \max\left\{\left|Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D_1(s-u)ds\right|,\right. \\
& \left.\left|Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D_2(s-u)ds\right|\right\} := m\,(u,t)\,,
\end{aligned}
$$

and so, in view of (2.3) and (2.2), we have

$$
\begin{aligned}
|x(t)| &\le |z(t)| + |F(t)| \\
&\quad + \int_0^t \left|R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds\frac{g(x(u))}{x(u)}\right| du \\
&\le \left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt} + \int_0^t m\,(u,t)\,du \\
&\le 1.
\end{aligned}
$$

Consequently, for the natural operator P defined by (2.1) we have $PM \subset M$. A routine proof of continuity of A and application of Theorem 1.1 complete the proof. □

Lemma 1, below, attempts to give a more handy form of the inequality (2.2). For our convenience, we set

$$T(u,t) = e^{-Jt}\int_0^{t-u} e^{J(s+u)}D(s)\,ds, \quad 0 \le u \le t,$$

$$T_i(u,t) = G_i T(u,t), \; 0 \le u \le t \;., \quad (i=1,2), \tag{2.4}$$

and

$$R(u,t) = Je^{-J(t-u)}, \quad 0 \le u \le t.$$

As it will be shown in the next section, in the special case of the linear forced equation (1.6) (i.e., when (1.5) holds), Lemma 1, reduces to Lemma A (Lemma 1 in [4]).

First, we note that by (1.2) one easily sees that $\int_0^\infty e^{Jv}D(v)\,dv$ is either a positive real number or $+\infty$, hence

$$D_0 := \begin{cases} \frac{J}{\int_0^\infty e^{Jv}D(v)dv}, & \int_0^\infty e^{Jv}D(v)\,dv < +\infty \\ 0, & \int_0^\infty e^{Jv}D(v)\,dv = +\infty \end{cases},$$

is a nonnegative real number. If G_1, G_2 are positive numbers with $G_1 \neq G_2$, then G_1, G_2 and $\frac{G_1+G_2}{2}$ define four disjoint semi-closed intervals on the half-axis $[0,+\infty)$, namely the intervals $[0,G_1)$, $\left[G_1, \frac{G_1+G_2}{2}\right)$, $\left[\frac{G_1+G_2}{2}, G_2\right)$, and (G_2,∞). In Lemma 1 we study the inequality (2.2) according to the position of D_0 regarding to these intervals. It turns out that, for the first two intervals, inequality (2.2) takes the same form (case (iii), in Lemma 1). If $G_1 = G_2 = G$, then $g(x) = Gx$, and (1.1) reduces to an equation of the type of (1.6), which has been studied in [5]. The proof of Lemma 1 is cited in the Appendix.

Lemma 1. *Let $J > 0$, D satisfy (1.2) and assume that*

$$\left|\int_0^t e^{Js}f(s)ds\right| \le G_1 \int_0^t \int_u^t e^{Js}D(s-u)ds du, \quad t \ge 0. \tag{2.5}$$

(i) If $D_0 \in [G_2,\infty)$ then (2.2) is always true.
(ii) If $D_0 \in \left[\frac{G_1+G_2}{2}, G_2\right)$, then (2.2) is

$$\begin{aligned}&\left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt} \\ &+ \int_0^t \left[R(u,t) - T_1(u,t)\right]du \le 1, \quad t \ge 0.\end{aligned} \tag{2.6}$$

(iii) If $D_0 \in \left[0, \frac{G_1+G_2}{2}\right)$ then (2.2) is

(2.7)

$$\begin{cases} \left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt} + \int_0^t \left[R(u,t) - T_1(u,t)\right]du \le 1, & t \le t_0, \\ \left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt} + \int_0^{t-t_0} \left[T_2(u,t) - R(u,t)\right]du & \\ + \int_{t-t_0}^t \left[R(u,t) - T_1(u,t)\right]du \le 1, & t > t_0. \end{cases}$$

where t_0 *the unique solution of the equation*

$$\frac{G_1+G_2}{2}\int_0^{t_0} e^{Js}D\left(s\right)ds=J. \tag{2.8}$$

3. DISCUSSION

In this section we are concerned, mainly, with four issues: (I) we show that if g is linear then Theorem 2.1 reduces to Theorem A and Lemma 1(ii) coincides with Lemma A (i) (see, also [5]); (II) we give a modified version of Theorem 2.1 which still yields bounded solutions of (1.1) under slightly weaker assumptions than inequality (2.2); (III) we apply our results to the case of a nonlinear unforced equation with $G_1=0$, and (IV) we discuss the inequality (2.7) in Lemma 1 (iii).

(I) As already noticed, for $G_1=1=G_2$ equation (1.1) becomes the linear forced equation (1.6). Furthermore, we have $\frac{G_1+G_2}{2}=1$, so the interval $\left[\frac{G_1+G_2}{2},G_2\right)$ (case (ii) in Lemma 1) reduces to the empty set. This explains why, in the linear case, only two cases are considered (Lemma A), in contrast with the three cases in Lemma 1.

By (1.4), from the definitions of D_i $(i=1,2)$ and m we have that

$$D_i(t)=D\left(t\right),\quad t\geq 0,\ \ (i=1,2),$$

and

$$m\left(u,t\right)=\left|Je^{-J(t-u)}-\int_u^t e^{-J(t-s)}D(s-u)ds\right|,\quad 0\leq u\leq t.$$

Then, inequality (2.2) becomes

$$\begin{aligned}&\left(|x(0)|+\left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt}\\&\qquad+\int_0^t\left|Je^{-J(t-u)}-\int_u^t e^{-J(t-s)}D(s-u)ds\right|du\leq 1,\quad t\geq 0,\end{aligned}$$

which, is (1.8), so Theorem 2.1 reduces to Theorem A.

Next, as $G_1=1=G_2$, we see that (2.5) becomes (1.9), (2.2) becomes (1.8), while $D_0\in[G_2,\infty)$ becomes (1.10). It follows that Lemma 1 (i) coincides with Lemma A (i). We note that, Lemma 1(iii) is not the analog of Lemma A (ii). It is possible that the technique used in [5] might also be of use to obtain results parallel to Lemma A (ii) in [5] for the (more complicated) inequality (2.2).

Concerning the first inequality in Lemma 1(iii), one may see that for $t\leq t_0$ we have $\int_0^t e^{Jv}D\left(v\right)dv\leq J$, and this is identical to the case discussed in (i), so we may conclude that *for* $t\leq t_0$ (1.10) *is satisfied, thus,* (2.2) *holds true* (this case coincides with Lemma 1 (i)). Thus, in the linear case, Lemma 1 (iii) reduces to

Corollary 1. *Let $J > 0$, D satisfy (1.2) and assume that (1.9) holds true. If (1.11) is true, then (1.8) is equivalent to*

$$\left(|x(0)| + \left|\int_0^t e^{Js} f(s)ds\right|\right) e^{-Jt} + \int_0^{t-t_0} \left[T(u,t) - R(u,t)\right] du$$
$$+ \int_{t-t_0}^t \left[R(u,t) - T(u,t)\right] du \leq 1, \quad t > t_0,$$

where $t_0 \equiv k$ is the unique solution of (1.12).

II. Theorem 2.1 yields bounded solutions to (1.1) provided that (1.8) holds. In order to have (1.8) satisfied, in particular, it must be true that

$$\sup_{t \geq 0} \left(|x(0)| + \left|\int_0^t e^{Js} f(s)ds\right|\right) e^{-Jt} \leq 1. \tag{3.1}$$

For $t = 0$, we see that it is necessary that $|x(0)| \leq 1$, which is natural as we seek solutions bounded by one. In fact, the initial values $x(0)$ may be forced to be very close to 0 as their values are dictated by the behavior of f so that the inequality (1.8) holds on the whole half axis. Moreover, from (3.1) it follows that $\sup_{t \geq 0} \left|\int_0^t e^{Js} f(s)ds\right| e^{-Jt} \leq 1$, also, another necessary condition so that (1.8) holds. Clearly inequality (3.1) shows that, the space for suitable $|x(0)|$ and f so that (1.8) holds is rather narrow. We would like to enlarge this space so that it includes large initial values and functions f with $\sup_{t \geq 0} \left|\int_0^t e^{Js} f(s)ds\right| e^{-Jt} > 1$. The following modification of Theorem 2.1 states that it is possible to obtain existence of bounded solutions of (1.1) for any initial value $x(0)$ as well as for functions f with $\sup_{t \geq 0} \left|\int_0^t e^{Js} f(s)ds\right| e^{-Jt} > 1$, provided that a weaker type of (1.8) holds.

Theorem 3.1. *If there exist an $\alpha > 0$ and a $c > 0$ such that for the function*

$$\phi(t) := \left(\alpha + \left|\int_0^t e^{Js} f(s)ds\right|\right) e^{-Jt}, \; t \geq 0, \tag{3.2}$$

we have

$$\int_0^t m(u,t)\, du \leq 1 - c\phi(t), \quad t \geq 0, \tag{3.3}$$

then, for any $x(0) \in \mathbb{R}$, equation (1.1) has a bounded solution.

Proof. Let $x(0) \in \mathbb{R}$ be given and set $\delta := max\left\{1, \frac{|x(0)|}{\alpha}\right\}$. Choose a $\mu > 0$ so that $\frac{\delta}{\mu} \leq c$ and denote by M_μ the set of all bounded by μ continuous functions, i.e.,

$$M_\mu = \{x : [0,\infty) \to \Re : \|x\| \leq \mu\}.$$

Let P be the natural mapping defined by (2.1) on the set M_μ. In view of (3.3), for an arbitrary $x \in M_\mu$ we have for $t \geq 0$

$$\begin{aligned}
|(Px)(t)| &\leq \left(|x(0)| + \left|\int_0^t e^{Js} f(s)ds\right|\right) e^{-Jt} + \int_0^t m(u,t)\,|x(u)|\,du \\
&= \left(\frac{|x(0)|}{\alpha}\alpha + \left|\int_0^t e^{Js} f(s)ds\right|\right) e^{-Jt} + \int_0^t m(u,t)\,|x(u)|\,du \\
&\leq \delta\left(\alpha + \left|\int_0^t e^{Js} f(s)ds\right|\right) e^{-Jt} + \mu\int_0^t m(u,t)\,du \\
&= \delta\phi(t) + \mu\int_0^t m(u,t)\,du \\
&= \mu\left\{\frac{\phi(t)\,\delta}{\mu} + \int_0^t m(u,t)\,du\right\} \\
&\leq \mu\left\{\frac{\phi(t)\,\delta}{\mu} + 1 - c\phi(t)\right\} \\
&\leq \mu\left[1 - \phi(t)\left(c - \frac{\delta}{\mu}\right)\right] \\
&\leq \mu,
\end{aligned}$$

from which we obtain $\|Px\| \leq \mu$. This implies that the natural mapping P defined by (2.1) on the set M_μ maps $M_\mu \to M_\mu$.

The rest of the proof is the same as in Theorem 2.1. We conclude that P has a fixed point x in M_μ. □

Remark 1. Taking $\alpha = |x(0)|$ in (3.2) and $c = 1$, we immediately see that inequality (3.3) becomes (1.8), so Theorem 3.1 reduces to Theorem 2.1.

Remark 2. From the proof of Theorem 3.1, we see that the bound μ for the solutions yielded satisfies $\frac{\delta}{\mu} \leq c$, i.e., $\frac{\delta}{c} \leq \mu$ This means that the smallest bound for the solutions is achieved by taking $\mu = \frac{\delta}{c}$. For a given $|x(0)|$, by

$$max\left\{\frac{1}{c}, \frac{|x(0)|}{c\alpha}\right\} = \frac{\delta}{c},$$

it follows that the smallest value of μ is either $\frac{|x(0)|}{\alpha c}$ or $\frac{1}{c}$, where $ac \leq 1$ (by setting $t = 0$ in (3.3)). We may conclude that the solution yielded by Theorem 3.1 is bounded either by $\frac{1}{c}$ when $|x(0)| \leq \alpha$, or by $\frac{|x(0)|}{c\alpha}$ when $|x(0)| \geq \alpha$. Note that, for $c = 1$ and $\alpha = |x(0)|$, we take $\mu = 1$, as in Theorem 2.1.

III. As pointed out in [5], condition (2.5) forces us to choose that $f(0) = 0$, which is not a necessity in (3.3). Hence, Lemma 1 does not apply for f with $f(0) \neq 0$. Furthermore, we see that if $G_1 = 0$, then

(2.5) holds only if $f \equiv 0$. However, in such a case, i.e., in the case of the nonlinear unforced equation

$$x'(t) = -\int_0^t D(t-s)g(x(s))ds, \quad t \geq 0, \tag{3.4}$$

with g satisfying

$$0 \leq \frac{g(x)}{x} \leq G_2, \quad x \neq 0,$$

condition (2.5) holds by itself, so both, Theorem 2.1 and Lemma 1 apply. By (2.4) we have $T_1 = 0$, while m becomes

$$m_0(u,t) :=$$
$$\max\left\{ Je^{-J(t-u)}, \left| Je^{-J(t-u)} - G_2 \int_u^t e^{-J(t-s)} D(s-u)ds \right| \right\}, 0 \leq u \leq t.$$

From Theorem 2.1 and Theorem 3.1 we have the following results.

Corollary 2. *If*

$$|x(0)|\, e^{-Jt} + \int_0^t m_0(u,t)\, du \leq 1, \quad t \geq 0, \tag{3.5}$$

then equation (3.4) *has a bounded solution.*

Corollary 3 *If there exist a* $c_0 > 0$ *such that*

$$\int_0^t m_0(u,t)\, du \leq 1 - c_0 e^{-Jt}, \quad t \geq 0,$$

then, equation (3.4) *has a bounded solution , for any* $x(0) \in \mathbb{R}$.

We note that as $T_1 = 0$, the inequality (2.6) reduces to

$$|x(0)|\, e^{-Jt} + \int_0^t R(u,t)\, du \leq 1 \quad t \geq 0. \tag{3.6}$$

As $\int_0^t R(u,t)\, du = \int_0^t Je^{-Ju} du = 1 - e^{-Jt}$, $t \geq 0$, we see that (3.6) becomes

$$|x(0)|\, e^{-Jt} + 1 - e^{-Jt} \leq 1 \quad t \geq 0,$$

hence (2.6) is satisfied for $|x(0)| \leq 1$, which is always true since solutions yielded by Theorem 2.1 are bounded by 1. Thus, from Lemma 1 (i) and (ii) we have the following result.

Corollary 4. *Let* $J > 0$, *D satisfy (1.2). If* $D_0 \in [\frac{G_2}{2}, \infty)$ *then (3.5) is always true and Theorem 2.1 yields a bounded solution to equation* (3.4).

IV. The most complicated form of the inequality (2.2) appears in the case where $D_0 \in [0, \frac{G_1+G_2}{2})$, i.e., when (2.2) is (2.7) in Lemma 1 (iii). In view of the proof of Lemma 1 (iii), we see that

(3.7)
$$\int_0^t m(u,t)\,du = \begin{cases} \int_0^t [R(u,t) - T_1(u,t)]\,du \leq 1, & t \leq t_0, \\ \int_0^{t-t_0} [T_2(u,t) - R(u,t)]\,du & \\ + \int_{t-t_0}^t [R(u,t) - T_1(u,t)]\,du \leq 1, & t > t_0. \end{cases}$$

With this in hand, we will show that (3.3) is not unrealistic. For convenience, we consider the simple case of $D(t) = e^{-qt}$, $t \geq 0$, with $q > J$ and assume that $\left|\int_0^t e^{Js} f(s) ds\right| \leq f^*$, $t \geq 0$. Note that, if $f \in C^1[0, +\infty)$, then (1.1) takes the form

$$x''(t) + qx'(t) = -g(x(t)) + h(t), \quad t > t_0,$$

with $h \in C[0, +\infty)$, so our results also hold for the above o.d.e..

The following is a summary of the calculations. Observing that for $t \geq 0$ we have $1 - c\phi(t) \geq 1 - cf^* e^{-Jt}$, it follows that, in order to establish (3.3), it suffices to show that

$$\int_0^t m(u,t)\,du \leq 1 - c^* e^{-Jt}, \quad t \geq 0, \tag{3.8}$$

for some $c^* > 0$. For $t \in [0, t_0]$, in view of (3.7) we have $\int_0^t m(u,t)\,du \leq \int_0^t R(u,t)\,du = 1 - e^{-Jt}$, so (3.8) holds true with $c^* = 1$. For $t > t_0$ we have $D_0 = (q - J)J$, so $D_0 \in [0, \frac{G_1+G_2}{2})$ implies that $(q-J)J < \frac{G_1+G_2}{2}$, while by (2.8),we obtain

$$e^{Jt_0} = \left(1 - \frac{2J|q-J|}{G_1+G_2}\right) e^{qt_0}. \tag{3.9}$$

Calculating the integral of $m(u,t)$, then estimating, we find

$$\int_0^t m(u,t)\,du \leq 1 - e^{-Jt_0} + \frac{G_2 e^{-Jt_0}}{|J-q|\,J} + \frac{G_1}{|J-q|\,q} - \frac{G_2}{|J-q|\,J} e^{-Jt}, \; t > t_0.$$

We see that if q is taken so that

$$-e^{-Jt_0} + \frac{1}{|J-q|}\left(\frac{G_2 e^{-Jt_0}}{J} + \frac{G_1}{q}\right) \leq 0, \tag{3.10}$$

then (3.7) is satisfied with $c^* = \frac{G_2}{|J-q|J}$. By (3.9), (3.10) becomes

$$\left(1 - \frac{2J|q-J|}{G_1+G_2}\right) \frac{1}{|J-q|}\left(\frac{G_2 e^{-Jt_0}}{J} + \frac{G_1}{q}\right) < e^{-qt_0}. \tag{3.11}$$

The left-hand-side of (3.11) is less than $\frac{G_1+G_2}{|J-q|J} - 2$, so, in order that (3.11) hold true, it suffices to have

$$\frac{G_1+G_2}{|J-q|\,J} < 2 + e^{-qt_0}. \tag{3.12}$$

Recalling that $(q-J)\,J < \frac{G_1+G_2}{2}$ and $q > J$, i.e., that

$$J < q < J + \frac{G_1+G_2}{2J}, \tag{3.13}$$

we observe that for values of q close to $J + \frac{G_1+G_2}{2J}$, the left hand side of (3.11) comes very close to 2 while, by (3.13), the right-hand-side is always larger than $e^{-\left(J+\frac{G_1+G_2}{2J}\right)t_0}$. We conclude that there exist values of q so that (3.12) is satisfied, and this proves that, for such values of q, (3.8) holds true, hence so does (3.3).

It is worth noticing here that if q, G_1, G_2 are given, then $J \in (0, q)$ can always be chosen so that (3.13) is satisfied.

4. APPENDIX

For convenience, we state and prove Lemma 1 by considering four cases corresponding to the four intervals mentioned in the paragraph preceding Lemma 1 in Section 2. Cases (iii) and (iv), below, are both included in case (iii) in the statement of Lemma 1 in Section 2.

Lemma 1. **(i)** *If $D_0 \in [G_2, \infty)$, then (2.2) is always true.*

Proof. Write $D_0 \in [G_2, \infty)$ as $\int_0^\infty e^{Jv} D(v)\,dv \le \frac{J}{G_2}$ or

$$\int_0^\infty e^{Jv} D_2(v)\,dv \le J,$$

and note that for $0 \le u \le t < \infty$ then

(4.1)

$$0 \le Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D_2(s-u)ds \Longleftrightarrow \int_0^\infty e^{Jv} D_2(v)\,dv \le J.$$

Indeed, for $0 \le u \le t$, $t \in [0, \infty)$ we have

$$\begin{aligned} & Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D_2(s-u)ds \\ = \;& e^{-J(t-u)} \left[J - \int_u^t e^{J(t-u)} e^{-J(t-s)} D_2(s-u)ds \right] \\ = \;& e^{-J(t-u)} \left[J - \int_u^t e^{J(s-u)} D_2(s-u)ds \right] \\ = \;& e^{-J(t-u)} \left[J - \int_0^{t-u} e^{Js} D_2(s)ds \right], \end{aligned}$$

which proves our assertion.

Thus, in view of our assumption on D_0, from (4.1) we have that $0 \le Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D_2(s-u)ds, \;\; 0 \le u \le t.$

Consequently, in view of $D_1(t) \le D_2(t)$, $t \ge 0$ we have

$$0 \le Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D_2(s-u)ds \le Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D_1(s-u)ds$$

and so,

$$\begin{aligned}
m(u,t) &= \max\left\{\left|Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D_1(s-u)ds\right|,\right.\\
&\qquad \left.\left|Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D_2(s-u)ds\right|\right\}\\
&= \max\left\{Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D_1(s-u)ds,\right.\\
&\qquad \left.Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D_2(s-u)ds\right\}\\
&= Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D_1(s-u)ds.
\end{aligned}$$

Then, (2.2) becomes

$$\begin{aligned}
&\left(|x(0)| + \left|\int_0^t e^{Js}f(s)ds\right|\right)e^{-Jt}\\
&\qquad + \int_0^t \left[Je^{-J(t-u)} - \int_u^t e^{-J(t-s)}D_1(s-u)ds\right]du \le 1, \quad t \ge 0,
\end{aligned}$$

which is inequality (1.8) for $D = D_1$. As $G_1 \le G_2$ we have

$$\begin{aligned}
\int_0^\infty e^{Jv}D(v)\,dv &\le \frac{J}{G_2} \le \frac{J}{G_1};\\
\int_0^\infty e^{Jv}G_1 D(v)\,dv &\le J
\end{aligned}$$

and

$$\int_0^\infty e^{Jv}D_1(v)\,dv \le J. \tag{4.2}$$

Now observe that (2.5) and (4.2) here, are (1.9) and (1.10) in Lemma A, respectively, with D_1 in place of D. Noting that the proof of Lemma A is idependent of the equation (1.6) we may conclude that our result follows from Lemma A (i) with D_1 in place of D.

For the rest of the proof we fix an arbitrary $t \ge 0$. By the definitions of the functions R and T_i, $(i = 1, 2)$ we have

$$\begin{aligned}
&R(0,t) = Je^{-Jt}, \quad R(t,t) = R(0) = J,\\
&T_i(0,t) = e^{-Jt}\textstyle\int_0^t e^{Js}D_i(s)\,ds, \quad T_i(t,t) = 0, \quad (i = 1,2),
\end{aligned} \tag{4.3}$$

and note that as $G_1 \le G_2$, it holds

$$\begin{aligned}
&0 \le T_1(u,t) < T_2(u,t),\ 0 \le u \le t, \quad \text{if } G_1 < G_2,\\
&0 \le T_1(u,t) = T_2(u,t),\ 0 \le u \le t, \quad \text{if } G_1 = G_2.
\end{aligned} \tag{4.4}$$

Clearly if $G_1 = G_2$ then $T_1 \equiv T_2$ which leads to the linear case, so for the rest of the proof we assume that $G_1 < G_2$.

Denote by $\mathcal{C}_R, \mathcal{C}_{T_i}$, the corresponding graphs of R and T_i, respectively $(i = 1, 2)$.

If C_{T_i} and C_R meet at some point $u_i \in (0, t)$ then

$$\begin{aligned} T(u_i, t) - R(u_i, t) &= 0; \\ e^{-J(t-u_i)} \left[\int_0^{t-u_i} e^{Js} D(s)\, ds - J \right] &= 0, \end{aligned}$$

or

$$J = \int_0^{t-u_i} e^{Js} D_i(s)\, ds, \quad i = 1, 2, \tag{4.5}$$

and we note that these meeting points (whenever they exist) have constant distance k_i from t, i.e., we have

$$u_i = t - k_i, \quad i = 1, 2,$$

where k_i is the unique positive number satisfying

$$J = \int_0^{k_i} e^{Js} D_i(s)\, ds, \quad i = 1, 2.$$

Due to the positivity of the integrand, this may happen if and only if $J < \int_0^t e^{Js} D_i(s)\, ds$; *i.e.*,

$$C_{T_i} \text{ and } C_R \text{ meet in } (0, t) \Leftrightarrow J \leq G_i \int_0^t e^{Js} D(s)\, ds, \;\; i = 1, 2,$$

and in such case, *the meeting points* $u_i, (i = 1, 2)$ *are unique.*

Let $h_i(u) := T_i(u, t) - R(u, t)$, $u \in [0, t]$, $(i = 1, 2)$, i.e.,

$$h_i(u) = e^{-J(t-u)} \left[\int_0^{t-u} e^{Js} D_i(s)\, ds - J \right], \; u \in [0, t], (i = 1, 2), \tag{4.6}$$

and note that

$$\begin{aligned} h_i(0) &= T_i(0, t) - R(0, t) = e^{-Jt} \left[\textstyle\int_0^t e^{Js} D_i(s)\, ds - J \right], \\ h_i(t) &= T_i(t, t) - R(t, t) = 0 - J < 0, \quad (i = 1, 2). \end{aligned}$$

We observe that the function m can be written as

$$m(u, t) = \max\{|h_1(u)|, |h_2(u)|\}, \quad 0 \leq u \leq t. \tag{4.7}$$

Due to (4.4) we have that

$$\begin{aligned} h_1(u) &< h_2(u), u \in [0, t), \quad \text{if } G_1 < G_2 \\ h_1(u) &= h_2(u), u \in [0, t], \quad \text{if } G_1 = G_2. \end{aligned} \tag{4.8}$$

In view of (4.6) and the positivity of the integrand we see that h_i may change sign at most once on $[0, t]$, and this may happen if and only if $J < \int_0^t e^{Js} D_i(s)\, ds$, hence

$$\begin{aligned} &\textstyle\int_0^t e^{Js} D_i(s)\, ds \leq J \Leftrightarrow h_i(u) \leq 0, \;\; u \in [0, t] \\ &J < \textstyle\int_0^t e^{Js} D_i(s)\, ds \Leftrightarrow \begin{cases} 0 \leq h_i(u), \; u \in [0, u_i] \\ h_i(u) \leq 0, \;\; u \in [u_i, t] \end{cases} \end{aligned} \tag{4.9}$$

where u_i satisfies $0 = h_i(u_i)$ $(i = 1, 2)$, i.e., u_i are the unique meeting points of $\mathcal{C}_R$, $\mathcal{C}_{T_i}$ $(i = 1, 2)$ given by (4.5).

Aiming to have a closer look at (4.7), we observe that as $G_1 < G_2$ by (4.8) it holds $h_1(u) \neq h_2(u)$ on $[0, t)$. Therefore, if there exists $u_0 \in (0, t)$ with $|h_1(u_0)| = |h_2(u_0)|$ then we must have $-h_1(u_0) = h_2(u_0) > 0$, from which we find

$$\begin{aligned} -h_1(u_0) &= h_2(u_0) \\ -T_1(u_0, t) + R(u_0, t) &= T_2(u_0, t) - R(u_0, t) \\ 2R(u_0, t) &= T_2(u_0, t) + T_1(u_0, t) \\ 2Je^{-J(t-u_0)} &= (G_1 + G_2)\, e^{-Jt} \int_0^{t-u_0} e^{J(s+u_0)} D(s)\, ds, \end{aligned}$$

i.e.,

$$J = \frac{G_1 + G_2}{2} \int_0^{t-u_0} e^{Js} D(s)\, ds. \tag{4.10}$$

It follows that

$$\exists u_0 \in (0, t) : |h_1(u_0)| = |h_2(u_0)| \Leftrightarrow J < \frac{G_1 + G_2}{2} \int_0^{t} e^{Js} D(s)\, ds. \tag{4.11}$$

Due to the positivity of the integrand in (4.10) the point u_0 is unique. By the definitions of u_1, u_0, u_2 by (4.5) and (4.10) it is not difficult to see

$$u_1 \le u_0 \le u_2. \tag{4.12}$$

provided that u_1, u_0, u_2 exist. Moreover, we may also see that *if u_1 exists in $(0, t)$ then so do u_0 and u_2, while existence of u_0 implies existence of u_2*. Consequently, by (4.11) we have that: if $\frac{G_1+G_2}{2} \int_0^t e^{Js} D(s)\, ds \le J$ then the functions $|h_1(u)|$ and $|h_2(u)|$ do not meet, so the continuous function $|h_1(u)| - |h_2(u)|$, $u \in (0, t)$ cannot change sign on $u \in (0, t)$; hence *either* $|h_1(u)| < |h_2(u)|$, $u \in (0, t)$ *or* $|h_1(u)| > |h_2(u)|$, $u \in (0, t)$, in other words,

$$\begin{gathered} \tfrac{G_1+G_2}{2} \int_0^t e^{Js} D(s)\, ds \le J \Longleftrightarrow \\ \begin{cases} \text{either} & m(u, t) = |h_1(u)|, u \in (0, t), \\ \text{or} & m(u, t) = |h_2(u)|, u \in (0, t). \end{cases} \end{gathered} \tag{4.13}$$

We note that if $h_2(u) \le 0$, $u \in [0, t]$ then (4.8) implies $|h_2(u)| \le |h_1(u)|$, $u \in [0, t]$, and so

$$h_2(u) \le 0, u \in [0, t] \Longrightarrow m(u, t) = |h_1(u)|.$$

Now let us assume that $t > 0$ is such that there exists $u_0 \in (0, t)$ with $|h_1(u_0)| = |h_2(u_0)|$. Since this implies the existence of u_2, in view of (4.9) it follows that for $u \in [u_2, t]$ we have $h_1(u) \le h_2(u) \le 0$, so the relations $|h_2(u)| \le |h_1(u)|$, $u \in [u_2, t]$ hold. As this inequality does not change on $[u_0, t]$ we have that $|h_2(u)| \leq |h_1(u)|, u \in [u_0, t]$. By uniqueness of u_0 we conclude that for $u \in [0, u_0]$ we must have $|h_1(u)| \leq |h_2(u)| = h_2(u)$. Thus, *in order that there exists a* $u_0 \in (0, t)$ with $|h_1(u_0)| = |h_2(u_0)|$, *it is necessary, in particular, that* $0 < h_2(0)$, i.e.,

$$\exists u_0 \in (0, t) : |h_1(u_0)| = |h_2(u_0)| \Longrightarrow 0 < h_2(0).$$

From the above discussion we have the following result:

$$\exists u_0 \in (0, t) : |h_1(u_0)| = |h_2(u_0)| \Longrightarrow m(u, t) = \begin{cases} |h_2(u)|, & u \in [0, u_0] \\ |h_1(u)|, & u \in [u_0, t] \end{cases},$$

or, more precisely,

$$\text{(4.14)} \qquad \begin{aligned} \exists u_0 \quad \in \quad & (0, t) : |h_1(u_0)| = |h_2(u_0)| \\ \Longrightarrow \quad & m(u, t) = \begin{cases} h_2(u), & u \in [0, u_0] \\ -h_1(u), & u \in [u_0, t] \end{cases}. \end{aligned}$$

Lemma 1 (ii) *If* $D_0 \in \left[\frac{G_1+G_2}{2}, G_2\right)$, *then (2.2) is,*

$$\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_0^t \left[Je^{-J(t-u)} - G_1 \int_u^t e^{-J(t-s)} D(s-u) ds\right] du \leq 1.$$

Proof. It is not difficult to see that all we have to prove is that

$$m(u, t) = R(u, t) - T_1(u, t), \quad 0 \leq u \leq t < \infty.$$

Write $D_0 \in \left[\frac{G_1+G_2}{2}, G_2\right)$ as

$$\frac{G_1 + G_2}{2} \int_0^\infty e^{Jv} D(v) dv \leq J < G_2 \int_0^\infty e^{Jv} D(v) dv.$$

First we note that, by (4.13), from the inequality on the left we have $\frac{G_1+G_2}{2} \int_0^t e^{Js} D(s) ds \leq J$ for all $t \geq 0$; thus, either $m(u, t) = |h_1(u)|, u \in [0, t]$ or $m(u, t) = |h_2(u)|, u \in [0, t]$, independently of t. We claim that

$$\text{(4.15)} \qquad m(u, t) = |h_1(u)|, u \in [0, t] \quad \text{for any } t \geq 0.$$

In view of the inequality on the right we have that there exists a $t_0 > 0$ such that $G_2 \int_0^t e^{Jv} D(v) dv \leq J$, $t \in [0, t_0]$ and $J < G_2 \int_0^t e^{Jv} D(v) dv$, $t \in (t_0, \infty)$. We consider two cases.

(I) For $t \in [0, t_0]$, as $\int_0^t e^{Jv} D_2(v) dv \leq J$, by (4.9) we have that $h_2(u) \leq 0$, $u \in [0, t]$, hence by (4.8) we take

$$h_1(u) \leq h_2(u) \leq 0, \quad u \in [0, t]$$

which gives

$$\text{(4.16)} \qquad m(u, t) = |h_1(u)|, \ u \in [0, t] \quad \text{for } t \in [0, t_0].$$

(II) Let $t \in (t_0, \infty)$. From $\frac{G_1+G_2}{2}\int_0^t e^{Js} D(s)\,ds < J$ we take $G_1 \int_0^t e^{Js} D(s)\,ds < J$ so in view of (4.9) we have that

$$h_1(u) \le 0, \quad u \in [0,t]$$

from which we have

$$|h_1(u)| = -h_1(u), \quad u \in [0,t]. \tag{4.17}$$

As $t \in (t_0, \infty)$, in view of by (4.9) from $J < G_2 \int_0^t e^{Jv} D(v)\,dv$, $t \in (t_0, \infty)$ we have that $h_2(0) \ge 0$ so

$$0 < |h_2(0)| = h(0) = T_2(0,t) - R(0,t).$$

But then, engaging (4.3) and (4.17) we find

$$\begin{aligned} |h_2(0)| - |h_1(0)| &= T_2(0,t) - R(0,t) - [R(0,t) - T_1(0,t)] \\ &= T_2(0,t) + T_1(0,t) - 2R(0,t) \\ &= e^{-Jt}\left[\int_0^t e^{Js} D(s)\,ds\,(G_1+G_2) - 2J\right] \le 0, \end{aligned}$$

i.e., $|h_2(0)| \le |h_1(0)|$, so by (4.13) it follows that

$$m(u,t) = |h_1(u)|, \; u \in [0,t] \quad \text{for } t \ge t_0. \tag{4.18}$$

From (4.16) and (4.18) we conclude that (4.15) holds true. Thus, in view of (4.17) it follows that for $u \in [0,t]$, $t \ge 0$ we have

$$m(u,t) = |h_1(u)| = -h_1(u) = R(u,t) - T_1(u,t)$$

so (2.2) becomes

$$\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_0^t [R(u,t) - T_1(u,t)]\,du \le 1,$$

as asserted.

Lemma 1 (iii) *If $D_0 \in \left[G_1, \frac{G_1+G_2}{2}\right)$ then (2.2) is*

$$\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_0^t [R(u,t) - T_1(u,t)]\,du \le 1, \quad t \le t_0,$$

$$\begin{aligned} \left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} &+ \int_0^{t-t_0} [T_2(u,t) - R(u,t)]\,du \\ &+ \int_{t-t_0}^t [R(u,t) - T_1(u,t)]\,du \le 1 \quad t > t_0. \end{aligned}$$

Proof. First let us write the condition $D_0 \in \left[G_1, \frac{G_1+G_2}{2}\right)$ as

$$G_1 \int_0^\infty e^{Jv} D(v)\,dv \le J < \frac{G_1+G_2}{2}\int_0^\infty e^{Jv} D(v)\,dv.$$

From the inequality on the left it follows that

$$\int_0^t e^{Jv} D_1(v)\,dv < J \quad \text{for all } t \ge 0,$$

so from (4.9) we have that $h_1(u) \le 0$, $u \in [0,t]$ for all $t \ge 0$, and so

$$|h_1(u)| = -h_1(u), \quad u \in [0,t], \quad t \ge 0.$$

From the inequality on the right,one can see that there exist a $t_0 > 0$ satisfying (2.8). Note that, by the definitions of u_0 and t_0 (in (2.8) and (4.10), respectively), it holds

$$t - u_0 = t_0.$$

It follows that

$$\frac{G_1+G_2}{2}\int_0^t e^{Jv} D(v)\, dv \le J, t \in [0,t_0],$$

and

$$J < \frac{G_1+G_2}{2}\int_0^t e^{Jv} D(v)\, dv, t > t_0.$$

We consider two cases.

(I) For $t \in [0,t_0]$ we have that $\frac{G_1+G_2}{2}\int_0^t e^{Jv} D(v)\, dv \le J$ so from (4.13) we see that *either* $|h_1(u)| < |h_2(u)|$, $u \in (0,t)$, *or* $|h_1(u)| > |h_2(u)|$, $u \in (0,t)$, so it suffices to check the difference $|h_1(0)| - |h_2(0)|$.

If $h_2(0) < 0$ then as $h_1(u) \le h_2(u)$ we have $h_1(0) \le h_2(0)$ which implies that $|h_2(0)| \le |h_1(0)|$. If $h(0) \ge 0$, then taking (4.3) into consideration from $\frac{G_1+G_2}{2}\int_0^t e^{Jv} D(v)\, dv \le J$ we have

$$\begin{aligned} T_2(0,t) + T_1(0,t) &\le 2R(0,t) \\ T_2(0,t) - R(0,t) &\le R(0,t) - T_1(0,t) \end{aligned}$$

i.e.,

$$0 \le h_2(0) \le |h_1(0)| = -h_1(0)$$

which again leads to $|h_2(0)| \le |h_1(0)|$. We onclude that

$$m(u,t) = |h_1(u)| = -h_1(u), \quad u \in [0,t], \quad t \in [0,t_0]. \tag{4.19}$$

(II) If $t > t_0$, then $J < \frac{G_1+G_2}{2}\int_0^t e^{Jv} D(v)\, dv$. By (4.11) and the discussion after (4.12) there exists a $u_0 \in [0,u_2]$ with $|h_1(u_0)| = |h_2(u_0)|$. From (4.14) it follows that

$$m(u,t) = \left\{ \begin{array}{ll} h_2(u), & u \in [0,t-t_0] \\ -h_1(u), & u \in [t-t_0,t] \end{array} \right., \quad t > t_0. \tag{4.20}$$

From (4.19) and (4.20) we have that

$$m(u,t) = \left\{ \begin{array}{ll} -h_1(u), & 0 \le u \le t \le t_0 \\ h_2(u), & 0 \le u \le t - t_0, \ t > t_0 \\ -h_1(u), & t - t_0 \le u \le t, \ t > t_0. \end{array} \right.$$

and

$$m(u,t) = \left\{ \begin{array}{ll} R(u,t) - T_1(u,t), & 0 \le u \le t \le t_0 \\ T_2(u,t) - R(u,t), & 0 \le u \le t - t_0 \\ R(u,t) - T_1(u,t), & t - t_0 \le u \le t \end{array} \right., \quad t > t_0 \quad ,$$

With this in hand we see that (2.2) becomes

$$\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_0^t \left[R(u,t) - T_1(u,t)\right] du \leq 1, \quad t \leq t_0,$$

$$\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} + \int_0^{t-t_0} \left[T_2(u,t) - R(u,t)\right] du$$
$$+ \int_{t-t_0}^t \left[R(u,t) - T_1(u,t)\right] du \leq 1 \quad t > t_0,$$

as asserted.

Lemma 1 (iv) *If $D_0 \in [0, G_1)$ then (2.2) becomes*

$$\left(|x(0)| + \left|\int_0^t e^{Js} f(s) ds\right|\right) e^{-Jt} +$$
$$\int_0^{t-t_0} \left[T_2(u,t) - R(u,t)\right] du + \int_{t-t_0}^t \left[R(u,t) - T_1(u,t)\right] du \leq 1.$$

with T_i, $(i = 1, 2)$ and t_0 defined in (2.4) and (2.8), respectively.

Proof. Condition $D_0 \in [0, G_1)$ gives

$$0 < J < G_1 \int_0^\infty e^{Jv} D(v)\, dv,$$

so existence of t_1 with $J = G_1 \int_0^{t_1} e^{Jv} D(v)\, dv$ is established.

(I) For $t \in [0, t_1]$ we have $G_1 \int_0^t e^{Jv} D(v)\, dv \leq J$ from which, in view of $D_0 < G_1$ and $G_1 < G_2$ we have

$$G_1 \int_0^t e^{Jv} D(v)\, dv \leq J < \frac{G_1 + G_2}{2} \int_0^\infty e^{Jv} D(v)\, dv, \quad t \in [0, t_1].$$

Arguing exactly as in the preceding case *(iii)*, we may conclude that

$$m(u,t) = \begin{cases} -h_1(u), & 0 \leq u \leq t \leq t_0 \\ h_2(u), & 0 \leq u \leq t - t_0 \\ -h_1(u), & t - t_0 \leq u \leq t \end{cases}, \quad t \in [0, t_1]. \tag{4.21}$$

(II) If $t > u_1$ then by the discussion following (4.12) we have that existence of $u_1 \in (0, t)$ implies existence of u_0, u_2 with (4.12), thus from (4.14) we have that

$$m(u,t) = \begin{cases} h_2(u), & u \in [0, t - t_0] \\ -h_1(u), & u \in [t - t_0, t] \end{cases}, \quad t > u_1. \tag{4.22}$$

From (4.21) and (4.22) we see that in both relations, m has the same expression for $t > t_0$, so we may conclude that

$$m(u,t) = \begin{cases} -h_1(u), & 0 \leq u \leq t \leq t_0 \\ h_2(u), & 0 \leq u \leq t - t_0 \\ -h_1(u), & t - t_0 \leq u \leq t \end{cases}, \quad t > t_0 \quad ,$$

so the expression for (2.2) is the same as in the previous case.

References

[1] M. A. Aizerman, On a problem concerning the stability "in the large" of dynamical systems, *Usp. mat Nauk* **4**, No. 4 (1949), 187-188.

[2] Wolfgang Hahn, *Theory and Application of Liapunov's Direct Method*, Translated by Siegfried H. Lehnigk, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963

[3] Solomon Lefschetz, *Stability of Nonlinear Control Systems*, Academic Press, New York, 1965.

[4] T. A. Burton, and I. K. Purnaras, A unification theory of Krasnoselskii for differential equations, *Nonlin. Anal.: TMA* **89** (2013), 121-133.

[5] T. A. Burton, and I. K. Purnaras, Volterra's linear equation and Krasnoselskii's hypothesis, *Communications in Applied Analysis* **17** (2013), 529-554.

[6] T. A. Burton, Fractional equations and a theorem of Brouwer-Schauder type, *Fixed Point Theory* **14** (2013), 91-96.

[7] T. A. Burton, Correction to "Fractional equations and a theorem of Brouwer-Schauder type", *Fixed Point Theory*, submitted.

[8] Krasnoselskii, M. A., *Amer. Math. Soc. Transl. (2)* **10** (1958), 345–409.

[9] Krasovskii, N. N., *Stability of Motion*, Translated by J. L. Brenner, Stanford Univ. Press, Stanford, 1963.

[10] J. J. Levin, The asymptotic behavior of a Volterra equation, *Proc. Amer. Math. Soc.* **14** (1963), 434-451.

[11] J. J. Levin, The qualitative behavior of a nonlinear Volterra equation, *Proc. Amer. Math. Soc.* **16** (1965), 711-718.

[12] J. J. Levin, On a nonlinear Volterra equation, *J. Math. Anal. Appl.* **39** (1972), 458-476.

[13] J. J. Levin and J. A. Nohel, Note on a nonlinear Volterra equation, *Proc. Amer. Math. Soc.* **14** (1963), 924-929.

[14] Stig-Olof Londen, On the solution of a nonlinear Volterra equation, *J. Math. Anal. Appl.* **39** (1972), 564-573.

[15] J. J. Levin and J. A. Nohel, On a system of integrodifferential equations occurring in reactor dynamics, *J. Math. Mech.* **9** (1960), 347-368.

[16] V. Volterra, Sur la théorie mathématique des phénomès héréditaires. *J. Math. Pur. Appl.***7** (1928) 249–298.

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