Krasnoselskii's Fixed Point Theorem and Stability

T.A. Burton
Northwest Research Institute
732 Caroline Street
Port Angeles, WA 98362
taburton at olypen.com

Tetsuo Furumochi
Department of Mathematics
Shimane University
Matsue, Japan 690-8504
furumochi at math.shimane-u.ac.jp

ABSTRACT. In this paper we use a fixed point theorem of Krasnoselskii to prove that the zero solution of a nonlinear ordinary differential equation is asymptotically stable. The result is applied to an equation

$$x'' + f(x)x' + g(x) = Kh(t, x, x').$$

Although the discussion concerns ordinary differential equations, it can be applied equally well to functional differential equations.

Key words and phrases: fixed points, stability, Liénard equation, Krasnoselskii

1. Introduction

This note represents a part of a continuing investigation of the use of fixed point theory in stability. One motivation for our work here comes from Perron's theorem [7] which states that if

(1)
$$x' = Dx + G(t, x)$$

with D a matrix, all of whose characteristic roots have negative real parts, and $\lim_{x\to 0} |G(t, x)|/|x| = 0$ uniformly for $0 \le t < \infty$, then $x = 0$ is uniformly asymptotically stable. Coddington and Levinson [4; p. 314 and 327] as well as Lakshmikantham and Leela [6; p. 115] use other methods to show that solutions with small initial conditions tend to zero provided that $G(t,x)$ tends to zero in a uniform way for small x. Those methods depend strongly on the unperturbed linear system $y' = Dy$ and they can be well motivated by solving the Bernoulli equation

$$x' + 2x = e^{-t}x^{3/5} \tag{2}$$

and readily concluding that solutions tend to zero.

And many nice results along the same lines given by Bellman [1] for

$$x' = Dx + E(t)x \tag{3}$$

where D has all characteristic roots with negative real parts, while E is small either in norm or in integral.

In this note we conjecture that there is a general theorem concerning asymptotic stability of the zero solution of

$$x' = f(t,x) + G(t,x) \tag{4}$$

when f satisfies a Lipschitz condition with $y' = f(t,y)$ uniformly asymptotically stable and, for example, when $|G(t,x)| \leq q(t)|x|^{\alpha}$ where $0 < \alpha < 1$ and q is small in some sense. Moreover, it seems that the following modification of Krasnoselskii's fixed point theorem may be a proper vehicle for the proof. It can be found in Burton (1998).

THEOREM. Let M be a closed, convex, and nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that $A : M \to S$ and $B : S \to S$ such that :

(i) B is a contraction with constant $\alpha < 1$,

(ii) A is continuous, AM resides in a compact subset of S,

(iii) $[x = Bx + Ay, y \in M] \implies x \in M$.

Then there is a $y \in M$ with $Ay + By = y$.

This result differs from the one of Krasnoselskii in that the former requires that $Bx + Ay$ always resides in M. We will see that this is a crucial change in the present application.

2. The main result

We begin the construction with a simple equation to guide us in the construction of our theorem and then return to a similar problem as an example. Consider the scalar equation

$$x' = -2x + G(t,x) \tag{5}$$

where G is continuous,

$$|G(t,x)| \le Ke^{-t}|x^{3/5}|, \tag{6}$$

and K is a positive constant. Let

$$M = \{\psi : [0,\infty) \to R | \psi \in C, |\psi(t)| \le e^{-t}\}, \tag{7}$$

where C denotes the set of continuous functions, and let $(S, \|\cdot\|)$ be the Banach space of bounded continuous function on $[0,\infty) \to R$ with the supremum norm.

LEMMA 1. If $|x_0| + (5/2)K < 1$ and if $x(t) = x(t,0,x_0)$ is the solution of

$$x' = -2x + G(t,\psi(t)), \psi \in M, \tag{8}$$

then $x \in M$.

Proof. We have

$$\begin{aligned}|x(t)| &\le |x_0|e^{-2t} + \int_0^t e^{-2(t-s)}Ke^{-s}e^{-(3/5)s}ds\\ &\le |x_0|e^{-2t} + Ke^{-2t}\int_0^t e^{(2/5)s}ds\\ &\le |x_0|e^{-2t} + (5/2)Ke^{-t} < e^{-t}.\end{aligned}$$

Hence, $x \in M$.

LEMMA 2. If for $\psi \in M$ we define

$$(A\psi)(t) = \int_0^t G(s,\psi(s))ds, t \ge 0, \tag{9}$$

then AM resides in a compact subset of S.

Proof. It is clear that the integrals exist and we readily verify that AM is an equicontinuous set. Moreover, AM is bounded. If we have a sequence $\{A\psi_n\}$, then by Ascoli's theorem and a diagonalization process there is a subsequence, say $\{A\psi_n\}$ again, converging uniformly on compact subsets of $[0,\infty)$. We will now show that $\{A\psi_n\}$ is a Cauchy sequence on $[0,\infty)$.

Given $\epsilon > 0$, fix $T > 0$ so that $\int_T^\infty 2Ke^{-s}ds < \epsilon/2$. Then find N such that $n, m > N$ implies that

$$\sup_{0\leq p\leq T} |\int_0^p [G(s,\psi_n(s) - G(s,\psi_m(s))]ds| < \epsilon/2.$$

Thus, if $n, m > N$ then

$$\sup_{0\leq t<\infty} |\int_0^t [G(s,\psi_n(s)) - G(s,\psi_m(s))]ds|$$

$$\leq \sup_{0\leq p\leq T} |\int_0^p [G(s,\psi_n(s)) - G(s,\psi_m(s))]ds| + \int_T^\infty [|G(s,\psi_n(s))| + |G(s,\psi_m(s))|]ds$$

$$< \epsilon.$$

As AM is contained in S and S is complete, AM is contained in a compact subset of S.

The following result is known, but we supply the details for reference.

LEMMA 3. Let $b : R^{d+1} \to R^d$ be continuous and suppose there is an $L > 0$ so that $|b(t,x) - b(t,y)| \leq L|x-y|$. With the norm

$$|\phi|_L = \sup_{0\leq s<\infty} \{|e^{-2Ls}\phi(s)|\}$$

on the Banach space U of bounded countinuous functions $\phi : [0,\infty) \to R^d$ then the operator H defined by

$$(Hx)(t) = x_0 + \int_0^t b(s,x(s))ds, t \geq 0,$$

is a contraction with contraction constant $1/2$.

Proof. We have

$$
\begin{aligned}
|Hx_1 - Hx_2|_L = & \sup_{0\le s<\infty} |e^{-2Ls} \int_0^s (b(u, x_1(u)) - b(u, x_2(u))du| \\
\le & \sup_{0\le s<\infty} \int_0^s e^{-2Ls} L|x_1(u) - x_2(u)|du \\
= & \sup_{0\le s<\infty} \int_0^s e^{-2Ls} L|x_1(u) - x_2(u)|e^{-2Lu} e^{2Lu} du \\
\le & |x_1 - x_2|_L \sup_{0\le s<\infty} \int_0^s e^{-2Ls} L e^{2Lu} du \\
\le & (1/2)|x_1 - x_2|_L
\end{aligned}
$$

a contraction.

In the proof of Lemma 2, the norm $|\cdot|_L$ works as well as the supremum norm.

With this example in mind we now consider a general theorem. Let $a, b : [0,\infty)\times R^d \to R^d$ be continuous and consider

$$x' = b(t,x) + a(t,x), x(0) = x_0 \tag{10}$$

where

$$|b(t,x) - b(t,y)| \le L|x-y| \text{ on } [0,\infty) \times R^d. \tag{11}$$

Thus, (10) has a solution.

Let $(U, \|\cdot\|)$ denote a Banach space of bounded continuous functions $\phi : [0,\infty) \to R^d$ and M denote a closed convex nonempty subset of U. Let the operator $A : M \to U$ defined by $\psi \in M$ implies that

$$(A\psi)(t) = \int_0^t a(s, \psi(s))ds, t \ge 0, \tag{12a}$$

be continuous and define the operator B by

$$(B\phi)(t) = x_0 + \int_0^t b(s, \phi(s))ds, t \ge 0, \tag{12b}$$

for each $\phi \in U$.

THEOREM 1. Let B be a contraction with constant $\alpha < 1$ on the space $(U, \|\cdot\|)$ and suppose that AM resides in a compact subset of that space. Suppose also that for each $\psi \in M$ the unique solution ϕ of

$$\phi'(t) = b(t, \phi(t)) + a(t, \psi(t)), \phi(0) = x_0 \tag{13}$$

is in M. Then a solution of (10) is in M.

PROOF. Notice first that if $\phi \in M$ is a fixed point of P, where P is defined by

$$(P\phi)(t) = x_0 + \int_0^t b(s, \phi(s))ds + \int_0^t a(s, \phi(s))ds, t \geq 0, \tag{14}$$

then ϕ is a solution of (10).

Now, for fixed $\psi \in M$ and all $\phi \in U$, define Q by

$$(Q\phi)(t) = x_0 + \int_0^t b(s, \phi(s))ds + \int_0^t a(s, \psi(s))ds, t \geq 0. \tag{15}$$

If $Q\phi = \phi$ for some $\phi \in U$, then ϕ is the unique solution of

$$\phi' = b(t, \phi) + a(t, \psi(t)), \phi(0) = x_0. \tag{16}$$

By assumption, that unique solution of (16) is in M. By the above stated revision of Krasnoselskii's theorem, P itself has a fixed point ϕ in M.

COROLLARY. If, in addition to the assumptions of Theorem 1, all functions in M tend to 0 as $t \to \infty$, then a solution of (10) tends to *zero* as $t \to \infty$.

The following example is parallel in content, but different in technique, to the results in [4; pp. 314, 327] and [6; p. 115]. In the next section we give a nonlinear example.

EXAMPLE. Let D be a $d \times d$ constant matrix, all of whose characteristic roots have negative real parts; thus, there exist $\alpha > 0$ and $k > 0$ with

$$|e^{Dt}| \leq ke^{-\alpha t}, t \geq 0. \tag{17}$$

Next, let $G : [0, \infty) \times R^d \to R^d$ be continuous and suppose there is a constant $\gamma > 0$, a continuous function $q : [0, \infty) \to [0, \infty)$ with $q(t) \to 0$ as $t \to \infty$ and $q \in L^1[0, \infty)$ so that

$$|G(t, x)| \leq Kq(t)|x|^\gamma. \tag{18}$$

We will show that the conditions of Theorem 1 are satisfied for

$$x' = Dx + G(t, x)$$

when K is sufficiently small.

To this end, if we let

$$r(t) := \int_0^t e^{-\alpha(t-s)} q(s) ds \tag{19}$$

then $r(t) \to 0$ as $t \to \infty$ and $r \in L^1[0, \infty)$ since r is the convolution of appropriate functions.

Define

$$h(t) = \max[r(t), e^{-\alpha t}] \tag{20}$$

and note that $h(t) \leq |r(t)| + e^{-\alpha t} \in L^1[0, \infty)$; moreover $h(t) \to 0$ as $t \to \infty$. By redefining q and K we may assume without loss of generality that

$$h(t) \leq 1, t \geq 0. \tag{21}$$

Define

$$M = \{\psi : [0, \infty) \to R^n | \psi \in C, |\psi(t)| \leq h(t)\}. \tag{22}$$

Thus, M is closed and convex.

For arbitrary $\psi \in M$, consider

$$x' = Dx + G(t, \psi(t)), x(0) = x_0. \tag{23}$$

Then

$$\begin{aligned} |x(t)| &\leq |x_0| k e^{-\alpha t} + \int_0^t k K e^{-\alpha(t-s)} q(s) |\psi(s)|^\gamma ds \\ &\leq |x_0| k h(t) + k K r(t) \\ &\leq [|x_0| k + k K] h(t) \leq h(t) \end{aligned} \tag{24}$$

provided that

$$[|x_0| + K]k \le 1.$$

Hence, $x(t) \in M$.

Exactly as in the proof of Lemma 2, if A is defined by (9) then any sequence $\{A\psi_n\}$ with $\psi_n \in M$ is equicontinuous and so we obtain a subsequence converging uniformly on compact sets. The norm $|\cdot|_L$ works just like the supremum norm in the convergence proof.

3. A perturbed Liénard equation

Consider the scalar equation

$$x'' + f(x)x' + g(x) = Kh(t, x, x') \tag{25}$$

which we write as the system

$$\begin{aligned} x' &= y \\ y' &= -f(x)y - g(x) + Kh(t, x, y) \end{aligned} \tag{26}$$

or in vector form as

$$X' = b(X) + a(t, X) \tag{27}$$

where

$$a(t, X) = (0, Kh(t, x, y))^T.$$

We assume that for any $\alpha > 0$ and for any $J > 0$, if $\psi : [0, \infty) \to R^2$ and $|\psi(t)| \le Je^{-\alpha t}$ then

$$a(t, \psi(t)) \in L^1[0, \infty), \tag{28}$$

that $\forall J > 0\ \forall \alpha > 0\ \exists D > 0$ such that $|\psi(t)| \le Je^{-\alpha t}$ implies that

$$\left|\int_{t_1}^{\infty} a(s, \psi(s))ds - \int_{t_2}^{\infty} a(s, \psi(s))ds\right| = \left|\int_{t_1}^{t_2} a(s, \psi(s))ds\right| \le D|t_1 - t_2|, \tag{29}$$

and that there are positive L_1, L_2, L_3, L_4 so that if $X_i \in R^2$ then

$$|b(X_1) - b(X_2)| \le L_1|X_1 - X_2|, L_4 \ge f(x) \ge L_2, \text{ and } g(x)\int_0^x f(s)ds \ge L_3x^2. \tag{30}$$

Now, for J, α to be determined, let

$$M = \{\psi : [0, \infty) \to R^2 | \psi \in C, |\psi(t)| \le Je^{-\alpha t}\}$$

and for each $\psi \in M$ consider the system

$$\begin{aligned} x' &= y \\ y' &= -f(x)y - g(x) + e(t) \end{aligned} \tag{31}$$

where $e(t) = Kh(t, \psi(t))$.

LEMMA. If (27)-(30) hold and if we define

$$V(x, y) = (1/2)y^2 + 2\int_0^x g(s)ds + (1/2)(y + \int_0^x f(s)ds)^2 \tag{32}$$

then there is an $\eta > 0$ so that the derivative of V along a solution of (31) satisfies

$$V'(x(t), y(t)) \le -\eta V(x, y) + 2\sqrt{V(x, y)}|e(t)| \tag{33}$$

and there is a $k_1 > 0$ with

$$k_1(x^2 + y^2) \le V(x, y). \tag{34}$$

Proof. We have

$$V'(x, y) = 2g(x)y - f(x)y^2 - yg(x) + ye(t) + (y + \int_0^x f(s)ds)(f(x)y - f(x)y - g(x) + e(t))$$

$$= -f(x)y^2 + ye(t) - g(x)\int_0^x f(s)ds + (y + \int_0^x f(s)ds)e(t)$$

$$\le -f(x)y^2 - g(x)\int_0^x f(s)ds + |y||e(t)| + |y + \int_0^x f(s)ds||e(t)|$$

$$\le -L_2y^2 - L_3x^2 + [\sqrt{2}(|y|/\sqrt{2}) + \sqrt{2}(|y + \int_0^x f(s)ds|)/\sqrt{2}]|e(t)|$$

$$\leq -L_2y^2 - L_3x^2 + 2|e(t)|\sqrt{V(x,y)}.$$

But if we use (30), in particular g is Lipschitz, then we have

$$V(x,y) \leq (1/2)y^2 + (L_1)x^2 + y^2 + (\int_0^x f(s)ds)^2$$

$$\leq (3/2)y^2 + (L_1)x^2 + L_4^2x^2$$

and so there is an $\eta > 0$ with

$$V'(x,y) \leq -\eta V(x,y) + 2|e(t)|\sqrt{V(x,y)}.$$

To find k_1, we have

$$L_3x^2 \leq g(x)\int_0^x f(s)ds \leq |g(x)|L_4|x|$$

or

$$|g(x)| \geq L_3|x|/L_4$$

and so

$$\int_0^x g(s)ds \geq L_3x^2/(2L_4).$$

From these we can find k_1.

THEOREM 2. Suppose there are α, β, J, and S with $0 < \alpha \leq \beta < \eta/2$ so that

$$|\psi(t)| \leq Je^{-\alpha t} \implies |h(t,\psi(t))| \leq Se^{-\beta t}, t \geq 0 \tag{35}$$

and

$$J((\eta/2) - \beta)\sqrt{k_1} > SK.$$

If

$$M = \{\psi : [0,\infty) \to R^2 | \psi \in C, |\psi(t)| \leq Je^{-\alpha t}\}$$

and if $|(x_0, y_0)|$ is small, then the solution of (31) through (x_0, y_0) for any $t_0 \geq 0$ is in M.

Proof. Select $\psi \in M$ and (x_0, y_0, t_0) so that $(x(t), y(t))$ is fixed, and hence, $V(t) := V(x(t), y(t))$ is determined in (32). In

$$V'(t) \leq -\eta V(t) + 2|e(t)|\sqrt{V(t)},$$

we first obtain

$$V(t) \leq V(0)e^{-\eta t} + 2\int_0^t e^{-\eta(t-s)}|e(s)|\sqrt{V(s)}ds$$

or

$$e^{\eta t}V(t) \leq V(0) + 2\int_0^t e^{(1/2)\eta s}|e(s)|\sqrt{e^{\eta s}V(s)}ds$$

which we write as

$$u(t) \leq u(0) + 2\int_0^t e^{(1/2)\eta s}|e(s)|\sqrt{u(s)}ds.$$

By Bihari's inequality ([2] and [5; p. 29]) we have $u(t) \leq w(t)$ where $w(t)$ is the maximal solution of

$$w(t) = u(0) + 2\int_0^t e^{(1/2)\eta s}|e(s)|\sqrt{w(s)}ds.$$

Thus, letting $v(t) = \sqrt{w(t)e^{-\eta t}}$ we obtain $2v'(t) + \eta v(t) = 2|e(t)|$ or $v' + (\eta/2)v = |e(t)|$. We then have

$$\begin{aligned} v(t) &= v_0 e^{-(\eta/2)t} + \int_0^t e^{-(\eta/2)(t-s)}|e(s)|ds \\ &\leq v_0 e^{-(\eta/2)t} + \int_0^t SKe^{-(\eta/2)(t-s)-\beta s}ds \\ &= v_0 e^{-(\eta/2)t} + SKe^{-(\eta/2)t}[(\eta/2)-\beta]^{-1}e^{[(\eta/2)-\beta]s}|_0^t \\ &\leq (v_0 + [(\eta/2)-\beta]^{-1}SK]e^{-\beta t}. \end{aligned}$$

Hence,

$$\sqrt{k_1(x^2(t)+y^2(t))} \leq \sqrt{V(t)}$$

$$\leq \left[\sqrt{V(x_0,y_0)} + [(\eta/2)-\beta]^{-1}SK\right]e^{-\beta t}. \tag{36}$$

Thus, $(x(t), y(t))$ is in M provided that

$$J_0 := \sqrt{V(x_0,y_0)/k_1} + [((\eta/2)-\beta)\sqrt{k_1}]^{-1}SK < J, \tag{37}$$

as required.

REMARK. Notice that (35) is an interesting relation. For example, let $h(t,x,y) = Kp(t)x^n$. Thus, if $|\psi(t)| < Je^{-\alpha t}$, then

$$|h(t,\psi(t))| \leq KJp(t)e^{-\alpha nt} < Se^{-\alpha t}$$

provided that

$$p(t) < (S/KJ)e^{-\alpha(1-n)t} :$$

(i) If $n = 1$, $p(t)$ must be bounded.

(ii) If $n > 1$, then $p(t)$ can be exponentially unbounded.

(iii) If $n < 1$, then $p(t)$ must tend to 0 exponentially.

Now for a local result we look at (36) and (37). Let D be the set of (x_0, y_0) for which (37) holds. For any such (x_0, y_0) and any $t_0 \geq 0$, the solution $(x(t), y(t))$ remains in a set

$$\Omega(J_0) = \{(x, y) | x^2 + y^2 \leq J_0^2\}.$$

THEOREM 3. If (30) holds in $\Omega(J_0)$ and if (x_0, y_0) satisfies (37) then the solution of (31) through (x_0, y_0) for $t_0 \geq 0$ is in M and the corresponding solution of (26) is in M.

Proof. Notice that $\Omega(J_0)$ is convex. Write (31) as

$$x' = F(X) + E(t) \tag{31}$$

with $E(t) = (0, Kh(t, \psi(t)))^T$ and define a new system

$$X' = G(X) + E(t) \tag{31*}$$

by $G(X) = F(X)$ for $X \in \Omega(J_0)$ and if X is in the complement of $\Omega(J_0)$ then the line from (0,0) to X intersects the boundary of $\Omega(J_0)$ at a unique point X^*. In the latter case, define $G(X) = F(X^*)$. Then G is continuous and globally Lipschitz. Any solution of (31*) with initial values in $\Omega(J_0)$ lies in M. Krasnoselskii's theorem will now say that (27) has a solution in M.

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