INTEGRAL EQUATIONS, PERIODICITY, AND FIXED POINTS

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ABSTRACT. In this paper we are led to the conjecture that if there is a $T > 0$ with $a(t+T) = a(t)$ and $D(t+T, s+T, x) = D(t, s, x)$ and if D is fairly smooth then the integral equation $x(t) = a(t) + \int_{-\infty}^{t} D(t, s, x)ds$ has a T-periodic solution. Several results are offered in defense of the conjecture, but the problem is far from being solved. We use Schaefer's fixed point theorem and a variety of Liapunov functionals to prove the results. The most striking feature of the paper is the fact that we can prove that there is a periodic solution either by differentiating D or by integrating D. It is a very attractive problem for further study.

1. INTRODUCTION

Numerous problems in ordinary and partial differential equations lead us to seek a periodic solution of the scalar equation

$$x(t) = a(t) + \int_{-\infty}^{t} D(t, s, x(s))ds \tag{1}$$

with $T > 0$ so that

$$a(t+T) = a(t), \quad D(t+T, s+T, x) = D(t, s, x), \tag{2}$$

and with a continuous. All of the results will be easily illustrated from the single function

$$D(t, s, x) = m[(t-s)+1]^{-k}g(x)$$

and the algebra is simple. We have studied this problem in [1-7] under considerably stronger assumptions.

The literature is replete with differential and integral equations related to (1) with proofs of the existence of periodic solutions which usually require extensive sign and growth conditions of a very detailed nature. Our conjecture here is that if $\int_{-\infty}^{t} D(t, s, x(s))ds$ converges for any continuous and periodic function x and if D is reasonably smooth, then there always is a periodic solution. This is suggested in an old result, which we offer as Theorem 1.1. But it is offered more strongly in Theorem 3.1 in which we do ask that the equation be of sublinear

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type; however, once the periodic solution is established, then all the action is taking place in a strip of $|x| \le K$ for some $K > 0$. It is then totally immaterial what the behavior of D is with respect to x outside that strip.

We study this problem by looking at examples when D is globally Lipschitz, locally Lipschitz, and non-Lipschitz. In Section 5 we place it in a general framework and offer the details of the first of two steps which would be required for a general solution.

Let $(\mathcal{P}_T, \|\cdot\|)$ denote the Banach space of continuous scalar T-periodic functions with the supremum norm and assume that for $\phi \in \mathcal{P}_T$ then

$$\int_{-\infty}^{t} D(t,s,\phi(s))ds \in \mathcal{P}_T. \tag{3}$$

This will allow problems with mild singularities.

The following simple result is well-known.

Theorem 1.1. *Let (2) and (3) hold. Suppose there is a function $B(t,s)$ with*

$$|D(t,s,x) - D(t,s,y)| \le B(t,s)|x-y| \tag{4}$$

for $-\infty < s \le t < \infty$, $x, y \in \Re$ and $\alpha < 1$ with $\int_{-\infty}^{t} B(t,s)ds$ defined and

$$\int_{-\infty}^{t} B(t,s)ds \le \alpha. \tag{5}$$

Then (1) has a solution in $\mathcal{P}_T$.

Proof. Define a mapping $P : \mathcal{P}_T \to \mathcal{P}_T$ by $\phi \in \mathcal{P}_T$ implies that

$$(P\phi)(t) = a(t) + \int_{-\infty}^{t} D(t,s,\phi(s))ds$$

and notice that by (3) it is well-defined, while by (4) if $\phi, \eta \in \mathcal{P}_T$ then

$$|(P\phi)(t) - (P\eta)(t)| \le \int_{-\infty}^{t} B(t,s)|\phi(s) - \eta(s)|ds \le \alpha\|\phi - \eta\|$$

by (5). Thus, P is a contraction and there is a unique fixed point in $\mathcal{P}_T$. $\square$

Notice that there is no sign condition; everything depends on a global Lipschitz condition, (4), and smallness condition, (5). We will write $D(t,s,x) = C(t,s)g(x)$, drop the global Lipschitz condition, let $\int_{-\infty}^{t} |C(t,s)|ds$ be large, and show that by either making C smooth or g small we can still conclude that there is a periodic solution.

Notice that if g satisfies a global Lipschitz condition and if C is continuous with

$$a(t+T) = a(t), \quad C(t+T, s+T) = C(t,s)$$

then we could use a weighted norm, $|\phi|_k = \sup_{0\leq s\leq T} e^{-ks}|\phi(s)|$, on

$$(P\phi)(t) = a(t) - \int_{t-T}^{t} C(t,s)g(\phi(s))ds$$

and have a contraction with periodic solution regardless of the magnitude of $C(t,s)$. Thus, if there are nice convergence conditions then we can write

$$\begin{aligned}
x(t) &= a(t) - \int_{-\infty}^{t} C(t,s)g(x(s))ds \\
&= a(t) - \sum_{n=0}^{\infty}\int_{t-(n+1)T}^{t-nT} C(t,s)g(x(s))ds \\
&= a(t) - \sum_{n=0}^{\infty}\int_{t-T}^{t} C(t,s-nT)g(x(s-nT))ds
\end{aligned}$$

which has a periodic solution if

$$x(t) = a(t) - \sum_{n=0}^{\infty}\int_{t-T}^{t} C(t,s-nT)g(x(s))ds$$

does. But under strict convergence conditions we can interchange the order of summation and integration to obtain

$$\begin{aligned}
x(t) &= a(t) - \int_{t-T}^{t}\sum_{n=0}^{\infty} C(t,s-nT)g(x(s))ds \\
&=: a(t) - \int_{t-T}^{t} H(t,s)g(x(s))ds.
\end{aligned}$$

This suggests that with a global Lipschitz condition and a nice kernel then the magnitude of the integral of the kernel is immaterial. Thus, we proceed to work on reduction of the Lipschitz condition.

2. Schaefer's theorem: local Lipschitz

We now investigate whether loss of the global Lipschitz condition will affect the result. Consider

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)x^n(s)ds \tag{6}$$

where n is an odd positive integer and let

$$a(t+T) = a(t), \quad C(t+T, s+T) = C(t,s). \tag{7}$$

Assume that for $\phi \in \mathcal{P}_T$ then

$$\int_{-\infty}^{t} |C(t,s)\phi^n(s)|ds \quad \text{is continuous.} \tag{8}$$

Our work will be based on a Liapunov functional working together with the fixed point theorem of Schaefer [9] (see also Smart [10; p. 29]). Indeed, our main interest is in showing how fixed point theory and Liapunov's direct method work together in solving these problems for integral equations. The Liapunov functionals will satisfy very weak conditions and might more correctly be called guiding functions since we will work only with the derivative of the Liapunov functional and often not use the usual positive definite property.

Theorem 2.1. *(Schaefer) Let $(X, \|\cdot\|)$ be a normed space, P a continuous mapping of X into X which is compact on each bounded subset of X. Then either*

(i) *the equation $x = \lambda Px$ has a solution in X for $\lambda = 1$, or*
(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

CONSTRUCTION OF LIAPUNOV FUNCTIONALS I

We first give a Liapunov functional requiring much smoothness, but yields an exact fit with no inequalities required. In the classical theory of integral equations, if

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(s,x(s))ds$$

has a well-behaved kernel and if g has the sign of x, then the solution follows $a(t)$ in some broad sense. Thus, we write

$$(x(t) - a(t))^2 = \left(-\int_{-\infty}^{t} C(t,s)g(s,x(s))ds\right)^2$$

and we strive to prove that the left-hand-side remains small, in accordance with the classical theory. We now show how we arrive at a Liapunov functional, although the conditions here are stronger than will subsequently be needed. Assume that $C_s \geq 0$, that there is an $M > 0$ with $\int_{-\infty}^{t} C_s(t,s)ds \leq M$, and that $C(t,s)(t-s) \to 0$ as $s \to -\infty$. Our goal is to obtain a bound on periodic solutions.

If we integrate the right-hand-side by parts, use the Schwarz inequality, and assume that $g(t,x(t)) \in \mathcal{P}_T$ then we have

$$\begin{aligned}(x(t) - a(t))^2 &= \left(C(t,s)\int_s^t g(u,x(u))du\Big|_{-\infty}^{t}\right.\\ &\quad \left. - \int_{-\infty}^{t} C_s(t,s)\int_s^t g(u,x(u))du\right)^2\\ &\leq \int_{-\infty}^{t} C_s(t,s)ds \int_{-\infty}^{t} C_s(t,s)\left(\int_s^t g(u,x(u))du\right)^2 ds,\\ &\leq M\int_{-\infty}^{t} C_s(t,s)\left(\int_s^t g(u,x(u))du\right)^2 ds.\end{aligned}$$

We have arrived at the Liapunov functional

$$V(t,x(\cdot)) = \int_{-\infty}^{t} C_s(t,s)\left(\int_s^t g(u,x(u))du\right)^2 ds.$$

Notice that there is something of a "wedge" under V in the form of $(1/M)(x(t)-a(t))^2$. Moreover, if $C_s \geq 0$, then for $a(t)=0$ the function V becomes a positive definite Liapunov functional in the classical sense. But here, we will ask much less and still obtain very good bounds on solutions in terms of L^p.

We specialize the above work and use the Liapunov functional

$$V(t) = \lambda \int_{-\infty}^{t} C_s(t,s)\left(\int_s^t x^n(u)du\right)^2 ds \tag{9}$$

to prove that there is an *a priori* $L^p[0,T]$ bound on the norm of the solution of

$$x(t) = \lambda\left[a(t) - \int_{-\infty}^{t} C(t,s)x^n(s)ds\right] \tag{10}$$

and then parlay that into a supremum norm bound. For our mapping, define P in the usual way of $\phi \in \mathcal{P}_T$ implies that

$$(P\phi)(t) = \lambda\left[a(t) - \int_{-\infty}^{t} C(t,s)\phi^n(s)ds\right]. \tag{11}$$

Very roughly, the next two results say that if $\int_{-\infty}^{t}|C(t,s)s|ds$ exists and $C_{st} \leq 0$, then there is a periodic solution. So we need a bit more convergence than desired and the smoothness is a bit more pointed than hoped for, but it gets closer to the long term goal. It allows unlimited growth with x^n. We are surprised that we need n to be odd, yet place no sign condition on C; thus, we feel that the oddness of n can be reduced. In the next section we need $n = 1/m$ where m is odd, but that is to avoid questions of complex solutions.

We will see many derivatives of C here, but that can be misleading as examples will show. If we take $C(t,s) = C(t-s)$ with $C(t) = t(t-1)$ for $0 \leq t \leq 1$ and $C(t) = 0$ for $t \geq 1$, then the limits on the integral of the Liapunov functional will change and problems with derivatives will vanish. In this case, we would see C_s change sign and that is a property in which we will always be interested. To leave open a number of possibilities of the type just mentioned we will refrain from placing strict conditions on C, but ask that V can always be differentiated by Leibnitz rule. In later sections differentiability is reduced and even eliminated.

Theorem 2.2. *Let (7) and (8) hold. Assume that*

$$\int_{-\infty}^{t} C_s(t,s)(t-s)^2 ds \quad \text{is continuous,} \tag{12}$$

as is

$$\int_{-\infty}^{t} C_{st}(t,s)(t-s)^2 ds, \tag{13}$$

and that

$$C(t,s)(t-s) \to 0 \quad \text{as} \quad s \to -\infty. \tag{14}$$

If, in addition,

$$C_{st}(t,s) \leq 0, \tag{15}$$

then for any fixed point x of (11) in $\mathcal{P}_T$ we have

$$\int_0^T x^{n+1}(s)ds \leq \int_0^T a^{n+1}(s)ds. \tag{16}$$

Proof. Use the fixed point x and define $V(t)$ in (9) so that

$$\begin{aligned} V'(t) = \lambda &\int_{-\infty}^{t} C_{st}(t,s)\left(\int_s^t x^n(u)du\right)^2 ds \\ &+ 2\lambda x^n(t) \int_{-\infty}^{t} C_s(t,s) \int_s^t x^n(u)duds. \end{aligned}$$

We integrate the last term by parts obtaining

$$2\lambda x^n(t)\left[C(t,s)\int_s^t x^n(u)du\Big|_{-\infty}^{t} + \int_{-\infty}^{t} C(t,s)x^n(s)ds\right]$$

so that by (14) the first term is zero and we then have

$$V'(t) \leq 2\lambda x^n(t)\left[\int_{-\infty}^{t} C(t,s)x^n(s)ds\right]$$

by Leibnitz rule without any reference to the integral equation. But now we use the integral equation, (10), and write

$$\begin{aligned} V'(t) &\leq 2x^n(t)[\lambda a(t) - x(t)] \\ &\leq \frac{2}{n+1}[a^{n+1}(t) - x^{n+1}(t)]. \end{aligned}$$

Next, it is readily verified that since $x \in \mathcal{P}_T$, so is V. Thus, $V(T) = V(0)$ and

$$0 = V(T) - V(0) \leq \frac{2}{n+1}\left(\int_0^T a^{n+1}(s)ds - \int_0^T x^{n+1}(s)ds\right)$$

or

$$\int_0^T x^{n+1}(s)ds \leq \int_0^T a^{n+1}(s)ds,$$

as required. □

We will use the L^p bound and (6), the integral equation, to obtain a supremum norm bound. Here, (16) is the crucial condition. One readily supposes that there are many conditions which can replace (18) and we will see other possibilities as we proceed.

Theorem 2.3. *Let (7), (8), and (16) hold, suppose there is a constant Q such that*

$$\int_{-\infty}^{t_1} |C(t_1,s) - C(t_2,s)|ds \le Q|t_1 - t_2| \quad \text{if} \quad 0 \le t_1 \le t_2 \le T, \tag{17}$$

and suppose that

$$\sup_{0\le t\le T} \sum_{j=0}^{\infty} \left(\int_{t-T}^{t} C^{n+1}(t+jT,s)ds \right)^{\frac{1}{n+1}} < \infty. \tag{18}$$

Then (6) has a solution in $\mathcal{P}_T$.

Proof. In (11) we defined $P : \mathcal{P}_T \to \mathcal{P}_T$. We will find a number L such that if $x \in \mathcal{P}_T$ is a fixed point of that mapping then $\|x\| < L$, where $\|\cdot\|$ is the supremum norm.

Condition (17) readily shows that bounded sets are mapped into equicontinuous sets. Let the bounded set be fixed and let ϕ be any point in that set. There are positive constants J and Y with $\|\phi\| \le J$ and $\|\phi^n\| \le Y$. Thus, if $0 \le t_1 \le t_2 \le T$, then

$$\begin{aligned}
|(P\phi)(t_2) - (P\phi)(t_1)| &\le |a(t_2) - a(t_1)| \\
&\quad + \left| \int_{-\infty}^{t_1} [C(t_2,s) - C(t_1,s)]\phi^n(s)]ds \right| \\
&\quad + \left| \int_{t_1}^{t_2} C(t_2,s)\phi^n(s)ds \right| \\
&\le |a(t_2) - a(t_1)| + (YQ + YE)||t_1 - t_2|
\end{aligned}$$

where $E = \sup_{0\le s\le T, 0\le t_2\le T} |C(t_2,s)|$.

Continuity of P follows from (8) and the uniform continuity of ϕ^n when $\phi_n \in \mathcal{P}_T$ and $\phi_n \to \phi$.

For $x \in \mathcal{P}_T$ a solution of (10) we have

$$\begin{aligned}
|x(t) - \lambda a(t)| &\leq \left| \int_{-\infty}^{t} C(t,s)x^n(s)ds \right| \\
&= \left| \sum_{j=0}^{\infty} \int_{t-(j+1)T}^{t-jT} C(t,s)x^n(s)ds \right| \\
&= \left| \sum_{j=0}^{\infty} \int_{t-T}^{t} C(t,s-jT)x^n(s)ds \right| \\
&= \left| \sum_{j=0}^{\infty} \int_{t-T}^{t} C(t+jT,s)x^n(s)ds \right| \\
&\leq \sum_{j=0}^{\infty} \left(\int_{t-T}^{t} C^{n+1}(t+jT,s)ds \right)^{\frac{1}{n+1}} \left(\int_{t-T}^{t} a^{n+1}(s)ds \right)^{\frac{n}{n+1}},
\end{aligned}$$

which has a finite bound independent of λ. Referring now to Schaefer's theorem we see that the second alternative is ruled out and the conclusion holds. □

In the next section we will use $x^{1/m}$ where m is an odd positive integer instead of x^n and the same Liapunov functional would work in that case. But we opt for a different method of proof and move from the requirement of two derivatives on C to the condition that C be "twice" integrable on the whole line.

3. A sublinear problem

We have seen that absence of a global Lipschitz condition offers no difficulties. We now examine an example lacking a local Lipschitz condition. Consider the equation

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)x^{1/m}(s)ds \tag{19}$$

where m is an odd positive integer, $m > 1$, and where (7), periodicity, holds. We will first suppose that $C_s(t,s)$ exists and later suppose that $C(t,s)$ is bounded by a function, $E(t,s)$, for which $E_s(t,s)$ exists. A far stronger result will be proved in the first case.

CONSTRUCTION OF LIAPUNOV FUNCIONALS II

If we want to bound $x(t)$ in (19) then we write

$$|x(t)| \leq |a(t)| + \int_{-\infty}^{t} |C(t,s)||x^{1/m}(s)|ds.$$

We would like to exchange the functional on the right for a function so that we could get an algebraic relation and solve for $x(t)$. Such a technique may be found in the literature for integrodifferential equations as long ago as the 1970's. We define

$$V(t) := \int_{-\infty}^{t} \int_{t-s}^{\infty} |C(u+s,s)| du |x^{1/m}(s)| ds$$

and throughout the remainder of the paper we suppose that for $x \in \mathcal{P}_T$ then this can be differentiated by Leibnitz rule yielding

$$V'(t) = \int_{0}^{\infty} |C(u+t,t)| du |x^{1/m}(t)| - \int_{-\infty}^{t} |C(t,s)||x^{1/m}(s)| ds.$$

Hence, if x solves (19) then we have

$$V'(t) \leq \int_{0}^{\infty} |C(u+t,t)| du |x(t)| - |x(t)| + |a(t)|,$$

a totally algebraic relation which we exploit throughout the remainder of the paper.

We now come to a very interesting situation. In order to get an L^p bound we had asked that $C_{st}(t,s) \leq 0$. We now see that integrating C can be just as effective. In the next result we ask that the integral of $C(t,s)$ with respect to t be bounded. Then we differentiate $C(t,s)$ with respect to s, but partially restore it by multiplying by $(t-s)$; finally, then, we ask that the integral of $|C_s(t,s)|(t-s)$ with respect to s be bounded. A bounded integral with respect to both coordinates will do the same for us as asking that $C_{st}(t,s) \leq 0$. Moreover, both sets of conditions are used to make a Liapunov functional effective.

Consider the conditions of the following theorem. First, in order for (19) to be a well-defined problem we would expect

$$\int_{-\infty}^{t} |C(t,s)| ds < \infty; \tag{*}$$

thus, we would expect $C(t,s) \to 0$ as $s \to -\infty$ a bit faster than $1/s$. Hence, (20) is expected.

Moreover, in the convolution case, the first part of (21) would say that $\int_0^\infty |C(u)| du < \infty$, which is just (*) again. It is less than Theorem 1.1 asks since the integral can be large and a Lipschitz condition is not required. The second part of (21) would ask that $\int_t^\infty |C(u)| du \in L^1[0,\infty)$ and that is so much more than Theorem 1.1 asks; but this is only to make the Liapunov functional defined, so it may be completely extraneous.

Finally, with (20) holding we have

$$\int_{-\infty}^{t} C_s(t,s)(t-s) ds = C(t,s)(t-s)\Big|_{-\infty}^{t} + \int_{-\infty}^{t} C(t,s) ds = \int_{-\infty}^{t} C(t,s) ds$$

which is periodic and, hence, bounded if it is continuous.

For smooth kernels, this next result says essentially that if (21) holds and (19) is well-defined, then it has a periodic solution. That periodic solution lies in a strip $|x| \leq K$, for some $K > 0$, so the fact that (19) is sublinear is a device in the proof, but apparently it is not part of the essential nature of the problem.

Theorem 3.1. *Suppose that $n = 1/m$ and that (7), (8), and (17) hold. In addition, suppose that*

$$C(t,s)(t-s) \to 0 \quad \text{as} \quad s \to -\infty \quad \text{for fixed} \ \ t, \tag{20}$$

that there is an $\alpha < \infty$ with

$$\int_0^\infty |C(u+t,t)|du \le \alpha \text{ and } \int_{-\infty}^t \int_{t-s}^\infty |C(u+s,s)|duds \text{ exists,} \tag{21}$$

and that

$$\int_{-\infty}^t |C_s(t,s)|[(t-s)+1]ds \quad \text{is bounded.} \tag{22}$$

Then (19) has a solution in $\mathcal{P}_T$.

Proof. With a view to using Schaefer's theorem we start with

$$x(t) = \lambda\left[a(t) - \int_{-\infty}^t C(t,s)x^{1/m}(s)ds\right], \tag{19_λ}$$

define the corresponding mapping P from it, and then define the new Liapunov functional

$$V(t) = \lambda \int_{-\infty}^t \int_{t-s}^\infty |C(u+s,s)|du|x^{1/m}(s)|ds.$$

The derivative along the solution in $\mathcal{P}_T$ of the integral equation (19_λ) satisfies

$$\begin{aligned}
V'(t) &= \lambda \int_0^\infty |C(u+t,t)|du|x^{1/m}(t)| - \lambda \int_{-\infty}^t |C(t,s)x^{1/m}(s)|ds \\
&\le \alpha|x^{1/m}(t)| - \lambda \int_{-\infty}^t |C(t,s)x^{1/m}(s)|ds \\
&\le \alpha|x^{1/m}| + |a(t)| - |x(t)| \\
&\le -\beta|x^{1/m}(t)| + (\gamma + |a(t)|)
\end{aligned}$$

for some positive constants β and γ since $m > 1$.

As x is supposed to be a solution in $\mathcal{P}_T$ we see that $V \in \mathcal{P}_T$. Thus,

$$0 = V(T) - V(0) \le -\beta \int_0^T |x^{1/m}(s)|ds + \int_0^T (\gamma + |a(t)|)dt$$

so

$$\int_0^T |x^{1/m}(s)|ds \le (1/\beta) \int_0^T (\gamma + |a(t)|)dt =: J.$$

Moreover, if t is chosen so that $V(t)$ is the maximum of that periodic function, V, then for $s < t$ we have

$$0 \le V(t) - V(s) \le \int_s^t (\gamma + |a(u)|)du - \beta \int_s^t |x^{1/m}(u)|du.$$

For any t and for $s < t$ it follows that

$$\int_s^t |x^{1/m}(u)|du \le (1/\beta)\int_s^t (\gamma + |a(u)|)du + J.$$

Thus, there is a constant $K > 0$ with

$$\int_s^t |x^{1/m}(u)|du \le J + (t-s)K.$$

An integration by parts in (19_λ) yields

$$x(t) = \lambda a(t) + \lambda C(t,s)\int_s^t x^{1/m}(u)du\Big|_{-\infty}^{t} - \lambda\int_{-\infty}^t C_s(t,s)\int_s^t x^{1/m}(u)duds$$

or by (20)

$$\begin{aligned}|x(t)| &\le |a(t)| + \int_{-\infty}^t |C_s(t,s)|\int_s^t |x^{1/m}(u)|duds\\ &\le |a(t)| + \int_{-\infty}^t |C_s(t,s)|[K(t-s)+J]ds\end{aligned}$$

By (22) this is bounded. The compactness follows from (17) and the continuity follows from (8). By Schaefer's theorem the mapping has a fixed point. □

We now suppose that $C_s(t,s)$ fails to exist and that there is a continuous function $E(t,s)$ and a positive constant M with

$$\begin{aligned}&|C(t,s)| \le E(t,s), \int_{-\infty}^t E(t,s)ds \le M,\\ &\int_{-\infty}^t E_s(t,s)[1+(t-s)]^2 ds \quad \text{is bounded, and}\\ &E(t,s)(t-s) \to 0 \quad \text{as} \quad s \to -\infty \quad \text{for fixed} \quad t,\end{aligned} \tag{23}$$

Notice that this will still not allow a mild singularity in C; that must wait for the next section.

Theorem 3.2. *Let (23) hold. If $x(t)$ is a bounded solution of (19_λ) then*

$$(x(t) - \lambda a(t))^2 \le M\int_{-\infty}^t E_s(t,s)\left(\int_s^t |x^{1/m}(u)|du\right)^2 ds. \tag{24}$$

Proof. We have

$$\begin{aligned}
(x(t)-\lambda a(t))^2 &= \left|\int_{-\infty}^{t} C(t,s)x^{1/m}(s)ds\right|^2 \\
&\le \left|\int_{-\infty}^{t} E(t,s)|x^{1/m}(s)|ds\right|^2 \\
&= \Bigg(-E(t,s)\int_s^t |x^{1/m}(u)|du\Bigg|_{-\infty}^{t} \\
&+ \int_{-\infty}^{t} E_s(t,s)\int_s^t |x^{1/m}(u)|duds\Bigg)^2 \\
&\le \int_{-\infty}^{t} E_s(t,s)ds\int_{-\infty}^{t} E_s(t,s)\left(\int_s^t |x^{1/m}(u)|du\right)^2 ds \\
&\le M\int_{-\infty}^{t} E_s(t,s)\left(\int_s^t |x^{1/m}(u)|du\right)^2 ds
\end{aligned}$$

proving the result. □

Theorem 3.3. *For $n = 1/m$ let (7), (8), (17), (21) and (23) hold.*

(i) If $x \in \mathcal{P}_T$ solves (19_λ) then there is a $\beta > 0$, a $\gamma > 0$, and a $t \in [0,T]$ such that for $-\infty < s < t$ we have

$$\int_s^t |x^{1/m}(u)|du \le (1/\beta)\int_s^t (\gamma + |a(u)|)du. \tag{25}$$

(ii) If (23) holds so that (24) is satisfied then (19) has a solution in $\mathcal{P}_T$.

Proof. Following the proof of Theorem 3.1 we again start with

$$x(t) = \lambda\left[a(t) - \int_{-\infty}^{t} C(t,s)x^{1/m}(s)ds\right]$$

and define

$$V(t) = \lambda\int_{-\infty}^{t}\int_{t-s}^{\infty} |C(u+s,s)|du|x^{1/m}(s)|ds$$

whose derivative was shown to satisfy

$$V'(t) \le -\beta|x^{1/m}(t)| + (\gamma + |a(t)|).$$

As x is supposed to be a solution in $\mathcal{P}_T$ we see that $V \in \mathcal{P}_T$. Again, for $V(t)$ the maximum of that periodic function and for $s < t$ we have

$$0 \le V(t) - V(s) \le \int_s^t (\gamma + |a(u)|)du - \beta\int_s^t |x^{1/m}(u)|du$$

as in the proof of Theorem 3.1 so that

$$\int_s^t |x^{1/m}(u)|du \le (1/\beta)\int_s^t (\gamma + |a(u))du$$

and then we argued that there are fixed positive constants J and K with

$$\int_s^t |x^{1/m}(u)|du \leq J + (t-s)K$$

for any pair (s,t) with $s < t$. Using this in (24) yields

$$(x(t) - \lambda a(t))^2 \leq M \int_{-\infty}^t E_s(t,s)(J + (t-s)K)^2 ds,$$

and the right-hand-side is bounded by (23). Thus, the boundedness of $a(t)$ yields a suitable *a priori* bound on x. The equicontinuity and continuity of P are exactly as before. □

4. No derivatives of $C(t,s)$

One of our objectives is to consider problems originating as partial differential equations (See Miller [8; p. 60, p. 172, and p. 208].) which were then parlayed into integral equations and then into infinite delay problems by means of limiting processes. In some such problems we find mild singularities and, at least in the convolution case for the limiting process, C is to be an L^1-function. We now show how Theorem 3.1 can be changed to cover just such problems.

In case of mild singularities, (17) would not hold and the proof of equicontinuity given in the proof of Theorem 2.3 would not work. However, there are alternative methods as one readily sees in the case of

$$C(t,s) = e^{-(t-s)}(t-s)^{-1/2}$$

when we work out the left-hand-side of (17). Thus, in our result below we simply ask for the compactness.

Theorem 4.1. *Suppose that (7), (8), and (21) hold for $n = 1/m$ and that the mapping P defined by*

$$(P\phi)(t) = \lambda\left[a(t) - \int_{-\infty}^t C(t,s)x^{1/m}(s)ds\right]$$

maps bounded subsets of $\mathcal{P}_T$ into compact subsets and that P is continuous. Moreover, let

$$\sup_{0\leq t\leq T} \sum_{n=0}^{\infty} \left(\int_{t-T}^t C^2(t+nT,s)ds\right)^{1/2} < \infty. \tag{26}$$

Then (19) has a solution in $\mathcal{P}_T$.

Proof. We follow the proof of Theorem 3.1 with

$$x(t) = \lambda\left[a(t) - \int_{-\infty}^t C(t,s)x^{1/m}(s)ds\right]$$

and define the Liapunov functional

$$V(t) = \lambda \int_{-\infty}^{t} \int_{t-s}^{\infty} |C(u+s,s)|du|x^{1/m}(s)|ds.$$

Then we notice in the differentiation of V the last lines may be changed and we can have

$$\begin{aligned} V'(t) &= \lambda \int_{0}^{\infty} |C(u+t,t)|du|x^{1/m}(t)| - \lambda \int_{-\infty}^{t} |C(t,s)x^{1/m}(s)|ds \\ &\leq \alpha|x^{1/m}(t)| - \lambda \int_{-\infty}^{t} |C(t,s)x^{1/m}(s)|ds \\ &\leq \alpha|x^{1/m}| + |a(t)| - |x(t)| \\ &\leq -\beta|x^{2/m}(t)| + (\gamma + |a(t)|) \end{aligned}$$

for some positive constants β and γ. Recall that m is an odd positive integer, $m > 1$, so $m \geq 3$. That gives the change

$$0 = V(T) - V(0) \leq -\beta \int_{0}^{T} |x^{2/m}(s)|ds + \int_{0}^{T} (\gamma + |a(t)|)dt$$

so

$$\int_{0}^{T} |x^{2/m}(s)|ds \leq (1/\beta) \int_{0}^{T} (\gamma + |a(t)|)dt =: H.$$

Now from (19_λ) we have

$$\begin{aligned} |x(t)| - |a(t)| &\leq \sum_{n=0}^{\infty} \int_{t-(n+1)T}^{t-nT} |C(t,s)||x^{1/m}(s)|ds \\ &= \sum_{n=0}^{\infty} \int_{t-T}^{t} |C(t,s-nT)||x^{1/m}(s)|ds \\ &\leq \sum_{n=0}^{\infty} \left(\int_{t-T}^{t} C^2(t+nT,s)ds\right)^{1/2} \left(\int_{t-T}^{t} x^{2/m}(s)ds\right)^{1/2} \\ &\leq H^{1/2} \sum_{n=0}^{\infty} \left(\int_{t-T}^{t} C^2(t+nT,s)ds\right)^{1/2}. \end{aligned}$$

By (26) this yields the required bound and the conclusion follows from Schaefer's theorem. □

5. The general case

Let us now consider the general case of

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(x(s))ds \tag{27}$$

where a, C, g are all continuous, $xg(x) > 0$ for $x \neq 0$, and (7) holds. First, if we examine common kernels such as $(t-s+1)^{-2}$ or $e^{-(t-s)}$ then we notice that frequently there is a continuous function $\gamma : [0,\infty) \to (0,\infty)$ with

$$C_{ts}(t,s) \le -\gamma(t)C_s(t,s). \tag{28}$$

If we then define

$$V(t) = \lambda \int_{\infty}^{t} C_s(t,s)\left(\int_s^t g(x(u))du\right)^2 ds$$

we find that the derivative of V along the solution of the equation

$$x(t) = \lambda\left[a(t) - \int_{-\infty}^{t} C(t,s)g(x(s))ds\right] \tag{27_λ}$$

will satisfy

$$V'(t) \le -\gamma(t)V(t) + 2g(x)[\lambda a(t) - x(t)]. \tag{29}$$

When $|x| \le 2\|a\|$ then $2|g(x)a(t)| \le 2\|a\|g^*$ when $g^* = \sup_{0\le|x|\le 2\|a\|}|g(x)|$. Then for $|x| \ge 2\|a\|$ we have $2|g(x)a(t)| \le |g(x)||x|$. We therefore see that

$$V'(t) \le -\gamma(t)V(t) + 2\|a\|g^*. \tag{30}$$

Thus, Liapunov functionals for integral equations can satisfy the same kinds of differential inequalities widely seen for differential equations. In some cases that differential inequality will yield a bound suitable for Schaefer's theorem.

In our work to this point we have always taken two steps. First, we obtain an integral bound and then parlay it into a supremum bound. The following will show how the first part can be accomplished in very general cases. Suppose that we have the form (27) where

$$C_{st}(t,s) \le 0$$

and

$$g(x) =: xF(x)$$

where $F(x) > 0$ and continuous. We define V as above and obtain $V'(t) \le 2g(x)[\lambda a(t) - x(t)]$. As $a \in \mathcal{P}_T$ we can write

$$|a(t)| \le \|a\|.$$

Thus, when $|x| \ge 2\|a\|$ we have

$$2|g(x)a(t)| \le 2\|a\||xF(x)| \le x^2F(x)$$

so that

$$V'(t) \le -x^2F(x).$$

When $|x| \le 2\|a\|$ then for

$$F^* := \sup_{|x|\le 2\|a\|} F(x)$$

we have

$$2|a(t)xF(x)| \le (2\|a\|)^2F^*.$$

In any case, we will have

$$V'(t) \le -x^2F(x) + (2\|a\|)^2F^*.$$

This will yield

$$\int_0^T x^2(s)F(x(s))ds \le (2\|a\|)^2F^*T.$$

The second step of our problem is to parlay this into a supremum norm bound. This is a broad unsolved problem.

6. A choice of hypotheses

We are now going to combine Theorems 2.2, 2.3, and 4.1 in a way which gives us choices of hypotheses and those choices seem totally independent.

Theorem 6.1. *In Equation (19) take* $m = 3$ *and suppose that (7), (8), (17), and (26) hold. If either (21) or* $\{$*(12), (13), (14), and (15)*$\}$ *hold then (19) has a solution in* $\mathcal{P}_T$.

Proof. First, because (17) holds we can prove that the mapping P in Theorem 4.1 maps bounded subsets of $\mathcal{P}_T$ into compact subsets, as we did in the proof of Theorem 2.3. Thus, if (21) holds then our first choice becomes exactly Theorem 4.1 so there is a periodic solution.

Next, suppose that (12)–(15) hold. According to the proof of Theorem 2.2 the functional V in (9) will be defined for $n = 1/3$ and we will again have

$$V'(t) \le \frac{2}{n+1}\Big[a^{n+1}(t) - x^{n+1}(t)\Big],$$

together with (16) which now reads

$$\int_0^T x^{4/3}(s)ds \le \int_0^T a^{4/3}(s)ds =: K.$$

But $x^{4/3}(s) + 1 \ge x^{2/3}(s)$ and so we have

$$\int_0^T x^{2/3}(s)ds \le K + T =: H$$

where H will now be used again as the constant in the proof of Theorem 4.1. Thus, the proof of Theorem 4.1 can be completed with that H and the conclusion holds. □

We are left with an intriguing problem. Perhaps there is a great array of independent conditions such as the two sets illustrated in the theorem. On the other hand, it may be that (7), (8), (17), and a convergence condition are all that is needed and the two hypotheses offered here are totally extraneous. As mentioned earlier, it is interesting that we need n to be odd, but there is no sign condition on $C(t,s)$. We feel that this is a reducible condition.

Finally, compare Theorems 3.3 and 2.3. Theorem 3.3 contains nothing about derivatives of C. Except for (14), everything in Theorem 2.3 rests on derivatives of C. For smooth functions, (14) is closely related to (8).

7. References

1. Burton, T.A., Liapunov functionals and periodicity in integral equations, Tohoku Math. J. (1994), 207-220

2. Burton, T.A., Boundedness and periodicity in integral and integro-differential equations, Differential Equations and Dynamical Systems 1(1993), 161-172.

3. Burton, T. A. and Furumochi, Tetsuo, Periodic solutions of Volterra equations and attractivity, Dynamic Systems and Applications 3(1994), 583-598.

4. Burton, T. A. and Furumochi, Tetsuo, Periodic solutions of a Volterra equation and robustness, Nonlinear Analysis 25(1995), 1199-1219.

5. Burton, T. A. and Furumochi, Tetsuo, Periodic and asymptotically periodic solutions of Volterra integral equations, Funkcialaj Ekvacioj 39(1996), 87-107.

6. Burton, T. A. and Furumochi, Tetsuo, Existence theorems and periodic solutions of neutral integral equations, Nonlinear Analysis 43(2001), 527-546.

7. Burton, T. A. and Makay, G., Continuity, compactness, fixed points, and integral equations, Electron. J. Qual. Theory Differ. Equ., 2002 No. 14, pp. 1-13.

8. Miller, Richard K., Nonlinear Volterra Integral Equations, Benjamin, Menlo Park, California, 1971.

9. Schaefer, H., Über die Methode der a priori-Schranken, Math. Ann. 129(1955), 415–416.

10. Smart, D.R., Fixed Point Theorems, Cambridge, 1980.

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