Integral Equations, Volterra Equations, and the Remarkable Resolvent: Contractions

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1. Introduction. This paper concerns several variants of an integral equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds,$$

a resolvent

$$R(t,s),$$

and a variation-of-parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds$$

with special accent on the case in which $a(t)$ is unbounded. We use contration mappings to establish close relations between $a(t)$ and $\int_0^t R(t,s)a(s)ds$.

This work gives us a fundamental understanding of the nature of $R(t,s)$. It establishes numerous elementary boundedness results including some from a new point of view. And it tells us that one of our long-held basic assumptions is very incomplete. For more than one hundred years investigators have taken the view that, for a well behaved kernel $C(t,s)$, the solution follows $a(t)$: if $a(t)$ is bounded, the solution $x(t)$ is bounded; if $a(t)$ is L^p, then x is L^p; if $a(t)$ is periodic, x approaches a periodic function. Indeed, the author, himself, has formally stated this in a number of papers. A more accurate view may be that $\int_0^t R(t,s)a(s)ds$ follows $a(t)$ and, hence, there is the occasional appearance that x is following $a(t)$, particularly when $a(t)$ is bounded. But when $a(t)$ is unbounded, we have a much clearer perception.

Investigators spent much time in the 19th century devising methods of solving differential and integral equations in closed form. Although there are still vigorous areas of research along those lines today, the scope of the investigations was drastically reduced by the work of Ritt [14], Kaplansky [11], and others who used ideal theory to show that even the simple differential equation $x'' + tx = 0$ has solutions which are not composits of elementary functions and their integrals. Thus, we come to understand that solutions of linear differential and integral equations are arbitrarily complicated.

But if we solve the resolvent equation then there is a variation-of-parameters formula in the form of an integral of the resolvent and the forcing function, written $\int_0^t R(t,s)a(s)ds$, which gives the solution of the forced equation. This is true for a wide variety of integral equations and Volterra integrodifferential equations. That resolvent contains those arbitrarily complicated functions discussed above.

Here we raise and answer two questions. First, while $R(t,s)$ is very complicated, can it be that the integral, $\int_0^t R(t,s)a(s)ds$, is extremely simple? In fact, could that integral be well approximated by the forcing function, $a(t)$, even when $a(t)$ is unbounded? The resolvent behavior studied here is general. We consider three essentially different resolvents and all exhibit these same properties. The questions which we raise are answered by means of the contraction mapping principle, readily accessible to second or third year university students.

The literature on the resolvent is massive. Becker [1], Burton [2], Corduneanu [5, 6], Eloe *et al* [7], Gripenberg *et al* [9], Hino and Murkami [10], Miller [12], Zhang [15] all contain discussions from very different points of view. Gripenberg *et al* [9] have a large bibliography on it. Formulae for resolvents are found in Chapter 7 of [2] and Chapter 4 of [12], for example.

2. Old resolvents and new ones. Our discussion here will concern scalar equations, although much of it is true for vector systems. Indeed, simply ask that x, y, ϕ, ψ, a be n-vectors, while C, R, Z are $n \times n$-matrices, and $Z(t,t) = I$. The intent is to convey ideas. Given an integral equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds, \tag{1}$$

with $a(t)$ continuous for $t \geq 0$ and $C(t,s)$ continuous for $0 \leq s \leq t$, we define the resolvent equation by

$$R(t,s) = C(t,s) - \int_s^t R(t,u)C(u,s)du \tag{2}$$

with solution $R(t,s)$, called the resolvent, and the variation-of-parameters formula

$$x(t) = a(t) - \int_0^t R(t,u)a(u)du. \tag{3}$$

Thus, if we could find $R(t,s)$, then we could find $x(t)$ for an arbitrary continuous $a(t)$. Suppose that $(V,+,\cdot)$ is a vector space of certain continuous functions $\phi : [0,\infty) \to \Re$, the reals. For a given resolvent, $R(t,s)$, we may define a mapping $P : V \to W$ by $\phi \in V$ implies that

$$(P\phi)(t) = \phi(t) - \int_0^t R(t,u)\phi(u)du, \tag{4}$$

where $(W,+,\cdot)$ is the vector space of continuous functions $\psi : [0,\infty) \to \Re$.

Proposition 1. Let $C(t,s)$ be a given continuous function and let $R(t,s)$ be the resolvent defined by (2). Every solution of (1) is bounded for every bounded continuous function a if and only if

$$\sup_{t\geq 0} \int_0^t |R(t,s)|ds < \infty. \tag{5}$$

Proof. If (5) holds then from (3) it is trivial that $x(t)$ is bounded for every bounded continuous function a. Now, suppose that $x(t)$ is bounded for every bounded continuous function a and examine (3). Again, it is then trivial that $\int_0^t R(t,u)a(u)du$ is bounded for every bounded continuous a. Then (5) follows from Perron's theorem (cf. Perron [13] or Burton [3;p. 116]).

Remark 1. If (5) holds then from Proposition 1 we derive what we perceive to be the totally unremarkable fact that P, defined by (4), maps the vector space of bounded and continuous functions, denoted by BC, into itself. Our perception will change as we allow $a(t)$ to become unbounded.

Now the reason that we are slow to make a significant observation is that we have asked too little. Consider (3) and (4) under the assumption that $a(t)$ is unbounded. Imagine that we have placed conditions ensuring that the solution $x(t)$ of (1) is bounded for every function a having a bounded derivative. Then (4) maps a vector space of unbounded functions into BC. In other words, that integral $\int_0^t R(t,s)a(s)ds$ is a fair approximation to $a(t)$ so that it is reasonable to think of that integral as defining the identity map with a small perturbation. This is a remarkable property considering how complicated $R(t,s)$ may be. Continue, and under the same conditions suppose that the solution tends to zero. Then that integral has become very nearly the identity map for large t. We will state these simple observations as formal results and give examples in the next section.

The ideas just expressed are reminiscent of the method of undetermined coefficients, but on examination they are far more precise. Recall that for a linear second order differential equation with constant coefficients and with forcing function $a(t) = t$, for example, we would try for a solution $x(t) = \alpha t + \beta$ with α and β constants to be determined and seldom is $\alpha = 1$. But under the conditions of our work here that integral, $\int_0^t R(t,s)a(s)ds$, turns out to equal $\alpha a(t) + \beta(t)$ where α is invariably 1 and $\beta(t)$ is a bounded function.

Proposition 2. Suppose that for a given $C(t,s)$ the unique solution of (1) is bounded whenever $a'(t)$ is bounded and continuous. If $(V, +, \cdot)$ is the vector space of continuously differentiable functions $\phi : [0,\infty) \to R$ with $\phi'(t)$ bounded, then $P : V \to BC$, the space of bounded continuous functions, where P is defined by (4).

Proof. Obviously, if the right-hand-side of (3) is bounded for every function $a(t)$ with $a'(t)$ bounded and continuous, so is the right-hand-side of (4).

Remark 2. Under the condition of Proposition 2, we see from (3) that $\int_0^t R(t,u)\phi(u)du$ approximates $\phi(t)$ to within a bounded function. It does so for every ϕ in the vector space. And this is remarkable when $\phi(t)$ is unbounded.

Definition 1. Let P, defined by (4), map a vector space V into a vector space W. The resolvent $R(t,s)$ is said to generate an aproximate identity on V if $W = BC$.

Remark 3. If the same resolvent $R(t,s)$ generates an approximate identity on vector spaces V_1 and V_2, then it generates an approximate identity on $V_1 \cup V_2 =: V_3$ in the sense that if $\phi \in V_3$ and if we find $\phi_1 \in V_1$ and $\phi_2 \in V_2$ with $\phi = \phi_1 + \phi_2$, then $P\phi \in BC$. For example, $\phi(t) = t^{1/3}$ is neither bounded, as in Proposition 1, nor is ϕ' bounded and continuous, as in Proposition 2. However, we can write

$$\phi_1(t) = \begin{cases} t^{1/3} - (1/3)t, & \text{for } 0 \le t \le 1 \\ (2/3) & \text{for } t \ge 1, \end{cases}$$

and

$$\phi_2(t) = \begin{cases} (1/3)t & \text{for } 0 \le t \le 1 \\ t^{1/3} - (2/3) & \text{for } t \ge 1. \end{cases}$$

Definition 2. Let the resolvent $R(t,s)$ generate an approximate identity on V. Then $R(t,s)$ generates an asymptotic identity on V if $\phi \in V$ implies that for P defined by (4), then $(P\phi)(t) \to 0$ as $t \to \infty$.

Remark 4. We can give conditions on $C(t,s)$ so that if $a'(t) \to 0$ then the solution $x(t)$ of (1) tends to zero. Now it is often the case that $a'(t) \to 0$, but $a(t) \to \infty$ (e.g., $a(t) = \ln(t+1)$). In (3) we then see the remarkable fact that $a(t) \to \infty$ and yet $\int_0^t R(t,u)a(u)du \to a(t)$.

Definition 3. The resolvent $R(t,s)$ is said to generate an L^p approximate identity on V if for P defined by (4) there is a p with $P : L^p \to L^p$.

Our work here is entirely based on contractions and we do not prove any result about L^p mappings. But there is a parallel work in progress based on Liapunov theory for integral equations and in that context L^p properties are natural.

In the process of implementing Definition 2 the scope of our investigation expands and we consider integrodifferential equations and a new resolvent.

Let $A(t)$ and $a'(t)$ be continuous scalar functions for $t \ge 0$, $B(t,s)$ be continuous for $0 \le s \le t$, and consider

$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)ds + a'(t). \tag{6}$$

Now one resolvent equation for (6) is

$$Z_s(t,s) = -Z(t,s)A(s) - \int_s^t Z(t,u)B(u,s)du,\ Z(t,t) = 1, \tag{7}$$

with resolvent $Z(t,s)$. The variation-of-parameters formula is

$$x(t) = Z(t,0)x(0) + \int_0^t Z(t,s)a'(s)ds. \tag{8}$$

It can be shown using contractions that (7) has a unique continuous solution $Z(t,s)$ and, by (7), it follows that $Z_s(t,s)$ is continuous. Hence, we can integrate by parts in (8) and obtain

$$\begin{aligned} x(t) &= Z(t,0)x(0) + Z(t,s)a(s)\Big|_0^t - \int_0^t Z_s(t,s)a(s)ds \\ &= Z(t,0)x(0) + Z(t,t)a(t) - Z(t,0)a(0) - \int_0^t Z_s(t,s)a(s)ds \end{aligned}$$

or

$$x(t) = Z(t,0)[x(0) - a(0)] + a(t) - \int_0^t Z_s(t,s)a(s)ds \tag{9}$$

as a new variation-of-parameters formula with a new resolvent $Z_s(t,s)$. Moreover, the case $x(0) = a(0)$ is of special interest, yielding the principal variation-of-parameters formula

$$x(t) = a(t) - \int_0^t Z_s(t,s)a(s)ds \tag{10}$$

which is identical with (3) for (1) with $R(t,s)$ replaced by $Z_s(t,s)$. In particular, (4) becomes

$$(P\phi)(t) = \phi(t) - \int_0^t Z_s(t,s)\phi(s)ds \tag{11}$$

and the definitions may be repeated for $Z_s(t,s)$.

In our examples we will show that for our equation we have $Z(t,0)$ bounded because that will constitute the case $a'(t) = 0$. Thus, in our context the condition $Z(t,0)$ bounded in the next two propositions will be readily satisfied.

Proposition 3. Suppose that $Z(t,0)$ is bounded. Every solution of (6) is bounded for every bounded continuous $a'(t)$ if and only if

$$\sup_{t\geq 0} \int_0^t |Z(t,s)|ds < \infty. \tag{*}$$

The proof is exactly like that of Proposition 1 when we focus on (8) with the sure knowledge that $Z(t,0)$ is bounded independently of $a'(t)$.

We will obtain an equation in the form of (6) in two very different ways. The most direct way is to assume that a' and $C_t(t,s)$ in (1) are both continuous and differentiate (1) to obtain (6) which we interpret in terms of (1).

We may summarize our previous work as follows.

Proposition 4. Consider the equation

$$x' = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds \tag{12}$$

with $a'(t)$ and $C_t(t,s)$ continuous and with the resolvent equation from (7) being

$$Z_s(t,s) = Z(t,s)C(s,s) + \int_s^t Z(t,u)C_1(u,s)du, \; Z(t,t) = 1. \tag{12a}$$

Let the Z in (8), (9), (10), and (11) be from (12a). Suppose that for the resolvent, $Z(t,s)$, we have $Z(t,0)$ is bounded and that (*) holds. Thus, each solution $x(t)$ of (12) is bounded for every bounded continuous $a'(t)$ and so the solution $x(t)$ of (10) is bounded for every $a(t)$ with $a'(t)$ bounded and continuous. This means that $Z_s(t,s)$ generates an approximate identity on the vector space of functions $\phi : [0,\infty) \to \Re$ for which $\phi'(t)$ is bounded and continuous. If, in addition, every solution of (12) tends to zero for every function $a'(t)$ which tends to zero as $t \to \infty$, then $Z_s(t,s)$ generates an asymptotic identity on the vector space of functions $\phi : [0,\infty) \to \Re$ for which $\phi'(t) \to 0$.

There will be a parallel result, Proposition 5, for the case in which (6) is obtained from (1) in a different way than simple differentiation.

3. Some examples. Our first example is of the type which we called "unremarkable" in Remark 1. It is a very old and well-known result with many variations in the literature. Corduneanu [5 ; p. 129] adds condition to show that the solution tends to zero exponentially.

Example 1. If $a(t)$ is bounded and continuous and if there is a constant $\alpha < 1$ with

$$\sup_{t\geq 0} \int_0^t |C(t,s)|ds \leq \alpha \tag{13}$$

then the unique solution $x(t)$ of (1) is bounded, the resolvent $R(t,s)$ in (2) satisfies

$$\sup_{t\geq 0} \int_0^t |R(t,s)|ds < \infty, \tag{14}$$

and $R(t,s)$ generates an approximate identity on the vector space BC. Here, the solution of (1) is given by (3) so that when (13) holds then $x(t)$ is bounded for each bounded $a(t)$;

thus, in (4) when (13) holds then we have $(P\phi)(t)$ bounded for every bounded continuous ϕ.

Proof. Let $(M, \|\cdot\|)$ be the complete metric space of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum metric, $\phi(0) = a(0)$. Define $Q : M \to M$ by $\phi \in M$ implies

$$(Q\phi)(t) = a(t) - \int_0^t C(t,s)\phi(s)ds.$$

Clearly, Q is a contraction by (13) with unique fixed point $x \in M$, so that x is a bounded function. Also, (14) holds by Perron's result, as discussed in the proof of Proposition 1. Finally, Definition 1 is satisfied using (4) and (14) on the space BC.

Our second example is more pointed in that now we will have the right-hand-side of the mapping Q tending to zero as $t \to \infty$ so that $x(t) \to 0$ as $t \to \infty$; using this in (3) we see that as $t \to \infty$ then $\int_0^t R(t,s)a(s)ds$ converges to $a(t)$ showing that R generates an asymptotic identity. This will become more pronounced in later examples as we let $a(t)$ become unbounded.

Example 2. If $r : [0,\infty) \to (0,1]$ with $r(t) \downarrow 0$, with

(15) $$\sup_{t\geq 0} \int_0^t |C(t,s)r(s)/r(t)|ds \leq \alpha < 1,$$

and with

(*) $$|a(t)| \leq kr(t)$$

for some $k > 0$, then the unique solution $x(t)$ of (1) also satisfies $|x(t)| \leq k^* r(t)$ for some $k^* > 0$. Moreover, the resolvent $R(t,s)$ in (2) generates an asymptotic identity on the space of functions $\phi : [0,\infty) \to \Re$ with $\sup_{t\geq 0} \left|\frac{\phi(t)}{r(t)}\right| < \infty$.

Proof. The proof is based on a weighted norm. Let $(M, |\cdot|_r)$ denote the Banach space of continuous functions $\phi : [0,\infty) \to R$ with the property that

$$|\phi|_r := \sup_{t\geq 0} \frac{|\phi(t)|}{r(t)} < \infty.$$

Define $Q : M \to M$ by $\phi \in M$ implies that

$$(Q\phi)(t) = a(t) - \int_0^t C(t,s)\phi(s)ds.$$

We have

$$
\begin{aligned}
|(Q\phi)(t)/r(t)| &\leq |a(t)/r(t)| + \int_0^t |C(t,s)r(s)/r(t)||\phi(s)/r(s)|ds \\
&\leq k + |\phi|_r \int_0^t |C(t,s)r(s)/r(t)|ds \\
&\leq k + \alpha|\phi|_r
\end{aligned}
$$

so $Q\phi \in M$. To see that P is a contraction in that norm we have immediately that

$$|[(Q\phi)(t) - (Q\eta)(t)]/r(t)| \leq \alpha|\phi - \eta|_r$$

for $\phi, \eta \in M$. Hence, there is a fixed point in M and so it has the required properties. As $x(t)$ in (3) tends to zero for $a(t)$ satisfying (*), so does $P\phi$ in (4). This completes the proof.

A version of the Corduneanu result [5; p. 129] on exponential decay may be obtained by asking for positive numbers d, γ, λ with $\gamma < \lambda - d,\ \lambda > d,\ |a(t)| \leq e^{-dt}$, and $|C(t,s)| \leq \gamma e^{-\lambda(t-s)}$.

Our third example lets $\int_0^t a(s)ds$ be unbounded and still a resolvent, $Z_s(t,s)$, satisfies

$$\left| \int_0^t a(s)ds - \int_0^t Z_s(t,s) \int_0^s a(u)ds \right| \leq J$$

for some constant J so that $Z_s(t,s)$ generates an approximate identity on the space of functions ϕ with $|\phi(t)| \leq k(t+1)$ where k depends on ϕ and may be arbitarily large. It is very common to differentiate (1) and write

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds \tag{12}$$

from which results for (1) can be derived. We do this later, but if $a'(t)$ does not exist, then a parallel process is still possible which complements that work in more than one way. We illustrate the new way here.

If $C(t,s)$ contains an additive function of t, perhaps a constant, then (13) must often fail. But such problems can be effectively removed if $C_s(t,s)$ exists. In that case we write (1) as

$$\begin{aligned}
x(t) &= a(t) - \int_0^t C(t,s)x(s)ds \\
&= a(t) - C(t,s)\int_0^s x(u)du|_0^t + \int_0^t C_2(t,s)\int_0^s x(u)duds \\
&= a(t) - C(t,t)\int_0^t x(u)du + \int_0^t C_2(t,s)\int_0^s x(u)duds.
\end{aligned}$$

If we let $y(t) = \int_0^t x(u)du$ our equation becomes

$$y'(t) = a(t) - C(t,t)y(t) + \int_0^t C_2(t,s)y(s)ds. \tag{16}$$

There is good independent reason for studying $\int_0^t x(u)du$, as is discussed by Feller [4] concerning the renewal equation. The resolvent equation for (16) is (7) which becomes

$$Z_s(t,s) = Z(t,s)C(s,s) + \int_s^t Z(t,u)C_2(u,s)du,\ Z(t,t) = 1 \tag{16a}$$

with resolvent $Z(t,s)$ and with y satisfying the variation-of-parameters formula (8) which becomes

$$y(t) = Z(t,0)y(0) + \int_0^t Z(t,s)a(s)ds \tag{16b}$$

and by (10) (remembering that $y(0) = 0$ since $y(t) = \int_0^t x(u)du$) yields

$$y(t) = \int_0^t a(s)ds - \int_0^t Z_s(t,s)\int_0^s a(u)duds. \tag{16c}$$

In this example we will see $y(t)$ bounded even when $\int_0^t a(s)ds$ is unbounded, meaning that $Z_s(t,s)$ generates an approximate identity on a space of unbounded functions. In the next example we will get an asymptotic identity.

We now formulate the counterpart to Proposition 4 for (16).

Proposition 5. Let $Z(t,s)$ be the solution of (16a). Every solution $y(t) = \int_0^t x(u)du$ of (16) is bounded for every bounded continuous $a(t)$ if and only if

$$\sup_{t\geq 0}\int_0^t |Z(t,s)|ds < \infty. \tag{**}$$

Moreover, if (**) holds then $Z_s(t,s)$ generates an approximate identity on the vector space of continuous functions $\phi : [0,\infty) \to R$ for which $\phi'(t)$ is bounded. Finally, if in addition, $y(t) \to 0$ for every $a(t)$ which tends to zero, then $Z_s(t,s)$ generates an asymptotic identity on the vector space of continuous functions $\phi : [0,\infty) \to \Re$ for which $\phi'(t) \to 0$.

Proof. The proof of the first part is like that of Proposition 1 using (16b) with $y(0) = 0$. The next part, $Z_s(t,s)$ generates an approximate identity, follows from (16c) when we recall that $y(t)$ is bounded for bounded $a(t)$. The last conclusion follows in the same way.

Next, recall that $y(0) = 0$ and write

$$y(t) = \int_0^t e^{\int_u^t -C(s,s)ds} a(u)du + \int_0^t e^{\int_u^t -C(s,s)ds} \int_0^u C_2(u,s)y(s)dsdu. \tag{17}$$

Example 3. Suppose that $a(t)$ is bounded, that $\int_0^t e^{\int_u^t -C(s,s)ds}du$ is bounded, and that there exists $\alpha < 1$ with

$$\sup_{t\geq 0} \int_0^t e^{\int_u^t -C(s,s)ds} \int_0^u |C_2(u,s)|dsdu \leq \alpha. \tag{18}$$

Then for $x(t)$ the solution of (1) we have $\int_0^t x(s)ds$ bounded. Thus, using Proposition 5 we see that $Z_s(t,s)$ of (16a) generates an approximate identity on the space of function ϕ such that ϕ' is bounded.

Proof. Use (17) and the supremum norm to define a mapping $Q : BC \to BC$ by $\phi \in BC$ implies that

$$(Q\phi)(t) = \int_0^t e^{\int_u^t -C(s,s)ds} a(u)du + \int_0^t e^{\int_u^t -C(s,s)ds} \int_0^u C_2(u,s)\phi(s)dsdu.$$

If $\phi \in BC$, so is $Q\phi$ by assumption and (18). Also, Q is a contraction by (18). Hence, $y(t) = \int_0^t x(s)ds$ is bounded.

Example 4. Let the conditions of Example 3 hold. Suppose that for each $T > 0$

$$\int_0^T e^{\int_u^t -C(s,s)ds} \int_0^u |C_2(u,s)|dsdu \to 0$$

and

$$\int_0^T e^{\int_u^t -C(s,s)ds} du \to 0$$

as $t \to \infty$. If, in addition, $a(t) \to 0$ as $t \to \infty$ then every solution of (16) satisfies $y(t) = \int_0^t x(u)du \to 0$ as $t \to \infty$. Also, for the $Z(t,s)$ of (16a) we have $\int_0^t Z(t,s)\phi(s)ds \to 0$ as $t \to \infty$ for every continuous function ϕ which tends to zero as $t \to \infty$.

Proof. Use the mapping from the proof of Example 3, but replace BC by the complete metric space of continuous $\phi : [0,\infty) \to \Re$ such that $\phi(t) \to 0$ as $t \to \infty$. Use the assumptions and essentially the classical proof that the convolution of an L^1-function with a function tending to zero does, itself, tend to zero. This will show that $(Q\phi)(t) \to 0$ when $\phi(t) \to 0$. The mapping is a contraction as before with unique solution $y(t) = \int_0^t x(u)du \to 0$ as $t \to \infty$. The last conclusion is immediate.

We now turn to the more conventional technique of differentiation of (1) and consider

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_1(t,s)x(s)ds \tag{19}$$

under the assumption of continuity on a, C, C_1. The resolvent equation for (19) is obtained from (7) and is

$$Z_s(t,s) = Z(t,s)C(s,s) + \int_0^t Z(t,u)C_1(u,s)du,\ Z(t,t) = 1, \tag{19a}$$

while from (8) and (10) the variation-of-parameters formulae are

$$\begin{aligned} x(t) &= Z(t,0)x(0) + \int_0^t Z(t,s)a'(s)ds \\ &= Z(t,0)[x(0) - a(0)] + a(t) - \int_0^t Z_s(t,s)a(s)ds. \end{aligned} \tag{19b}$$

Suppose that

$$\int_0^t C(s,s)ds \to \infty$$

as $t \to \infty$ and use the variation-of-parameters formula to write (19) as

$$\begin{aligned} x(t) &= x(0)e^{\int_0^t -C(s,s)ds} + \int_0^t e^{\int_u^t -C(s,s)ds} a'(u)du \\ &\quad - \int_0^t e^{\int_u^t -C(s,s)ds} \int_0^u C_1(u,s)x(s)dsdu. \end{aligned} \tag{20}$$

Remark 4. Thus, our equation is again an integral equation and it will require the integral of the second coordinate of $C_1(t,s)$ to be small; C_1 is cleansed of any additive constants or additive functions of t which might have conflicted with the hypotheses of Proposition 1. But, perhaps more to the point, any such contants are now transferred to the exponential which will help the subsequent contraction condition.

In order to make (19b) and (8) more symmetric we begin with a proposition showing $Z(t,0)$ bounded.

Proposition 6. Suppose there is an $\alpha < 1$ with

$$\sup_{t\geq 0}\int_0^t e^{\int_u^t -C(s,s)ds}\int_0^u |C_1(u,s)|dsdu \leq \alpha. \tag{21}$$

Then $Z(t,0)$ in (19b) is bounded. Moreover:

(a) Every solution of (1) is bounded for every $a(t)$ with $a'(t)$ bounded and continuous if and only if

$$\sup_{t\geq 10}\int_0^t |Z(t,s)|ds < \infty.$$

(b) Every solution of (1) is bounded for every bounded and continuous $a(t)$ if and only if

$$\sup_{t\geq 0}\int_0^t |Z_s(t,s)|ds < \infty.$$

Proof. In (20) let $a'(t) = 0$ and use (20) to define a mapping $Q : BC \to BC$. It is a contraction by (21) with fixed point $x(t) = Z(t,0)x(0)$ which is bounded for each $x(0)$. Parts (a) and (b) now follow exactly as in the proof of Proposition 1 using (19b).

Example 5. Suppose that (21) holds and that

$$\int_0^t e^{\int_u^t -C(s,s)ds}du \text{ is bounded for } t \geq 0. \tag{***}$$

Then the unique solution of (1) is bounded for each continuous function $a(t)$ with $a'(t)$ bounded and continuous; thus, from (19b) we see that $Z_s(t,s)$ generates an approximate identity on the space of functions $\phi : [0,\infty) \to \Re$ for which $\phi'(t)$ is bounded, while $Z(t,s)$ generates an approximate identity on BC.

Proof. Use (20), as before, to define a mapping $Q : BC \to BC$ to prove that the solution of (19) (and, hence, of (1)) is bounded for every bounded and continuous $a'(t)$. The last conclusion follows from (19b) since $x(t)$ is bounded when $a'(t)$ is bounded.

Example 6. Let the conditions of Example 5 hold, $a'(t) \to 0$ as $t \to \infty$, and suppose there is a constant $\lambda > 0$ with $-C(t,t) \le -\lambda$. Suppose also that there is a continuous function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi \in L^1[0,\infty)$ and $\Phi(u-s) \ge |C_1(u,s)|$. Then the solution $x(t)$ of (1) tends to zero as $t \to \infty$ and so does $Z(t,0)$. Finally, under these additional conditions the conclusions of Example 5 change to asymptotic identity.

Proof. In our mapping we add to the mapping set BC the condition that $\phi(t) \to 0$ as $t \to \infty$. Then notice that

$$\int_0^u |C_1(u,s)\phi(s)|ds \le \int_0^u |\Phi(u-s)|\phi(s)|ds,$$

is the convolution of an L^1-function with a function tending to zero so it tends to zero. Then

$$\int_0^t e^{\int_u^t -C(s,s)ds} \int_0^u |C_1(u,s)\phi(s)|ds \le \int_0^t e^{-\lambda(t-u)} \int_0^u |C_1(u,s)\phi(s)|ds$$

which tends to zero for the same reason. Finally,

$$\int_0^t e^{\int_u^t -C(s,s)ds}|a'(u)|du \le \int_0^t e^{-\lambda(t-u)}|a'(u)|du$$

which tends to zero. This will then show that the modified BC will be mapped into itself.

We conclude this section with an example containing several thought provoking relations.

Example 7. Let $g(x)$ be positive, bounded, and locally Lipschitz on $(-\infty,\infty)$ into the reals and consider the integral equation

$$x(t) = t + \int_0^t g(x(s))ds - \int_0^t C(t,s)x(s)ds$$

where C satisfies (***) and (21). Standard existence theory will yield a unique solution on $[0,\infty)$ so it is possible to define a unique continuous function

$$a(t) = t + \int_0^t g(x(s))ds$$

with $a'(t)$ being bounded and $a(t) \ge t$. The conditions of Example 5 are satisfied and we have then a list of properties.

(a) The solution $x(t)$ is bounded. This means that $a(t)$ and $\int_0^t C(t,s)x(s)ds$ differ by at most a bounded function. Recall that $a(t) \geq t$.

(b) The variation of parameters formula for the solution is

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds$$

where R is the resolvent from (2). That integral differs from $a(t)$ by at most a bounded function and, again, $a(t) \geq t$.

(c) From the second equation in (19b) we have

$$x(t) = t + \int_0^t g(x(s))ds - \int_0^t Z_s(t,s)[s + \int_0^s g(x(u))du]ds$$

and that quantity is bounded since Z_s generates an approximate identity on functions with bounded derivative. Again, $a(t) \geq t$ and $a(t) - \int_0^t Z_s(t,s)a(s)ds$ is bounded.

(d) From the first equation in (19b) we have

$$x(t) = Z(t,0)x(0) + \int_0^t Z(t,s)[1 + g(x(s))]ds$$

where the last term is bounded. That integral differs from $1+g(x(t))$ by at most a bounded function, while $Z(t,0)$ is bounded.

The resolvents may be arbitrarily complicated, as the work of Ritt shows, but their operation on a forcing function is almost an identity map. The resolvent is, indeed, remarkable.

4. Conclusions. We have used simple contraction mappings to obtain the basic result that various resolvents have bounded integrals, thereby establishing necessary and sufficient conditions for boundedness of solutions. These boundedness results have then led us to understand that, however complicated the resolvent is, when it is applied in a variation of parameters formula the net effect is that it closely approximates the identity map and it does so on an entire vector space.

In a recent monograph [4] we have examined approximately 100 classical and modern problems in functional differential equations, mainly by means of the simplest contractions, obtaining stability results which we contrast with results using Liapunov's direct method.

In the same way, a paper parallel to this one is in preparation which treats integral equations and their integrodifferential equations counterparts using Liapunov functionals and establishing conditions under which solutions are bounded and the resolvents generate approximate identities, asymptotic identities, and L^p-identities. The great contrast lies in the fact that here we focus on integration of the second component of $C(t, s)$, while Liapunov functionals focus more on integration of the first component. This is also a contrast seen in the techniques of Razumikhin versus Liapunov.

These identity properties are fundamental to understanding integral equations and the methods are fully accessible to second or third year university students. Our continuing thesis is that fixed point theory yields simple and concrete answers to a great many of our questions in differential and integral equations without much of the drugery seen in so many other methods. If we integrate it into our courses early on, it will do much to advance understanding and give new life to one of our most useful, important, and beautiful subjects.

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