QUALITATIVE PROPERTIES OF SOLUTIONS OF INTEGRAL EQUATIONS

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Abstract. In this paper we study a linear integral equation $x(t) = a(t) - \int_0^t C(t,s)x(s)ds$ in which the kernel fails to satisfy standard conditions yielding qualitative properties of solutions. Thus, we begin by following the standard idea of differentiation to obtain $x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds$. The investigation so frequently depends on $x'(t) + C(t,t)x(t) = 0$ being uniformly asymptotically stable. When that property fails to hold, then the investigator must turn to ad hoc methods. We show that there is a way out of this dilemma. We note that if $C(t,t)$ is bounded, then for $k > 0$ the equation resulting from $x' + kx$ will have a uniformly asymptotically stable ODE part and the remainder can often be shown to be a harmless perturbation. The study is also continued to the pair $x'' + kx'$.

1. Introduction

This paper represents the development of an idea, briefly introduced in the appendix of Burton [2], concerning the possibility of determining qualitative properties of solutions of a scalar equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds \tag{1.1}$$

(together with nonlinear perturbations) by forming $x' + kx$ when it is possible to write

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds. \tag{1.2}$$

Here, k is a positive constant. An example was given, and is continued here, showing that there can be surprising large gains in two essentially different ways. It was also mentioned in [2] that more might be gained by studying $x'' + kx'$, but nothing further was developed.

The point of this paper is that, while x in (1.1) is uniquely determined by the function a and kernel C, there is a one-parameter family of functions $C(t,s,k)$, each member of which also generates that same unique solution x. As shall be demonstrated, those family members can differ so greatly from one another that one of them can fail to satisfy standard theorems about the properties of x, while a different member fits exactly.

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The idea is based on a well-known Liapunov functional for (1.1) when conditions K_1, are met, and another Liapunov functional for (1.2) when conditions K_2, are met. Is it possible that when we form $x' + kx$, then $K_1 \cup K_2$ are so greatly reduced that a new Liapunov functional can be constructed?

In the next section we lay the foundation for the theory showing that forming $x' + kx$ will always reduce the needed condition and can produce a Liapunov function(al) when K_1 and K_2 fail. We establish two inequalities, (2.5) and (2.6), which then are the main hypotheses of two theorems in subsequent sections. Along with those theorems are detailed examples showing precisely how we mean to proceed.

In the last section we address the formation of $x'' + kx'$. While we do establish a set of inequalities parallel to (2.5) and (2.6) for this case, we do not show that improvements are always accomplished. So much less is known about the second order case, and we do introduce a new idea dealing with it in a very concise and effective way. Again, we give two theorems concerning the qualitative properties and we give very detailed examples illustrating how we expect to proceed.

2. The Foundation

We focus here on two elementary lines of investigation into boundedness of solutions of

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds, \tag{2.1}$$

and we provide a third line which we can show to be less exacting than either of the others in the sense that its conditions can hold when neither of the first mentioned conditions hold.

Throughout the paper we are assuming $a : [0, \infty) \to \mathcal{R}$ and $C : [0, \infty) \times [0, \infty) \to \mathcal{R}$ are continuous. We define

$$\|a\| = \sup_{t \geq 0} |a(t)|.$$

So, by $\|a\| < \infty$ we simply mean that a is bounded on $[0, \infty)$.

Perhaps the oldest theorem in the theory of integral equations can be stated as follows. It is proved by a simple contraction mapping argument, as is seen, for example, in Burton [2; p. 54].

Theorem 2.1. *Suppose that $\|a\| < \infty$ and that there is an α, $0 \leq \alpha < 1$, with*

$$\sup_{t \geq 0} \int_0^t |C(t,s)|ds \leq \alpha,$$

then the solution x of (2.1) is bounded on $[0, \infty)$.

On the other hand, if a' and C_t are continuous, we can write

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds, \tag{2.2}$$

and the following is a standard theorem for boundedness. It is proved using a Razumikhin function, as may be seen in Burton [2; p. 85].

Theorem 2.2. *Suppose that $\|a'\| < \infty$ and that there is a $d > 0$ with*

$$C(t,t) - \int_0^t |C_t(t,s)|ds \geq d,$$

then all solutions of (2.2), and hence the solution x of (2.1), are bounded on $[0,\infty)$.

Suppose that the d of Theorem 2.2 exists. We then write the conditions of Theorems 2.1 and 2.2 as

$$\alpha \geq \int_0^t |C(t,s)|ds, \tag{2.3}$$

and

$$C(t,t) \geq \int_0^t |C_t(t,s)|ds + d. \tag{2.4}$$

If $0 \leq k$ and $0 \leq j$, then

$$\begin{aligned} k\alpha + jC(t,t) &\geq \int_0^t |kC(t,s)|ds + \int_0^t |jC_t(t,s)|ds + jd \\ &= \Big[\int_0^t |kC(t,s)|ds + \int_0^t |jC_t(t,s)|ds\Big] + jd \\ &\geq \int_0^t |kC(t,s) + jC_t(t,s)|ds + jd. \end{aligned}$$

In other words, neither (2.3) nor (2.4) needs to hold in order for

$$k\alpha + jC(t,t) - \int_0^t |kC(t,s) + jC_t(t,s)|ds \geq jd \tag{2.5}$$

to be true.

On the other hand, if (2.3) holds, then (2.5) holds for $k = 1$ *and* $j = 0$. If (2.4) holds, then (2.5) holds for $j = 1$ *and* $k = 0$.

Theorem 2.3. *Condition (2.5) holds if either (2.3) or (2.4) hold, but (2.5) can hold when both of the others fail.*

We remark that Theorem 2.1 is a natural result from a more fundamental level. We can show that $\int_0^t |C(t,s)|ds \leq \alpha$ implies the resolvent inequality

$$\int_0^t |R(t,s)|ds \leq \frac{\alpha}{1-\alpha}, \ 0 \leq t < \infty,$$

so

$$|x(t)| \leq |a(t)| + \|a\| \int_0^t |R(t,s)|ds \leq |a(t)| + \|a\| \frac{\alpha}{1-\alpha}.$$

We introduce the resolvent, R, and formally state the above result later in this section. Next, we turn to L^1 problems and, again, have two results. These can be proved by means of Liapunov functionals, as seen in [2; pp. 60, 78].

Theorem 2.4. *Suppose there exists* $\beta < 1$ *with* $\int_0^\infty |C(u+t,t)|du \le \beta$ *and* $a \in L^1[0,\infty)$. *Then* $x \in L^1[0,\infty)$.

Theorem 2.5. *If there exists* $d > 0$ *with* $C(t,t) - \int_0^\infty |C_t(u+t,t)|du \ge d$, *then* $a \in L^1[0,\infty)$ *implies* $x \in L^1[0,\infty)$.

We will now see that these conditions are stronger than a new one introduced in (2.6). Let $k \ge 0$ and $j \ge 0$ so that

$$\begin{aligned} k\beta + jC(t,t) &\ge \int_0^\infty k|C(u+t,t)|du + \int_0^\infty j|C_t(u+t,t)|du + jd \\ &= \Big[\int_0^\infty k|C(u+t,t)|du + \int_0^\infty j|C_t(u+t,t)|du\Big] + jd \\ &\ge \int_0^\infty |kC(u+t,t) + jC_t(u+t,t)|du + jd. \end{aligned}$$

If $k = 0$, we have Theorem 2.5, while $j = 0$ yields Theorem 2.4.

Theorem 2.6. *If the conditions of either Theorem 2.4 or 2.5 hold, then*

$$k\beta + jC(t,t) \ge \int_0^\infty |kC(u+t,t) + jC_t(u+t,t)|du + jd \tag{2.6}$$

holds. However, (2.6) can hold when both of the former fail.

Remark. There is more to recommend (2.5) and (2.6) than the theorems state. In 1928 Volterra [5] noted that common kernels satisfied

$$C(t,s) \ge 0,\ C_s(t,s) \ge 0,\ C_{st}(t,s) \le 0,\ C_t(t,s) \le 0.$$

If even the first and last hold, then in (2.5) and (2.6) β and $C(t,t)$ are positive and, hence, they add, while $C(t,s)$ and $C_t(t,s)$ subtract. Both properties tend to secure the inequalities.

Conditions (2.5) and (2.6) will be our basic assumptions for the first half of the paper. We have offered Theorems 2.1 and 2.4 as springboards for our present endeavor, and we need to show just how fundamental the assumptions in those theorems really are.

Along with integral equation (2.1), there is a (resolvent) function $R : [0,\infty) \times [0,\infty) \to \mathcal{R}$, defined by the kernel C, which solves the resolvent equation

$$R(t,s) = C(t,s) - \int_s^t C(t,u)R(u,s)du. \tag{2.7}$$

Likewise, there is a variation of parameters formula for which the unique solution x of (2.1) can be expressed as

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{2.8}$$

There is also the nonlinear equation

$$z(t) = a(t) - \int_0^t C(t,s)[z(s) - G(s,z(s))]ds, \tag{2.9}$$

which can be decomposed into (2.1) and then

$$z(t) = x(t) - \int_0^t R(t,s)G(s,z(s))ds \tag{2.10}$$

so that R is central. Moreover, Theorems 2.1 and 2.4 could have been expressed as follows.

Theorem 2.7. *If there exists $\alpha < 1$ with $\int_0^t |C(t,s)|ds \leq \alpha$, then*

$$\int_0^t |R(t,s)|ds \leq \frac{\alpha}{1-\alpha},\ 0 \leq t < \infty. \tag{2.11}$$

Theorem 2.8. *If there exists $\beta < 1$ with $\int_s^t |C(u,s)|du \leq \beta,\ 0 \leq s \leq t < \infty$, then*

$$\int_s^t |R(u,s)|du \leq \frac{\beta}{1-\beta},\ 0 \leq s \leq t < \infty. \tag{2.12}$$

Note that $\int_s^t |C(u,s)|du \leq \beta$ is equivalent to

$$\int_0^\infty |C(u+t,t)|du \leq \beta. \tag{2.13}$$

For a more detailed discussion of results related to the resolvent R, we refer to [2] and [4]. As R is so fundamental we would like to parlay (2.5) into a form of (2.11), but all we can do is parlay it into R bounded, a poor substitute. Instead, we can parlay (2.6) seamlessly into a form of (2.12). To that end we team Theorem 2.8 with

Theorem 2.9. *Suppose there is a $d > 0$ with*

$$C(t,t) - \int_0^\infty |C_t(u+t,t)|du \geq d.$$

Then there is a $\lambda > 0$ with

$$\int_s^t |R(u,s)|du \leq \frac{1}{\lambda}\Big[|C(s,s)| + \int_s^t |C_t(u,s)|du\Big].$$

Notice that the conditions in Theorems 2.8 and 2.9 are the same as in Theorems 2.4 and 2.5, so the conclusion of Theorem 2.6 will follow here as well.

To alleviate many of the difficulties, we present two simple ideas in this paper which seem to be new. These ideas are put to test in Sections 3 and 4 by examples in which (2.3), (2.4), and the conditions of Theorems 2.4 and 2.5 fail for the original kernel $C(t,s)$. The first idea is to combine x' and x to determine a one-parameter family of new kernels $C(t,s,k)$ by examining $x' + kx$. For instances in which this approach does not help, we frequently find it useful to differentiate (1.1) twice owing to the problems of failure of (2.3) and the conditions of Theorem 2.4 for finding an appropriate α or $\beta > 0$. In the case of differentiating twice, we need an idea which will allow us to handle a second order integrodifferential equation in a compact way. This idea is presented in Section 4 and involves parlaying a second order equation into two first order equations.

3. $x' + kx$ and Ramifications

In this section we begin our exploration of situations for which inequalities (2.3) and (2.4) fail. Along these lines, the equation

$$x(t) = a(t) - \int_0^t (1+t-s)^{-2} x(s)ds \tag{3.1}$$

will serve as a tour guide through this section. We note that, for $C(t,s) = (1+t-s)^{-2}$,

$$\sup_{t\geq 0} \int_0^t |C(t,s)|ds = \sup_{t\geq 0} \int_0^t (1+t-s)^{-2} ds = 1.$$

and

$$\sup_{t\geq 0} \int_0^\infty |C(u+t,t)|du = \int_0^\infty (1+u)^{-2} du = 1.$$

That is, neither (2.3) nor the hypothesis of Theorem 2.4 holds. In other words neither the contraction mapping approach nor the Liapunov functional approach which we have cited earlier will prove fruitful without additional efforts. This leads us to follow Miller [4] to differentiate (1.1) to obtain

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds \tag{3.2}$$

which, for the particular case (3.1), yields

$$x'(t) = a'(t) - x(t) + \int_0^t 2(1+t-s)^{-3} x(s)ds. \tag{3.3}$$

Our first thought might be to try our hand with a Razumikhin approach using the function $V(t) = |x(t)|$. Choosing any t for which

$$V(t) = |x(t)| = \sup_{0\leq s\leq t} |x(s)|,$$

we have

$$V'(t) \leq |a'(t)| - |x(t)|\Big(1 - \int_0^t 2(1+t-s)^{-3} ds\Big). \tag{3.4}$$

Since $\sup_{t\geq 0} \int_0^t 2(1+t-s)^{-3} ds = 1$, at this stage we are unable to extract information related to boundedness or L^p properties of solutions of (3.3) or of the unique solution of (3.1).

This brings us to the first of our two new strategies. The general idea is to combine x and x' in (3.1) and (3.3) to compose $x' + kx$ and obtain a one-parameter family of equations

$$x'(t)+kx(t) = a'(t)+ka(t)-x(t)-\int_0^t \Big[k(1+t-s)^{-2}-2(1+t-s)^{-3}\Big]x(s)ds. \tag{3.5}$$

or

$$(3.6)\quad x'(t) = a'(t) + ka(t) - (k+1)x(t) - \int_0^t \Big[k(1+t-s)^{-2} - 2(1+t-s)^{-3}\Big]x(s)ds.$$

Two things have happened with this construction. First, we have secured a uniformly asymptotically stable ODE $x'+(k+1)x = 0$. Next, the kernel $k(1+t-s)^{-2}$ is actually diminished by $2(1+t-s)^{-3}$. It is a significant advantage, and it is not at all unusual.

Inspired by the integrand, we set $k = 2$ so that (3.6) becomes

$$(3.7)\quad x'(t) = a'(t) + 2a(t) - 3x(t) - 2\int_0^t \Big[(1+t-s)^{-2} - (1+t-s)^{-3}\Big]x(s)ds.$$

This time, by letting $V(t) = |x(t)|$ and examining t for which

$$\sup_{0\le s\le t} V(s) = \sup_{0\le s\le t} |x(s)| = |x(t)| = V(t),$$

we have the most pleasant inequality

$$(3.8)\quad \begin{aligned} |x(t)|' = V'(t) &\le |a'(t) + 2a(t)| \\ &- |x(t)|\Big(3 - 2\int_0^t \Big[(1+t-s)^{-2} - (1+t-s)^{-3}\Big]ds\Big) \\ &\le |a'(t) + 2a(t)| - 2|x(t)|, \end{aligned}$$

from which we conclude that all solutions of (3.7) and the unique solution of (3.1) are bounded on $[0,\infty)$ whenever $|a'(t) + 2a(t)|$ is bounded there.

Returning now to (2.5), we assume that j is not zero (the trivial case), divide it out, and rename $k/j = k$.

The above calculations direct us to the following more general result. For the reader unfamiliar with the Razumikhin approach sketched above, the same approach with a few more details is provided in the proof below.

Theorem 3.1. *For equation (1.1), suppose $C(t,s)$ and $C_t(t,s)$ are continuous on $[0,\infty)\times[0,\infty)$ and suppose there exist $k,\gamma,\eta > 0$ such that, for all $t \ge 0$,*

$$(3.9)\quad |a'(t) + ka(t)| \le \gamma$$

and

$$(3.10)\quad k + C(t,t) - \int_0^t |kC(t,s) + C_t(t,s)|ds \ge \eta.$$

Then the unique solution of (1.1) and all solutions of

$$(3.11)\quad x'(t) + kx(t) = a'(t) + ka(t) - C(t,t)x(t) - \int_0^t \Big[kC(t,s) + C_t(t,s)\Big]x(s)ds$$

are bounded on $[0, \infty)$.

Proof. First, notice that the unique solution of (1.1) is also a solution of (3.11) with initial condition $x(0) = a(0)$. As in the previous calculations of (3.1), we set $V(t) = |x(t)|$. If a particular solution x of (3.11) were unbounded, then there must exist a $t^* > 0$ with

$$|x(t^*)| > \frac{\gamma}{\eta} \text{ and } |x(s)| \le |x(t^*)|,\ 0 \le s \le t^*.$$

Clearly for such t^*, $|x(t^*)|' \ge 0$. On the other hand,

$$V'(t^*) = |x(t^*)|' \le |a'(t^*)+ka(t^*)| - (k+C(t^*,t^*))|x(t^*)| + \int_0^{t^*} |kC(t^*,s)+C_t(t^*,s)||x(s)|ds$$
$$\le \gamma - |x(t^*)|\Big((k + C(t^*,t^*)) - \int_0^{t^*} |kC(t^*,s) + C_t(t^*,s)|ds\Big)$$
$$< \gamma - \frac{\gamma}{\eta}\eta = 0.$$

That is, $V'(t^*) < 0$, which is a contradiction.

□

Next, we move to a Liapunov functional approach which can often be used to establish L^p properties of solutions.

Theorem 3.2. *Suppose* $C(t,s)$, *and* $C_t(t,s)$ *are continuous on* $[0,\infty) \times [0,\infty)$ *and*

$$a \in L^1[0,\infty) \text{ and } a' \in L^1[0,\infty), \tag{3.12}$$

and further suppose that there exist $k, \delta > 0$ *such that*

$$k + C(t,t) - \int_0^\infty |kC(u+t,t) + C_t(u+t,t)|du \ge \delta. \tag{3.13}$$

Then the unique solution of (1.1) and all solutions of (3.11) satisfy: $\|x\| < \infty$ *and* $x \in L^1[0,\infty)$.

Proof. Define the Liapunov functional

$$V(t) = |x(t)| + \int_0^t \int_{t-s}^\infty |kC(u+s,s) + C_t(u+s,s)|du|x(s)|ds.$$

Then

$$V' \le |a'(t) + ka(t)| - (k + C(t,t))|x(t)| + \int_0^t |kC(t,s) + C_t(t,s)||x(s)|ds$$
$$+ \int_0^\infty |kC(u+t,t) + C_t(u+t,t)|du|x(t)| - \int_0^t |kC(t,s) + C_t(t,s)||x(s)|ds$$
$$= |a'(t) + ka(t)| - |x(t)|\Big(k + C(t,t) - \int_0^\infty |kC(u+t,t) + C_t(u+t,t)|du\Big).$$

So, from (3.13)

$$V' \le |a'(t) + ka(t)| - |x(t)|\delta.$$

Integrating V' we have

$$0 \le |x(t)| \le V(t) \le V(0) + \int_0^t |a'(s) + ka(s)|ds - \delta \int_0^t |x(s)|ds. \tag{3.14}$$

Now,

$$\delta \int_0^t |x(s)|ds \le \int_0^t |a'(s) + ka(s)|ds$$

and (3.12) establish that $x \in L^1[0, \infty)$ and this, in turn, implies x is bounded from (3.14).

□

For the choice of $k = 2$ in our opening discussion of this section, we witnessed the calculations unfolding to yield interesting qualitative properties of solutions of (3.1) and associated families (3.6). A key ingredient, of course, was the choice of k itself. In the next result, we apply Theorems 3.1 and 3.2 to obtain a rather easy, but quite interesting, generalization related to (3.1) and (3.6).

Corollary 3.3. *Consider the (nonconvolution) equation*

$$x(t) = a(t) - \int_0^t \varphi(s)(1 + t - s)^{-2}x(s)ds \tag{3.15}$$

with

$$\varphi : [0, \infty) \to [-\xi, 1] \text{ continuous and } 0 \le \xi < 1. \tag{3.16}$$

If $a'(t) + 2a(t)$ is bounded on $[0, \infty)$, then so is the solution x of (3.15). If $a \in L^1[0, \infty)$ and $a' \in L^1[0, \infty)$, then the solution x of (3.15) is bounded. In addition, $x \in L^1[0, \infty)$.

Proof. (3.15) generates a corresponding family of equations

(3.17)
$$x'(t)+kx(t) = a'(t)+ka(t)-\varphi(t)x(t)-\int_0^t \Big[k\varphi(s)(1+t-s)^{-2}-2\varphi(s)(1+t-s)^{-3}x(s)ds\Big].$$

Once again, choose $k = 2$. It is straightforward to show that (3.10) in Theorem 3.1 and (3.13) in Theorem 3.2 hold, where $C(t, s) = \varphi(s)(1 + t - s)^{-2}$. The result follows.

□

It is worthwhile noting that $\varphi(s)$ is allowed to change sign here. In fact, a function as $\varphi(s) = \xi \sin s,\ 0 < \xi < 1$ pertains. It is natural to ask if $\inf_{s\ge 0} \varphi(s) = -1$ is allowable. Can we obtain results, for instance, for $\varphi(s) = \sin s$, or more generally, for $\varphi : [0, \infty) \to [-1, 1]$ with φ periodic and φ not identically minus one? This remains to be determined.

4. $x'' + kx'$ and Ramifications

Section 3 was devoted to examining $x' + kx$ to consider examples for which standard techniques do not succeed. In this section we continue the process by studying $x'' + kx'$ to explore qualitative properties of solutions of the integral equation (1.1), which for the convenience of the reader, we re-write here as

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds. \tag{4.1}$$

By way of setting the stage, we will (as before) differentiate freely early in the section and then provide formal conditions in statements of theorems that give us permission to do so. From (4.1) we derive

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds \tag{4.2}$$

and

$$x''(t) = a''(t) - C'(t,t)x(t) - C(t,t)x'(t) - C_t(t,t)x(t) - \int_0^t C_{tt}(t,s)x(s)ds \tag{4.3}$$

or

$$\begin{aligned} x''(t) + kx'(t) &= a''(t) + ka'(t) - C(t,t)x'(t) \\ &-[C'(t,t) + C_t(t,t) + kC(t,t)]x(t) - \int_0^t \Big[kC_t(t,s) + C_{tt}(t,s)\Big]x(s)ds, \end{aligned} \tag{4.4}$$

which can be written

$$\begin{aligned} x''(t) + [k + C(t,t)]x'(t) + [C'(t,t) + C_t(t,t) + kC(t,t)]x(t) \\ = a''(t) + ka'(t) - \int_0^t \Big[kC_t(t,s) + C_{tt}(t,s)\Big]x(s)ds. \end{aligned} \tag{4.5}$$

To be definite we begin with the case

$$C(t,t) \equiv c_1 \textit{ and } C_t(t,t) \equiv c_2 \textit{ are identically constant.} \tag{4.6}$$

For this case, $C'(t,t) \equiv 0$ and (4.5) reduces to

$$\begin{aligned} x''(t) + [k + c_1]x'(t) + [c_2 + kc_1]x(t) \\ = a''(t) + ka'(t) - \int_0^t \Big[kC_t(t,s) + C_{tt}(t,s)\Big]x(s)ds \end{aligned} \tag{4.7}$$

which we write as

$$x''(t) + Ax'(t) + Bx(t) = f(t, x(\cdot)), \tag{4.8}$$

where

$$A = k + c_1,\ B = c_2 + kc_1 \textit{ are constant} \tag{4.9}$$

and

$$f(t, x(\cdot)) = a''(t) + ka'(t) - \int_0^t \Big[kC_t(t,s) + C_{tt}(t,s)\Big]x(s)ds. \tag{4.10}$$

For reasons to be made clear shortly, we wish to re-write (4.8) as

$$x'' + Lx' + M(x' + Lx) = f(t, x(\cdot)), \tag{4.11}$$

where L and M are real and constant. For instance, we have

$$L + M = A \text{ and } ML = B, \tag{4.12}$$

so that

$$M = \frac{B}{L} \text{ and } L + \frac{B}{L} = A$$

which, in turn, implies $L^2 - AL + B = 0$ and

$$L = \frac{A \pm \sqrt{A^2 - 4B}}{2}. \tag{4.13}$$

We wish to determine k so that L is real. That is,

$$A^2 \geq 4B \tag{4.14}$$

which, from (4.8) and (4.9), is

$$(k + c_1)^2 \geq 4(kc_1 + c_2)$$

or

$$k^2 + 2kc_1 + c_1^2 \geq 4kc_1 + 4c_2$$

or

$$k^2 - 2kc_1 + c_1^2 = (k - c_1)^2 \geq 4c_2.$$

Clearly, for any fixed $c_1 \equiv C(t,t)$ and $c_2 \equiv C_t(t,t)$, such a k can be found. Once k is chosen, A and B are automatically determined, from which L can be provided from (4.13), which then leads to $M = \frac{B}{L}$.

In (4.11) we now let

$$y = x' + Lx \tag{4.15}$$

so that (4.11) becomes

$$y' + My = f(t, x(\cdot)). \tag{4.16}$$

Noting that $A, B > 0$ implies

$$L > 0 \text{ and } M > 0, \tag{4.17}$$

we are now in position to apply a variation of parameters formula. From (4.10), (4.11), (4.15), and (4.16), we have

$$x'(t) + Lx(t) = y(t) = y(0)e^{-Mt} + \int_0^t e^{-M(t-s)} f(s, x(\cdot))ds \tag{4.18}$$

$$= y(0)e^{-Mt} + \int_0^t e^{-M(t-s)}[a''(s) + ka'(s)]ds - \int_0^t e^{-M(t-u)} \int_0^u \Big[kC_t(u,s) + C_{tt}(u,s)\Big]x(s)dsdu.$$

Our opening theorem of this section applies Razumikhin techniques - an approach that essentially represents integration with respect to s. For easy reference, we recall that we are considering the following.

$$C(t,t) \equiv c_1, C_t(t,t) \equiv c_2, A = k + c_1, B = c_2 + kc_1, \tag{4.19}$$

$$\textit{where } k \textit{ must be chosen so that } A^2 \geq 4B \textit{ in order for}$$

$$L = \frac{A \pm \sqrt{A^2 - 4B}}{2} \textit{ to be real, and } M = \frac{B}{L}, L + M = A = k + c_1,$$

where (4.19) is just an accumulation of (4.6), (4.9), and (4.12)-(4.14).

Theorem 4.1. *Suppose $C(t,s), C_t(t,s)$, and $C_{tt}(t,s)$ are continuous on $[0,\infty) \times [0,\infty)$ and k has been chosen so that (4.19) holds. Further, suppose there exist $\alpha > 0, \beta > 0$ with $\alpha < L$ such that*

$$\sup_{t \geq 0} \int_0^t e^{-M(t-u)} \int_0^u \Big| kC_t(u,s) + C_{tt}(u,s) \Big| ds du \leq L - \alpha \tag{4.20}$$

and

$$|a''(t) + ka'(t)| \leq \beta \textit{ for all } t \geq 0. \tag{4.21}$$

Then the solution x of (4.1) is bounded on $[0,\infty)$.

Proof. For the solution x of (4.1), define a Razumikhin function by $V(t) = |x(t)|$. We wish to examine V' at any value of t for which

$$\sup_{0 \leq s \leq t} V(s) = \sup_{0 \leq s \leq t} |x(s)| = |x(t)| = V(t). \tag{4.22}$$

For any such t, we have from (4.18), (4.20), and (4.21) that

$$V'(t) \leq -L|x(t)| + |y(0)|e^{-Mt} + \beta \int_0^t e^{-M(t-s)} ds$$

$$+|x(t)| \int_0^t e^{-M(t-u)} \int_0^u \Big| kC_t(u,s) + C_{tt}(u,s) \Big| ds du$$

$$\leq -L|x(t)| + |y(0)|e^{-Mt} + \frac{\beta}{M}[1 - e^{-Mt}] + (L - \alpha)|x(t)|$$

or

$$V'(t) \leq -\alpha|x(t)| + K,$$

where $K = |y(0)| + \frac{\beta}{M}$. If x were unbounded on $[0,\infty)$, then there would be a $t^* > 0$ and $N > 1$ for which

$$\frac{NK}{\alpha} > |x(0)| = |a(0)|, \tag{4.23}$$

$$|x(t^*)| = \sup_{0 \leq s \leq t^*} |x(s)| = \frac{NK}{\alpha},$$

$$\textit{and } |x(s)| \leq |x(t^*)| \textit{ for } 0 \leq s \leq t^*.$$

In other words, t^* is the first value of t for which $|x(t^*)| = \frac{NK}{\alpha}$. For such a t^*,

$$V'(t^*) \geq 0.$$

On the other hand,

$$V'(t^*) \le -\alpha|x(t^*)| + K = \frac{-\alpha NK}{\alpha} + K = -K(N-1) < 0.$$

With this contradiction the proof is complete.

□

Remark 4.2. A simple but useful observation. Consider the case (4.19) with

$$C(t,t) \equiv c_1 = 0 \textit{ and } C_t(t,t) \equiv c_2 > 0. \tag{4.24}$$

Then (4.5) simplifies to

$$x''(t) + kx'(t) + c_2 x(t) = \tag{4.25}$$
$$a''(t) + ka'(t) - \int_0^t \Big[kC_t(t,s) + C_{tt}(t,s)\Big]x(s)ds = f(t, x(\cdot)).$$

Recall that our charge is to produce k so that (4.25) can be written

$$x'' + kx' + c_2 x = (x'' + Lx') + M(x' + Lx) = f(t, x(\cdot)). \tag{4.26}$$

By combining the relations in (4.19), we see that

$$L + M = A = k + c_1 = k \textit{ and } ML = B = c_2 + kc_1 = c_2.$$

So

$$k = L + M = L + \frac{c_2}{L},$$

and, for $L > 0$, values of $k \ge 2\sqrt{c_2}$ are eligible.

We are now prepared for a few examples.

Example 4.3. Consider the scalar equation

$$x(t) = a(t) - \int_0^t \Big[1 - (1+t-s)^{-2}\Big]x(s)ds \tag{4.27}$$

For this case $C(t,s) = 1 - (1+t-s)^{-2}, C_t(t,s) = 2(1+t-s)^{-3}, C_{tt}(t,s) = -6(1+t-s)^{-4}, C(t,t) \equiv 0, C_t(t,t) \equiv 2$ and (4.25) becomes

$$x''(t) + kx'(t) + 2x(t) = \tag{4.28}$$
$$a''(t) + ka'(t) - \int_0^t \Big[k2(1+t-s)^{-3} - 6(1+t-s)^{-4}\Big]x(s)ds$$

with eligible choices for k being $k \ge 2\sqrt{2}$. By choosing $k = 3$, we can write (4.28) as

$$x''(t) + 3x'(t) + 2x(t) = \tag{4.29}$$
$$a''(t) + 3a'(t) - \int_0^t 6\Big[(1+t-s)^{-3} - (1+t-s)^{-4}\Big]x(s)ds.$$

To apply Theorem 4.1, we need only check (4.20) for $M = 2$ and $L = 1$. Now,

$$\sup_{t\ge 0} \int_0^t e^{-2(t-u)} \int_0^u \Big|6(1+u-s)^{-3} - 6(1+u-s)^{-4}\Big|dsdu$$

$$\leq \sup_{t\geq 0}\int_0^t e^{-2(t-u)}\ 6\Big[-\frac{(1+u-s)^{-2}}{-2}\Big|_0^u-(-1)\frac{(1+u-s)^{-3}}{-3}\Big|_0^u\Big]du$$

$$\leq \sup_{t\geq 0}\int_0^t e^{-2(t-u)}\ 6\Big[\frac{(1+u-s)^{-2}}{2}\Big|_0^u-\frac{(1+u-s)^{-3}}{3}\Big|_0^u\Big]du$$

$$\leq \sup_{t\geq 0}\int_0^t e^{-2(t-u)}\ 6\Big[\frac{1}{2}-\frac{(1+u)^{-2}}{2}-\frac{1}{3}+\frac{(1+u)^{-3}}{3}\Big]du$$

$$\leq \sup_{t\geq 0}\int_0^t e^{-2(t-u)}\ 6\Big[\frac{1}{2}-\frac{1}{3}-\Big(\frac{1}{2(1+u)^2}-\frac{1}{3(1+u)^3}\Big)\Big]du$$

$$\leq \sup_{t\geq 0}\int_0^t e^{-2(t-u)}\ \Big[6\frac{1}{6}-6\Big(\frac{3(1+u)-2}{6(1+u)^3}\Big)\Big]du = \sup_{t\geq 0}\int_0^t e^{-2(t-u)}\Big[1-\frac{1+3u}{(1+u)^3}\Big]du$$

$$\leq \sup_{t\geq 0} e^{-2t}\Big[\frac{e^{2u}}{2}\Big|_0^t\Big] = \frac{1}{2}-\frac{e^{-2t}}{2}\leq \frac{1}{2} = L-\alpha.$$

Since $L=1$, we merely need to pick $\alpha=\frac{1}{2}$ in (4.20). We conclude that the solution x of (4.27) is bounded whenever $a''+3a'$ is, in turn, bounded.

Notice that the Razumikhin argument in Theorem 4.1 asked that $|a''+ka'|$ be bounded on $[0,\infty)$ and that we integrated the second coordinate of $C_t(t,s)$ and $C_{tt}(t,s)$ to obtain a boundedness result for the solution x of (4.1). We next give a Liapunov functional argument by asking that $a''+ka'$ is in $L^1[0,\infty)$ and integrating the first coordinate of $C_t(t,s)$ and $C_{tt}(t,s)$.

First, we prepare the kernel in (4.18). We have from (4.5) and (4.18) that

$$\int_0^t e^{-M(t-u)}\int_0^u\Big[kC_t(u,s)+C_{tt}(u,s)\Big]x(s)dsdu \tag{4.30}$$

$$=\int_0^t\int_s^t e^{-M(t-u)}\Big[kC_t(u,s)+C_{tt}(u,s)\Big]dux(s)ds =: \int_0^t D(t,s)x(s)ds.$$

Equation (4.18) is now

$$x' = -Lx+y(0)e^{-Mt}+\int_0^t e^{-M(t-s)}\Big(a''(s)+ka'(s)\Big)ds+\int_0^t D(t,s)x(s)ds. \tag{4.31}$$

Notice that $e^{-Mt}\in L^1[0,\infty)$. So, when $|a''+ka'|\in L^1[0,\infty)$, we have $\int_0^t e^{-M(t-s)}|a''(s)+ka'(s)|ds$ is the convolution of two L^1 functions and, therefore, the integral itself, $\int_0^t e^{-M(t-s)}|a''(s)+ka'(s)|ds$ is L^1. Thus,

$$q(t) =: y(0)e^{-Mt}+\int_0^t e^{-M(t-s)}\Big(a''(s)+ka'(s)\Big))ds\in L^1[0,\infty). \tag{4.32}$$

That is, (4.18) and (4.32) can be written

$$x'(t) = -Lx(t)+q(t)+\int_0^t D(t,s)x(s)ds. \tag{4.33}$$

For equations of the form (4.32), we define

$$V(t) = |x(t)| + \int_0^t \int_{t-s}^{\infty} |D(u+s,s)|du|x(s)|ds. \tag{4.34}$$

so that, for $L > 0$

$$V'(t) \leq |q(t)| - L|x(t)| + \int_0^t |D(t,s)||x(s)|ds$$

$$+ \int_0^{\infty} |D(u+t,t)|du|x(t)| - \int_0^t |D(t,s)||x(s)|ds$$

$$= |q(t)| + |x(t)|\Big(-L + \int_0^{\infty} |D(u+t,t)|du\Big)$$

This leads to

Theorem 4.4. *Suppose $|a'' + ka'| \in L^1$ and there exists $\alpha, 0 < \alpha < L$, such that*

$$\int_0^{\infty} D(u+t,t)du \leq L - \alpha,$$

where

$$D(t,s) = \int_s^t e^{-M(t-u)}\Big[kC_t(u,s) + C_{tt}(u,s)\Big]du$$

and k, L, M are defined as in (4.26). Then the solution x of (4.1) is in $L^1[0,\infty)$.

Proof. For

$$V(t) = |x(t)| + \int_0^t \int_{t-s}^{\infty} |D(u+s,s)|du|x(s)|ds,$$

we have

$$V'(t) \leq |q(t)| + |x(t)|\Big(-L + \int_0^{\infty} |D(u+t,t)|du\Big)$$

$$\leq |q(t)| + |x(t)|(-L + L - \alpha) = |q(t)| - \alpha|x(t)|,$$

where $\alpha > 0$ and

$$q(t) = e^{-Mt}y(0) + \int_0^t e^{-M(t-s)}|a''(s) + ka'(s)|ds \in L^1[0,\infty).$$

Thus,

$$0 \leq V(t) \leq V(0) + \int_0^t |q(s)|ds - \alpha \int_0^t |x(s)|ds$$

is bounded. That is, $|x(t)| \in L^1[0,\infty)$ since $\int_0^t |q(s)|ds \in L^1[0,\infty)$. This is the conclusion we were seeking.

□

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